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Some frank aggregation operators based on the interval-valued intuitionistic fuzzy numbers

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Abstract

In this article, we introduce the interval-valued intuitionistic fuzzy set (**IVIFS**), which are generalized forms of intuitionistic fuzzy set (**IFS**) and fuzzy set, this is because in intuitionistic fuzzy sets the non-membership function also applies to evaluations, and these sets are useful for modelling ambiguous concepts that abound in real problems. Here we try to look for new methods for more practical solutions in optimization problems for various sciences such as computer science, mathematics, engineering, medicine, psychology, climate and etc. First, with the introduction of t-norm Frank, an action we construct some Frank aggregation operators on interval-valued intuitionistic fuzzy numbers (**IVIFN**s), including the Frank weighted averaging operator, Frank-ordered weighted averaging operator, Frank hybrid weighted averaging operator, Frank geometric weighted averaging operator, Frank geometric-ordered weighted averaging operator, and Frank geometric hybrid weighted averaging operator. Also, examine some of the characteristics of these operators. In the following, we introduce two multiple attribute group decision-making methods (**MAGDM**) based on such operators. Finally, we provide illustrative examples of these methods.

Keywords: decision-making sciences, aggregation operators, nonlinear integrals, intuitionistic fuzzy set, t-conorm and t-norm Frank.
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1. Introduction

In decision-making sciences, aggregation operators can be used to investigate issues generated by information sources during the decision-making process. Aggregation operators are special functions defined on a subset of the generalized real number system. The simplest aggregation operators are arithmetic and geometric averages, these operators are not only widely used in various sciences but are

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also widely used in our daily lives, as mentioned at the end of this article. Researchers have recently introduced new aggregation operators in computer science and decision science using powerful tools in fuzzy theory. Geometric mean is one of the basic operators of aggregation, because it has special and wider characters than arithmetic mean, such as weighted geometric operators, continuous weighted geometric operators, linguistic weighted geometric operators, etc. The fuzzy set was first proposed and founded by Zadeh in 1965 [46], and others began to develop different types, such as Saadati [13, 18, 21, 22, 28, 29, 34, 36, 32, 31, 35], and Allahviranloo [1, 2, 3, 4, 5, 6, 16, 19, 26, 37] ,that they have done a lot of extensive activities in various fields of fuzzy and intuitive fuzzy, which can be very useful resources for those interested in these fields.

There are several methods for defining aggregation functions, one of which is the efficient use of integrals. Researchers have recently introduced new operators in decision science using powerful tools in fuzzy theory. It should be noted that to provide this paper, nonlinear integrals can be used in combination in the structure of these operators. Sugeno and choquet integrals are widely used examples of this generalization, [20]. Such integrals combine information from multiple sources in order to achieve a final fuzzy classification.

Based on fuzzy set theory, Atanassov [7], [8] proposed the concept of intuitionistic fuzzy set (IFS), which is composed by a membership function and a nonmembership function. IFS overcomes a disadvantage of the fuzzy set which can only have a membership. Later, Atanassov and Gargov [9] and Atanassov [10] further proposed the interval-valued intuitionistic fuzzy set (IVIFS) in which the membership function and nonmembership function are extended to interval numbers, and defined some operations and relations of IVIFS. Liu et al, [25] proposed the interval-valued intuitionistic fuzzy entropy. Zhang et al, Xu and Yager [45] developed a new similarity measure between IVIFSs and utilized it to the group decision-making problems with interval-valued intuitionistic fuzzy numbers (IVIFNs). Wang et al, [39] defined a new score function for the IVIFNs based on the prospect value function, and applied it to the interval-valued intuitionistic fuzzy multicriteria decision making problems. Tan and Zhang [38] presented an extended technique for order of preference by similarity to ideal solution(TOPSIS) method for multiple attribute decision-making (MADM) problems with IVIFNs. Wang and Xu [42] presented a fractional programming method for interval-valued intuitionistic fuzzy multiattribute decision making. Gomathi Nayagam et al,[17] proposed a new accuracy function for IVIFNs.

All above aggregation operators are based on the algebraic operational rules of intuitionistic fuzzy numbers (IFNs) or IVIFNs, and the key of the algebraic operations is algebraic product and algebraic sum, which are one type of operations that can be chosen to model the intersection and union of IFNs or IVIFNs. In general, a general t-norm and t-conorm can be used to model the intersection and union of IFNs or IVIFNs [44]. Xia et al, [44] gave some operations of IFSs based on Archimedean t-conorm and t-norm which are generalizations of a lot of other t-conorms and t-norms, such as algebraic, Einstein, and Hamacher t-conorms and t-norms, and proposed the Archimedean t-conorm and t-norm-based intuitionistic fuzzy weighted averaging operator, and the Archimedean t-cornorm Hamacher t-conorm and t-norm, which are the generalization of algebraic and Einstein t-conorm and t-norm [11], are more general and more flexible.

As we know, the advantage of choosing an intuitive fuzzy number set over fuzzy numbers is that, in addition to the degree of membership, they also include the degree of non-membership, in other words, the degree of skepticism is also included in the information. This in turn leads to better and more desirable results in decision-making to solve problems, especially when there are more ambiguities and complexities of the problem. Now selecting the set of interval-valued intuitionistic fuzzy numbers, which is an extension of the set of intuitive fuzzy numbers, makes it possible for us to choose more problems. Also, because the t-norm Frank is logarithmic, it has a nonlinear structure; therefore, it is preferable to be selected compared to some other t-norm, especially those that have a linear shape. So the general conclusion is that building a aggregation operator based on t-norm Frank on interval-valued intuitionistic fuzzy numbers in solving various problems and multi-attribute decisions, especially those with more ambiguity and complexity, is a very useful tool. It will be more efficient and reliable, especially issues related to profits and losses in stock exchanges and transactions, issues related to artificial intelligence, etc.

The first, due to the importance of interval- value intuitionistic fuzzy numbers, in multiple attribute decision making sciences, in this paper, introducing t-conorm and t-norm Frank . We constract the various aggregation operators on such a set of numbers. In section second, we briefly review some basic concepts of IVIFSs and Frank t-conorm and t-norm. In Section third, we establish Frank operations of IVIFNs and their characteristics and, furthermore, develop some Frank arithmetic aggregation operators and Frank geometric aggregation operators based on IVIFNs, such as IVIFFWA operator, IVIFFOWA operator, IVIFFHWA operator, IVIFFGWA operator, IVIFFGOWA operator, and IVIFFGHWA operator. We also study some desirable properties of these operators, such as commutativity, idempotency, monotonicity, and boundedness, and some special cases in these operators. In Section fourth, based on those operators introduced in Section third, we propose two methods for MAGDM problems in which attribute values take the form of IVIFNs. In Section fifth, we give examples to illustrate the application of these methods and compare the developed methods with the existing methods.

2. Preliminaries

Definition 2.1. *[8, 10] Let $X = \{x_1, x_2, \cdots, x_n\}$ be a universe of discourse; an $IVIFS$ $\tilde{A}$ in X is given by*

$$\tilde{A} = \{< x, \tilde{u}_{\tilde{A}}(x), \tilde{v}_{\tilde{A}}(x) > x \in X\} \tag{2.1}$$

where $\tilde{u}_{\tilde{A}(x)} \subseteq [0,1]$ and $\tilde{v}_{\tilde{A}(x)} \subseteq [0,1]$ are interval numbers, under the condition $0 \le \sup(\tilde{u}_{\tilde{A}}(x)) + \sup(\tilde{v}_{\tilde{A}}(x)) \le 1$, for all $x \in X$. The numbers $\tilde{u}_{\tilde{A}(x)}$ and $\tilde{v}_{\tilde{A}(x)}$ represent the membership degree and nonmembership degree of the element x to the set $\tilde{A}$, respectively. For convenience, let $\tilde{u}_{\tilde{A}}(x_i) = [a,b]$, $\tilde{v}_{\tilde{A}}(x_i) = [c,d]$, then $\tilde{a} = ([a,b],[c,d])$ is called an $IVIFN$.

Definition 2.2. *[24] Let $\tilde{a} = ([a,b],[c,d])$ be an $IVIFN$; a score function S of $IVIFN$ $\tilde{a}$ can be represented as follows:*

$$S(\tilde{a}) = \frac{a+b-c-d}{2}. \tag{2.2}$$

Obviously, $S(\tilde{a}) \in [-1,1]$.

Definition 2.3. *[24] Let $\tilde{a} = ([a,b],[c,d])$ be an $IVIFN$; an accuracy function H of the $IVIFN$ $\tilde{a}$ can be represented as follows:*

$$H(\tilde{a}) = \frac{a+b+c+d}{2}. \tag{2.3}$$

Gomathi Nayagam et al. [17] further analyzed the deficiencies of the above accuracy function H and proposed a new accuracy function L shown as follows:

$$L(\tilde{a}) = \frac{a+b-d(1-b)-c(1-a)}{2}. \tag{2.4}$$

Based on the score function S and the accuracy function L, we can give an order relation between two $IVIFN$s, which is defined as follows.

Definition 2.4. *[24] If $\tilde{a}_1 = ([a_1, b_1], [c_1, d_1])$ and $\tilde{a}_2 = ([a_2, b_2], [c_2, d_2])$ are any two $IVIFNs$, then*

1. *If $S(\tilde{a}_1) > S(\tilde{a}_2)$, then $\tilde{a}_1 > \tilde{a}_2$.*
2. *If $S(\tilde{a}_1) = S(\tilde{a}_2)$, then $\tilde{a}_1 = \tilde{a}_2$.*
3. *If $L(\tilde{a}_1) > L(\tilde{a}_2)$, then $\tilde{a}_1 > \tilde{a}_2$.*
4. *If $L(\tilde{a}_1) = L(\tilde{a}_2)$, then $\tilde{a}_1 = \tilde{a}_2$.*

The notions of t-norm and t-conorm are important notions in fuzzy set theory, which are used to define a generalized union and intersection of fuzzy sets [15]. Based on t-norm (T) and t-conorm (T^*), a generalized union and a generalized intersection of IFSs were introduced by Deschrijver and Kerre [14].

Definition 2.5. *[14] Let A and B be any two $IFSs$; then, the generalized intersection and union of A and B are defined as follows:*

$$\begin{aligned} &A \cap_{T,T^*} B \\ &\quad =< x, T(u_A(x), u_B(x)), T^*(v_A(x), v_B(x)) > \quad x \in X \\ &A \cup_{T,T^*} B \\ &\quad =< x, T^*(u_A(x), u_B(x)), T(v_A(x), v_B(x)) > \quad x \in X \end{aligned}$$

where T denotes a t-norm and T^* a t-conorm.

Definition 2.6. *Let $\tilde{a}_1 = ([a_1, b_1], [c_1, d_1])$ and $\tilde{a}_2 = ([a_2, b_2], [c_2, d_2])$ be any two $IVIFNs$; then the generalized intersection and union of $\tilde{a}_1$ and $\tilde{a}_2$ are defined as follows:*

$$\begin{aligned} &\tilde{a}_1 \otimes_{T,T^*} \tilde{a}_2 \\ &\quad =([T(a_1, a_2), T(b_1, b_2)], [T^*(c_1, c_2), T^*(d_1, d_2)]) \\ &\tilde{a}_1 \oplus_{T,T^*} \tilde{a}_2 \\ &\quad =([T^*(a_1, a_2), T^*(b_1, b_2)], [T(c_1, c_2), T(d_1, d_2)]). \end{aligned}$$

For instance, the algebraic products $\tilde{a}_1 \otimes \tilde{a}_2$ and the algebraic sum $\tilde{a}_1 \oplus \tilde{a}_2$ on two $IVIFN$s $\tilde{a}_1$ and $\tilde{a}_2$ can be obttained by defining t-norm and t-conorm. When $T(x, y) = x \cdot y$ and $T^*(x, y) = x + y - xy$, we can get

$$\tilde{a}_1 \oplus \tilde{a}_2 =([a_1 + a_2 - a_1 a_2, b_1 + b_2 - b_1 b_2], [c_1 c_2, d_1 d_2]) \tag{2.5}$$

$$\tilde{a}_1 \otimes \tilde{a}_2 =([a_1 a_2, b_1 b_2], [c_1 + c_2 - c_1 c_2, d_1 + d_2 - d_1 d_2]) \tag{2.6}$$

$$n\tilde{a}_1 =([1 - (1 - a_1)^n, 1 - (1 - b_1)^n], [c_1^n, d_1^n]) \ n > 0 \tag{2.7}$$

$$\tilde{a}_1^n =([a_1^n, b_1^n], [1 - (1 - c_1)^n, 1 - (1 - d_1)^n]) \ n > 0. \tag{2.8}$$

Obviously, the above operational laws are the same as those given by Atanassov [10].

Definition 2.7. *[23] The Frank t-norms (T_F) and t-conorm (T_F^*) are defined for all $\lambda > 1$, by*

$$T_F(x, y) = \log_\lambda^{(1+\frac{(\lambda^x - 1)(\lambda^y - 1)}{\lambda - 1})}, \tag{2.9}$$

$$T_F^*(x, y) =1 - \log_\lambda^{(1+\frac{(\lambda^{1-x} - 1)(\lambda^{1-y} - 1)}{\lambda - 1})} \tag{2.10}$$

3. Frank operations of interval-valued intuitionistic fuzzy numbers

Frank $t-$norm and $t-$conorm, we can establish the Frank product and Frank sum of two IVIFNs, respectively. Let $\tilde{a}_1 = ([a_1, b_1], [c_1, d_1])$ and $\tilde{a}_2 = ([a_2, b_2], [c_2, d_2])$ be any two IVIFNs, and $\lambda > 1$; then, the operationd rules based on Frank $t-$norm and $t-$conorm are defined as follows:

$$\begin{aligned}
\tilde{a}_1 \oplus_F \tilde{a}_2 = & ([1 - \log_\lambda^{(1+\frac{(\lambda^{1-a_1}-1)(\lambda^{1-a_2}-1)}{\lambda-1})}, 1 - \log_\lambda^{(1+\frac{(\lambda^{1-b_1}-1)(\lambda^{1-b_2}-1)}{\lambda-1})}], \\
& [\log_\lambda^{(1+\frac{(\lambda^{c_1}-1)(\lambda^{c_2}-1)}{\lambda-1})}, \log_\lambda^{(1+\frac{(\lambda^{d_1}-1)(\lambda^{d_2}-1)}{\lambda-1})}]) \qquad (3.1) \\
\tilde{a}_1 \otimes_F \tilde{a}_2 = & ([\log_\lambda^{(1+\frac{(\lambda^{a_1}-1)(\lambda^{a_2}-1)}{\lambda-1})}, \log_\lambda^{(1+\frac{(\lambda^{b_1}-1)(\lambda^{b_2}-1)}{\lambda-1})}], \\
& [1 - \log_\lambda^{(1+\frac{(\lambda^{1-c_1}-1)(\lambda^{1-c_2}-1)}{\lambda-1})}, 1 - \log_\lambda^{(1+\frac{(\lambda^{1-d_1}-1)(\lambda^{1-d_2}-1)}{\lambda-1})}]) \qquad (3.2) \\
\tilde{a}_1^n = & ([\log_\lambda^{(1+\frac{(\lambda^{a_1}-1)^n}{(\lambda-1)^{n-1}})}, \log_\lambda^{(1+\frac{(\lambda^{b_1}-1)^n}{(\lambda-1)^{n-1}})}], \\
& [1 - \log_\lambda^{(1+\frac{(\lambda^{1-c_1}-1)^n}{(\lambda-1)^{n-1}})}, 1 - \log_\lambda^{(1+\frac{(\lambda^{1-d_1}-1)^n}{(\lambda-1)^{n-1}})}]) \qquad (3.3) \\
n\tilde{a}_1 = & ([1 - \log_\lambda^{(1+\frac{(\lambda^{1-a_1}-1)^n}{(\lambda-1)^{n-1}})}, 1 - \log_\lambda^{(1+\frac{(\lambda^{1-b_1}-1)^n}{(\lambda-1)^{n-1}})}], \\
& [\log_\lambda^{(1+\frac{(\lambda^{c_1}-1)^n}{(\lambda-1)^{n-1}})}, \log_\lambda^{(1+\frac{(\lambda^{d_1}-1)^n}{(\lambda-1)^{n-1}})}]). \qquad (3.4)
\end{aligned}$$

It is easy to prove the formulas in the following Theorem, therefore, they are omitted here.

Theorem 3.1. *Let $\tilde{a}_1 = ([a_1, b_1], [c_1, d_1])$ and $\tilde{a}_2 = ([a_2, b_2], [c_2, d_2])$ be any two $IVIFNs$, and $\lambda > 1$; then*

$$\begin{aligned}
\tilde{a}_1 \oplus_F \tilde{a}_2 = & \tilde{a}_2 \oplus_F \tilde{a}_1 \qquad (3.5) \\
\tilde{a}_1 \otimes_F \tilde{a}_2 = & \tilde{a}_2 \otimes_F \tilde{a}_1 \qquad (3.6) \\
\eta(\tilde{a}_1 \oplus_F \tilde{a}_2) = & \eta\tilde{a}_1 \oplus_F \eta\tilde{a}_2, \eta \geq 0 \qquad (3.7) \\
\eta_1\tilde{a}_1 \oplus_F \eta_2\tilde{a}_1 = & (\eta_1 + \eta_2)\tilde{a}_1, \eta_1, \eta_2 \geq 0 \qquad (3.8) \\
\tilde{a}_1^{\eta_1} \otimes_F \tilde{a}_1^{\eta_2} = & (\tilde{a}_1)^{\eta_1+\eta_2}, \eta_1, \eta_2 \geq 0 \qquad (3.9) \\
\tilde{a}_1^{\eta} \otimes_F \tilde{a}_2^{\eta} = & (\tilde{a}_1 \otimes_F \tilde{a}_2)^{\eta}, \eta \geq 0 \qquad (3.10)
\end{aligned}$$

We can give the definition of the interval-valued intuitionistic fuzzy Frank averaging operations.

Definition 3.2. *Let $\tilde{a}_j = ([a_j, b_j], [c_j, d_j])(j = 1, 2, \cdots, n)$ be a collection of the IVIFNs, and IVIFFWA:$\Omega^n \to \Omega$, if*

$$IVIFFWA(\tilde{a}_1, \tilde{a}_2, \cdots, \tilde{a}_n) = \oplus_{F}{}_{j=1}^{n}(w_j\tilde{a}_j) \qquad (3.11)$$

where Ω is the set of all $IVIFNs$, and $w = (w_1, w_2, \cdots, w_n)^T$ is weight vector of $(\tilde{a}_1, \tilde{a}_2, \cdots, \tilde{a}_n)$, such that $w_j > 0$ and $\sum_{j=1}^{n} w_j = 1$. Then, $IVIFFWA$ is called the interval-valuead intutionistic fuzzy Frank weighted averaging operator.

Theorem 3.3. *Let $\tilde{a}_j = ([a_j, b_j], [c_j, d_j])(j = 1, 2, \cdots, n)$ be a collection of the $IVIFNs$; then the results aggregated from Definition 3.2 is still an $IVIFNs$, and even,*

$IVIFFWA(\tilde{a_1}, \tilde{a_2}, \cdots, \tilde{a_n}) =$

$$([1-\log_\lambda^{(1+\frac{\prod_{j=1}^n(\lambda^{1-a_j}-1)^{w_j}}{\prod_{j=1}^n(\lambda-1)^{w_j-1}})}, 1-\log_\lambda^{(1+\frac{\prod_{j=1}^n(\lambda^{1-b_j}-1)^{w_j}}{\prod_{j=1}^n(\lambda-1)^{w_j-1}})}], [\log_\lambda^{(1+\frac{\prod_{j=1}^n(\lambda^{c_j}-1)^{w_j}}{\prod_{j=1}^n(\lambda-1)^{w_j-1}})}, \log_\lambda^{(1+\frac{\prod_{j=1}^n(\lambda^{d_j}-1)^{w_j}}{\prod_{j=1}^n(\lambda-1)^{w_j-1}})}]) \tag{3.12}$$

$$=([1-\log_\lambda^{(1+\frac{\prod_{j=1}^n(\lambda^{1-a_j}-1)^{w_j}}{(\lambda-1)^{1-n}})}, 1-\log_\lambda^{(1+\frac{\prod_{j=1}^n(\lambda^{1-b_j}-1)^{w_j}}{(\lambda-1)^{1-n}})}], [\log_\lambda^{(1+\frac{\prod_{j=1}^n(\lambda^{c_j}-1)^{w_j}}{\prod_{j=1}^n(\lambda-1)^{1-n}})}, \log_\lambda^{(1+\frac{\prod_{j=1}^n(\lambda^{d_j}-1)^{w_j}}{\prod_{j=1}^n(\lambda-1)^{1-n}})}]). \tag{3.13}$$

Proof . This Theorem can be proved by mathematical induction shown as follows:

1) When $n = 1$, $w_1 = 1$, for the left side of the (3.13), $IVIFFWA(\tilde{a_1}, \tilde{a_2}, \cdots, \tilde{a_n}) = \tilde{a_1} = ([a_1, b_1], [c_1, d_1])$ and for the right side of the (3.13), we have

$$([1-\log_\lambda^{(1+\lambda^{1-a_1}-1)}, 1-\log_\lambda^{1+\lambda^{1-b_1}-1}], [\log_\lambda^{(1+\lambda^{c_1}-1)}, \log_\lambda^{1+\lambda^{d_1}-1}]) = ([a_1, b_1], [c_1, d_1])$$

Therfore, (3.13) holds for $n = 1$.
2) Assume that (3.13) holds for $n = k$, we have $IVIFFWA(\tilde{a}_1, \tilde{a}_2, \cdots, \tilde{a}_k) =$

$$([1-\log_\lambda^{(1+\frac{\prod_{j=1}^k(\lambda^{1-a_j}-1)^{w_j}}{\prod_{j=1}^k(\lambda-1)^{w_j-1}})}, 1-\log_\lambda^{(1+\frac{\prod_{j=1}^k(\lambda^{1-b_j}-1)^{w_j}}{\prod_{j=1}^k(\lambda-1)^{w_j-1}})}], [\log_\lambda^{(1+\frac{\prod_{j=1}^k(\lambda^{c_j}-1)^{w_j}}{\prod_{j=1}^k(\lambda-1)^{w_j-1}})}, \log_\lambda^{(1+\frac{\prod_{j=1}^k(\lambda^{d_j}-1)^{w_j}}{\prod_{j=1}^k(\lambda-1)^{w_j-1}})}]).$$

When $n = k + 1$,

$IVIFFWA(\tilde{a}_1, \tilde{a}_2, \cdots, \tilde{a}_{k+1}) = IVIFFWA(\tilde{a}_1, \tilde{a}_2, \cdots, \tilde{a}_k) \oplus_F (w_{k+1}\tilde{a}_{k+1})$

$$= \quad ([1-\log_\lambda^{(1+\frac{\prod_{j=1}^k(\lambda^{1-a_j}-1)^{w_j}}{\prod_{j=1}^k(\lambda-1)^{w_j-1}})}, 1-\log_\lambda^{(1+\frac{\prod_{j=1}^n(\lambda^{1-b_j}-1)^{w_j}}{\prod_{j=1}^k(\lambda-1)^{w_j-1}})}], [\log_\lambda^{(1+\frac{\prod_{j=1}^k(\lambda^{c_j}-1)^{\omega_j}}{\prod_{j=1}^k(\lambda-1)^{w_j-1}})}, \log_\lambda^{(1+\frac{\prod_{j=1}^k(\lambda^{d_j}-1)^{w_j}}{\prod_{j=1}^k(\lambda-1)^{w_j-1}})}])$$

$$\oplus_F ([1-\log_\lambda^{(1+\frac{(\lambda^{1-a_{k+1}}-1)^{w_{k+1}}}{(\lambda-1)^{w_{k+1}-1}})}, 1-\log_\lambda^{(1+\frac{(\lambda^{1-b_{k+1}}-1)^{w_{k+1}}}{(\lambda-1)^{w_{k+1}-1}})}], [\log_\lambda^{(1+\frac{(\lambda^{c_{k+1}}-1)^{w_{k+1}}}{(\lambda-1)^{w_{k+1}-1}})}, \log_\lambda^{(1+\frac{(\lambda^{d_{k+1}}-1)^{w_{k+1}}}{(\lambda-1)^{w_{k+1}-1}})}])$$

$$= ([1-\log_\lambda^{(1+\frac{\prod_{j=1}^{k+1}(\lambda^{1-a_j}-1)^{w_j}}{\prod_{j=1}^{k+1}(\lambda-1)^{w_j-1}})}, 1-\log_\lambda^{(1+\frac{\prod_{j=1}^{k+1}(\lambda^{1-b_j}-1)^{w_j}}{\prod_{j=1}^{k+1}(\lambda-1)^{w_j-1}})}], [\log_\lambda^{(1+\frac{\prod_{j=1}^{k+1}(\lambda^{c_j}-1)^{\omega_j}}{\prod_{j=1}^{k+1}(\lambda-1)^{w_j-1}})}, \log_\lambda^{(1+\frac{\prod_{j=1}^{k+1}(\lambda^{d_j}-1)^{w_j}}{\prod_{j=1}^{k+1}(\lambda-1)^{w_j-1}})}]).$$

Therefore, when $n = k + 1$, (3.13) holds.
3) According to steps 1 and 2, we can get (3.13) holds for any n. It is easy to prove that the $IVIFFWA$ operator has the following properties. □

Theorem 3.4. *(Monotonicity) Let* $(\tilde{a}'_1, \tilde{a}'_2, \cdots, \tilde{a}'_n), (\tilde{a}_1, \tilde{a}_2, \cdots, \tilde{a}_n)$ *be two collections of* $IVIFNs$*, if* $\tilde{a}'_j \leq \tilde{a}_j$ *for all* $j = 1, 2, \cdots, n$*; then* $IVIFFWA(\tilde{a}'_1, \tilde{a}'_2, \cdots, \tilde{a}'_n) \leq IVIFFWA(\tilde{a}_1, \tilde{a}_2, \cdots, \tilde{a}_n)$*.*

Theorem 3.5. *(Idempotency) Let* $\tilde{a}'_j = \tilde{a}, j = 1, 2, \cdots, n$*; then,* $IVIFFWA(\tilde{a}_1, \tilde{a}_2, \cdots, \tilde{a}_n) = \tilde{a}$*.*

Proof .

$$\begin{aligned} IVIFFWA(\tilde{a}_1, \tilde{a}_2, \cdots, \tilde{a}_n) =& \oplus_{F}{}_{j=1}^{n} w_j \tilde{a}_j \\ =& w_1\tilde{a}_1 \oplus w_2\tilde{a}_2 \oplus \cdots \oplus w_n\tilde{a}_n \\ =& w_1\tilde{a} \oplus w_2\tilde{a} \oplus \cdots \oplus w_n\tilde{a} \\ =& (w_1 \oplus w_2 \oplus \cdots \oplus w_n)\tilde{a} = \tilde{a}. \end{aligned}$$

□

Theorem 3.6. *(Boundedness)*

$$\min(\tilde{a}_1, \tilde{a}_2, \cdots, \tilde{a}_n) \leq IVIFFWA(\tilde{a}_1, \tilde{a}_2, \cdots, \tilde{a}_n) \leq \max(\tilde{a}_1, \tilde{a}_2, \cdots, \tilde{a}_n).$$

Proof . This theorem prove able by Theorem 3.4. □

Example 3.7. *Let*

$$\tilde{a}_1 = ([0.22, 0.31], [0.23, 0.54]), \tilde{a}_2 = ([0.04, 0.21], [0.35, 0.46]), \tilde{a}_3 = ([0.25, 0.27], [0.23, 0.40])$$

be three $IVIFNs$, and $w = (0.314, 0.355, 0.331)^T$ be the weight vector of $\tilde{a}_j$ $j = (1, 2, 3)$, i.e, $a_1 = 0.22$, $a_2 = 0.04$, $a_3 = 0.25$, $b_1 = 0.31$, $b_2 = 0.21$, $b_3 = 0.27$, $c_1 = 0.23$, $c_2 = 0.35$, $c_3 = 0.23$, $d_1 = 0.54$, $d_2 = 0.46$, $d_3 = 0.40$. Suppose that $\lambda = 2$; then

$$\begin{aligned} & IVIFFWA(\tilde{a}_1, \tilde{a}_2, \tilde{a}_3) \\ =& ([1 - \log_2^{(1+(2^{1-0.22}-1)^{0.314}(2^{1-0.04}-1)^{0.355}(2^{1-0.25}-1)^{0.331})}, \\ & \quad 1 - \log_2^{(1+(2^{1-0.31}-1)^{0.314}(2^{1-0.21}-1)^{0.355}(2^{1-0.27}-1)^{0.331})}], \\ & [\log_2^{(1+(2^{0.23}-1)^{0.314}(2^{0.35}-1)^{0.355}(2^{0.23}-1)^{0.331})}, \\ & \log_2^{(1+(2^{0.54}-1)^{0.314}(2^{0.46}-1)^{0.355}(2^{0.40}-1)^{0.331})}]) = ([0.170, 0.261], [0.268, 0.462]). \end{aligned}$$

Definition 3.8. *Let $\tilde{a}_j = ([a_j, b_j], [c_j, d_j])$ $(j = 1, 2, \cdots, n)$ be a collection of the $IVIFNs$, and $IVIFFOWA : \Omega^n \to \Omega$, if*

$$IVIFFOWA(\tilde{a}_1, \tilde{a}_2, \cdots, \tilde{a}_n) = \oplus_{F}{}_{j=1}^{n}(\omega_j \tilde{a}_{\sigma(j)}) \tag{3.14}$$

where Ω is the set of all $IVIFNs$ and $\omega = (\omega_1, \omega_2, \cdots, \omega_n)^T$ is the weighted vector associated with $IVIFFOWA$ such that $\omega_j > 0$ and $\sum_{j=1}^{n} \omega_j = 1$. $(\sigma(1), \sigma(2), \cdots, \sigma(n))$ is a permutation of $(1, 2, \cdots, n)$, such that $\tilde{a}_{\sigma(j-1)} \geq \tilde{a}_{\sigma(j)}$ for any j. Then $IVIFFOWA$ is called the interval-valued intuitionistic fuzzy Frank-ordered weighted averaging operator.

Theorem 3.9. *Let $\tilde{a}_j = ([a_j, b_j], [c_j, d_j])$ $(j = 1, 2, \cdots, n)$ be a collection of the $IVIFNs$; then, the result aggregated from Definition 3.8 is still an $IVIFN$, and even*

$$\begin{aligned} & IVIFFOWA(\tilde{a}_1, \tilde{a}_2, \cdots, \tilde{a}_n) \\ =& ([1 - \log_\lambda^{(1+\frac{\prod_{j=1}^{n}(\lambda^{1-a_{\sigma(j)}}-1)^{\omega_j}}{\prod_{j=1}^{n}(\lambda-1)^{\omega_j}})}, 1 - \log_\lambda^{(1+\frac{\prod_{j=1}^{n}(\lambda^{1-b_{\sigma(j)}}-1)^{\omega_j}}{\prod_{j=1}^{n}(\lambda-1)^{\omega_j}})}], \\ & [\log_\lambda^{(1+\frac{\prod_{j=1}^{n}(\lambda^{c_{\sigma(j)}}-1)^{\omega_j}}{\prod_{j=1}^{n}(\lambda-1)^{\omega_j}})}, \log_\lambda^{(1+\frac{\prod_{j=1}^{n}(\lambda^{d_{\sigma(j)}}-1)^{\omega_j}}{\prod_{j=1}^{n}(\lambda-1)^{\omega_j}})}]). \end{aligned} \tag{3.15}$$

Proof . Can be proved by mathematical induction, and it is omitted here. □

Remark 3.10. *Similar to the $IVIFFWA$ operator, the $IVIFFOWA$ operator also has the properties of monotonicity, idempotency and boundedness. In addition, it is easy to prove that the $IVIFFOWA$ operator has the commutativity.*

Theorem 3.11. *(Commutativity) Let $(\tilde{a}'_1, \tilde{a}'_2, \cdots, \tilde{a}'_n), (\tilde{a}_1, \tilde{a}_2, \cdots, \tilde{a}_n)$ be two collections of $IVIFNs$, and $(\tilde{a}'_1, \tilde{a}'_2, \cdots, \tilde{a}'_n)$ is any permutation of $(\tilde{a}_1, \tilde{a}_2, \cdots, \tilde{a}_n)$, then,*

$$IVIFFOWA(\tilde{a}'_1, \tilde{a}'_2, \cdots, \tilde{a}'_n) = IVIFFOWA(\tilde{a}_1, \tilde{a}_2, \cdots, \tilde{a}_n).$$

Example 3.12. *Let*

$$\tilde{a}_1 = ([0.22, 0.31], [0.23, 0.54]), \tilde{a}_2 = ([0.04, 0.21], [0.35, 0.46]), \tilde{a}_3 = ([0.25, 0.27], [0.23, 0.40])$$

be three $IVIFNs$, and $\omega = (0.314, 0.355, 0.331)^T$ be the position weighted vector. Similar to Example 3.7, we suppose $\lambda = 2$. The simple steps are shown as follows
1) Calculating score functions of $\tilde{a}_1, \tilde{a}_2$, and $\tilde{a}_3$ by (2.2), we can get

$$S(\tilde{a}_1) = \frac{(0.22 + 0.31 - 0.23 - 0.54)}{2} = -0.12$$
$$S(\tilde{a}_2) = \frac{(0.04 + 0.21 - 0.35 - 0.46)}{2} = -0.26$$
$$S(\tilde{a}_3) = \frac{(0.25 + 0.27 - 0.23 - 0.40)}{2} = -0.055$$

2) Rank the $IVIFNs$, $\tilde{a}_1, \tilde{a}_2$, and $\tilde{a}_3$ by Definition 2.4, we get $\tilde{a}_3 > \tilde{a}_1 > \tilde{a}_2$.
3) Get $\sigma(j)(j = 1, 2, 3)$ by the ranking of $\tilde{a}_1, \tilde{a}_2$ and $\tilde{a}_3$, we have $\sigma(1) = 3, \sigma(2) = 1, \sigma(3) = 2$.
4) Calulate for $\lambda = 2$

$$1 - \log_{\lambda}^{(1+\frac{\prod_{j=1}^{3}(\lambda^{1-a_{\sigma(j)}}-1)^{\omega_j}}{(\lambda-1)^{1-3}})} = 1 - \log_2^{(1+(2^{1-0.25}-1)^{0.314}(2^{1-0.22}-1)^{0.355}(2^{1-0.04}-1)^{0.331})} = 0.173$$
$$1 - \log_{\lambda}^{(1+\frac{\prod_{j=1}^{3}(\lambda^{1-b_{\sigma(j)}}-1)^{\omega_j}}{(\lambda-1)^{1-3}})} = 1 - \log_2^{(1+(2^{1-0.27}-1)^{0.314}(2^{1-0.31}-1)^{0.355}(2^{1-0.21}-1)^{0.331})} = 0.265$$
$$\log_{\lambda}^{(1+\frac{\prod_{j=1}^{3}(\lambda^{c_{\sigma(j)}}-1)^{\omega_j}}{(\lambda-1)^{1-3}})} = \log_2^{(1+(2^{0.23}-1)^{0.314}(2^{0.23}-1)^{0.355}(2^{0.35}-1)^{0.331})} = 0.265$$
$$\log_{\lambda}^{(1+\frac{\prod_{j=1}^{3}(\lambda^{d_{\sigma(j)}}-1)^{\omega_j}}{(\lambda-1)^{1-3}})} = \log_2^{(1+(2^{0.40}-1)^{0.314}(2^{0.54}-1)^{0.355}(2^{0.46}-1)^{0.331})} = 0.466$$

5) Calculating $IVIFFOWA(\tilde{a}_1, \tilde{a}_2, \tilde{a}_3)$,

$$IVIFFOWA(\tilde{a}_1, \tilde{a}_2, \tilde{a}_3) = ([0.173, 0.265], [0.265, 0.466]).$$

In order to improve the shortcomings of these operators, we introduce the Frank hybrid weighted averaging operator as follows.

Definition 3.13. *Let* $\tilde{a}_j = ([a_j, b_j], [c_j, d_j])(j = 1, 2, \cdots, n)$ *be a collection of the IVIFNs, and* $IVIFFHWA : \Omega^n \to \Omega$*, if*

$$IVIFFHWA(\tilde{a}_1, \tilde{a}_2, \cdots, \tilde{a}_n) = \oplus_F{}_{j=1}^{n}(\omega_j \tilde{b}_{\sigma(j)}) \tag{3.16}$$

Where Ω *is the set of all IVIFNs, and* $\omega = (\omega_1, \omega_2, \cdots, \omega_n)^T$ *is the weighted vector associated with IVIFFHWA, such that* $\omega_j > 0$ *and* $\sum_{j=1}^{n} \omega_j = 1$. $W = (w_1, w_2, \cdots, w_n)$ *is the weight vector of* $\tilde{a}_j (j = 1, 2, \cdots, n)$*, and* $w_j \in [0, 1], \sum_{j=1}^{n} w_j = 1$. *Let* $\tilde{b}_j = nw_j\tilde{a}_j = ([\dot{a}_j, \dot{b}_j], [\dot{c}_j, \dot{d}_j])$*; n is the adjustment factor. Suppose* $(\sigma(1), \sigma(2), \cdots, \sigma(n))$ *is a permutation of* $(1, 2, \cdots, n)$*, and then, function IVIFFHWA is called the interval-valued intuitionistic fuzzy Frank hybrid weighted averaging operator.*

Theorem 3.14. *Let* $\tilde{a}_j = ([a_j, b_j], [c_j, d_j])(j = 1, 2, \cdots, n)$ *be a collection of the IVIFNs; then, the result is aggregated from Definition 3.13 is still an IVIFN, and even*

$$\begin{aligned} &IVIFFHWA(\tilde{a}_1, \tilde{a}_2, \cdots, \tilde{a}_n) \\ &=([1-\log_\lambda^{(1+\frac{\prod_{j=1}^{n}(\lambda^{1-\dot{a}_{\sigma(j)}}-1)^{\omega_j}}{(\lambda-1)^{1-n}})}, 1-\log_\lambda^{(1+\frac{\prod_{j=1}^{n}(\lambda^{1-\dot{b}_{\sigma(j)}}-1)^{\omega_j}}{(\lambda-1)^{1-n}})}], \\ &[\log_\lambda^{(1+\frac{\prod_{j=1}^{n}(\lambda^{\dot{c}_{\sigma(j)}}-1)^{\omega_j}}{(\lambda-1)^{1-n}})}, \log_\lambda^{(1+\frac{\prod_{j=1}^{n}(\lambda^{\dot{d}_{\sigma(j)}}-1)^{\omega_j}}{(\lambda-1)^{1-n}})}]). \end{aligned} \tag{3.17}$$

Theorem 3.15. *The IVIFFWA operator and IVIFFOWA operator are the special case of the IVIFFHWA operator. It is easy to prove that when* $W = (\frac{1}{n}, \frac{1}{n}, \cdots, \frac{1}{n})$*, the IVIFFHWA operator will reduce to IVIFFOWA operator, and when* $\omega = (\frac{1}{n}, \frac{1}{n}, \cdots, \frac{1}{n})$ *will reduce to IVIFFWA operator.*

Example 3.16. *Let*

$$\tilde{a}_1 = ([0.22, 0.31], [0.23, 0.54]), \tilde{a}_2 = ([0.04, 0.21], [0.35, 0.46]), \tilde{a}_3 = ([0.25, 0.27], [0.23, 0.40])$$

be three IVIFNs, $W = (0.314, 0.355, 0.331)^T$ *be the weight vector of* $\tilde{a}_j (j = 1, 2, 3)$*, and* $\omega = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})^T$ *be the position weighted vector. Supposing* $\lambda = 2$*, the calculation steps of the IVIFFHWA operator are shown as follows.*
1) Calculating $\tilde{b}_j = 3w_j\tilde{a}_j = ([\dot{a}_j, \dot{b}_j], [\dot{c}_j, \dot{d}_j])$ *by (3.4), we can get*

$$\begin{aligned} \tilde{b}_1 &= (3w_1)\tilde{a}_1 = ([\dot{a}_1, \dot{b}_1], [\dot{c}_1, \dot{d}_1]) \\ &= ([1-\log_2^{1.731}, 1-\log_2^{1.630}], [\log_2^{1.191}, \log_2^{1.475}]) = ([0.208, 0.293], [0.253, 0.561]). \\ \tilde{b}_2 &= (3w_2)\tilde{a}_2 = ([\dot{a}_2, \dot{b}_2], [\dot{c}_2, \dot{d}_2]) \\ &= ([0.043, 0.222], [0.325, 0.435]). \\ \tilde{b}_3 &= (3w_3)\tilde{a}_3 = ([\dot{a}_3, \dot{b}_3], [\dot{c}_3, \dot{d}_3]) \\ &= ([0.248, 0.268], [0.233, 0.403]). \end{aligned}$$

2) Calculating score functions of $\tilde{b}_1, \tilde{b}_2$*, and* $\tilde{b}_3$ *by (2.2), we can get*

$$S(\tilde{b}_1) = \frac{(0.208 + 0.293 - 0.253 - 0.561)}{2} = -0.156,$$
$$S(\tilde{b}_2) = -0.247, S(\tilde{b}_3) = -0.060.$$

3) Rank $\tilde{b}_1, \tilde{b}_2$, and $\tilde{b}_3$ by Definition 2.4, we can get $\tilde{b}_3 > \tilde{b}_1 > \tilde{b}_2$.
4) Get $\sigma(j)(j = 1, 2, 3)$ by the ranking of $\tilde{b}_1, \tilde{b}_2$ and $\tilde{b}_3$, we have $\sigma(1) = 3, \sigma(2) = 1, \sigma(3) = 2$.
5) Calculate by Theorem 3.14,

$$1-\log_2^{(1+\prod_{j=1}^{3}(2^{1-\dot{a}_{\sigma(j)}}-1)^{\omega_j})} = 1-\log_2^{(1+(2^{1-0.248}-1)^{\frac{1}{3}}(2^{1-0.208}-1)^{\frac{1}{3}}(2^{1-0.043}-1)^{\frac{1}{3}})} = 0.170$$
$$1-\log_2^{(1+\prod_{j=1}^{3}(2^{1-\dot{b}_{\sigma(j)}}-1)^{\omega_j})} = 1-\log_2^{(1+(2^{1-0.268}-1)^{\frac{1}{3}}(2^{1-0.293}-1)^{\frac{1}{3}}(2^{1-0.222}-1)^{\frac{1}{3}})} = 0.261$$
$$\log_2^{(1+\prod_{j=1}^{3}(2^{\dot{c}_{\sigma(j)}}-1)^{\omega_j})} = \log_2^{(1+(2^{0.233}-1)^{\frac{1}{3}}(2^{0.253}-1)^{\frac{1}{3}}(2^{0.325}-1)^{\frac{1}{3}})} = 0.268$$
$$\log_2^{(1+\prod_{j=1}^{3}(2^{\dot{d}_{\sigma(j)}}-1)^{\omega_j})} = \log_2^{(1+(2^{0.0.403}-1)^{\frac{1}{3}}(2^{0.561}-1)^{\frac{1}{3}}(2^{0.435}-1)^{\frac{1}{3}})} = 0.462.$$

As a result

$$IVIFFHWA(\tilde{a}_1, \tilde{a}_2, \tilde{a}_3) = ([0.170, 0.261], [0.268, 0.462]).$$

Obviously, the result of IVIFFHWA$(\tilde{a}_1, \tilde{a}_2, \tilde{a}_3)$ is the same as one of IVIFFWA$(\tilde{a}_1, \tilde{a}_2, \tilde{a}_3)$ in Example 3.7; this is because the position weighted vector $\omega = (1/3, 1/3, 1/3)^T$. Therefore, we also with this example showed that the IVIFFWA operator is a special case of the IVIFFHWA operator.

3.1. Interval-valued intuitionistic fuzzy hybrid geometric operator

Definition 3.17. *Let $\tilde{a}_j = ([a_j, b_j], [c_j, d_j])(j = 1, 2, \cdots, n)$ be a collection of the $IVIFNs$, and $IVIFFGWA : \Omega^n \to \Omega$, if*

$$IVIFFGWA(\tilde{a}_1, \tilde{a}_2, \cdots, \tilde{a}_n) = \otimes_{F}{}_{j=1}^{n}(\tilde{a}_j^{\omega_j}) \tag{3.18}$$

$W = (w_1, w_2, \cdots, w_n)^T$ is weight vector of $(\tilde{a}_1, \tilde{a}_2, \cdots, \tilde{a}_n)$, such that $w_j > 0$ and $\sum_{j=1}^{n} w_j = 1$. Then, $IVIFFGWA$ is called the interval-valued intuitionistic fuzzy Frank geometric weighted averaging operator.

Theorem 3.18. *Let $\tilde{a}_j = ([a_j, b_j], [c_j, d_j])(j = 1, 2, \cdots, n)$ be a collection of the $IVIFNs$, then, the result aggregated from Definition 3.17 is still an $IVIFN$, and even*

$$\begin{aligned} &IVIFFGWA(\tilde{a}_1, \tilde{a}_2, \cdots, \tilde{a}_n) \\ &= ([\log_\lambda^{(1+\frac{\prod_{j=1}^{n}(\lambda^{a_j}-1)^{\omega_j}}{(\lambda-1)^{1-n}})}, \log_\lambda^{(1+\frac{\prod_{j=1}^{n}(\lambda^{b_j}-1)^{\omega_j}}{(\lambda-1)^{1-n}})}], \\ &[1-\log_\lambda^{(1+\frac{\prod_{j=1}^{n}(\lambda^{1-c_j}-1)^{\omega_j}}{(\lambda-1)^{1-n}})}, 1-\log_\lambda^{(1+\frac{\prod_{j=1}^{n}(\lambda^{d_j}-1)^{\omega_j}}{(\lambda-1)^{1-n}})}]). \end{aligned} \tag{3.19}$$

Similar to the $IVIFFWA$ operator, the $IVIFFGWA$ operator also has the propecties of monotonicity, idempotency, and boundedness.

Definition 3.19. *Let $\tilde{a}_j = ([a_j, b_j], [c_j, d_j])$ $(j = 1, 2, \cdots, n)$ be a collection of the $IVIFNs$, and $IVIFFGOWA : \Omega^n \to \Omega$, if*

$$IVIFFGOWA(\tilde{a}_1, \tilde{a}_2, \cdots, \tilde{a}_n) = \otimes_{F}{}_{j=1}^{n}(\tilde{a}_{\sigma(j)}^{\omega_j}) \tag{3.20}$$

where Ω is the set of all $IVIFNs$ and $\omega = (\omega_1, \omega_2, \cdots, \omega_n)^T$ is the weighted vector associated with $IVIFFGOWA$ such that $\omega_j > 0$ and $\sum_{j=1}^{n} \omega_j = 1$. $(\sigma(1), \sigma(2), \cdots, \sigma(n))$ is a permutation of $(1, 2, \cdots, n)$, such that $\tilde{a}_{\sigma(j-1)} \geq \tilde{a}_{\sigma(j)}$ for any j. Then $IVIFFGOWA$ is called the interval-valued intuitionistic fuzzy Frank geometric-orderd weighted averaging operator.

Theorem 3.20. *Let $\tilde{a}_j = ([a_j, b_j], [c_j, d_j])$ $(j = 1, 2, \cdots, n)$ be a collection of the IVIFNs; then, the result aggregated from Definition 3.19 is still an IVIFN, and even*

$$IVIFFGOWA(\tilde{a}_1, \tilde{a}_2, \cdots, \tilde{a}_n) = ([\log_\lambda^{(1+\frac{\prod_{j=1}^{n}(\lambda^{a_{\sigma(j)}}-1)^{\omega_j}}{(\lambda-1)^{1-n}})}, \log_\lambda^{(1+\frac{\prod_{j=1}^{n}(\lambda^{b_{\sigma(j)}}-1)^{\omega_j}}{(\lambda-1)^{1-n}})}], [1-\log_\lambda^{(1+\frac{\prod_{j=1}^{n}(\lambda^{1-c_{\sigma(j)}}-1)^{\omega_j}}{(\lambda-1)^{1-n}})}, 1-\log_\lambda^{(1+\frac{\prod_{j=1}^{n}(\lambda^{d_{\sigma(j)}}-1)^{\omega_j}}{(\lambda-1)^{1-n}})}]).$$

Similar to the IVIFFOWA operator, the IVIFFGOWA operator also has the properties of monotonicity, idempotency, boundedness and commutativity.

Definition 3.21. *Let $\tilde{a}_j = ([a_j, b_j], [c_j, d_j])(j = 1, 2, \cdots, n)$ be a collection of the IVIFNs, and $IVIFFGHWA : \Omega^n \to \Omega$, if*

$$IVIFFGHWA(\tilde{a}_1, \tilde{a}_2, \cdots, \tilde{a}_n) = \otimes_{F\,j=1}^{\,n}(\tilde{b}_{\sigma(j)}^{\omega_j}) \tag{3.21}$$

where Ω is the set of all IVIFNs, and $\omega = (\omega_1, \omega_2, \cdots, \omega_n)^T$ is the weighted vector associated with IVIFFGWA, such that $\omega_j > 0$ and $\sum_{j=1}^{n} \omega_j = 1$. $W = (w_1, w_2, \cdots, w_n)$ is the weight vector of $\tilde{a}_j(j = 1, 2, \cdots, n)$, and $w_j \in [0, 1], \sum_{j=1}^{n} w_j = 1$. Let $\tilde{b}_j = \tilde{a}_j^{nw_j} = ([\dot{a}_j, \dot{b}_j], [\dot{c}_j, \dot{d}_j])$; n is the adjustment factor. Suppose $(\sigma(1), \sigma(2), \cdots, \sigma(n))$ is a permutation of $(1, 2, \cdots, n)$, such that $\tilde{b}_{\sigma(j-1)} \geq \tilde{b}_{\sigma(j)}$ for any j, and then, IVIFFGHWA is called the interval-valued intuitionistic fuzzy Frank geometric hybrid weighted averaging operator.

Theorem 3.22. *Let $\tilde{a}_j = ([a_j, b_j], [c_j, d_j])(j = 1, 2, \cdots, n)$ be a collection of the IVIFNs; then, the result is aggregated from Definition 3.21 is still an IVIFN, and even*

$$IVIFFGHWA(\tilde{a}_1, \tilde{a}_2, \cdots, \tilde{a}_n) = ([\log_\lambda^{(1+\frac{\prod_{j=1}^{n}(\lambda^{\dot{a}_j}-1)^{\omega_j}}{(\lambda-1)^{1-n}})}, \log_\lambda^{(1+\frac{\prod_{j=1}^{n}(\lambda^{\dot{b}_j}-1)^{\omega_j}}{(\lambda-1)^{1-n}})}], [1-\log_\lambda^{(1+\frac{\prod_{j=1}^{n}(\lambda^{1-\dot{c}_j}-1)^{\omega_j}}{(\lambda-1)^{1-n}})}, 1-\log_\lambda^{(1+\frac{\prod_{j=1}^{n}(\lambda^{1-\dot{d}_j}-1)^{\omega_j}}{(\lambda-1)^{1-n}})}]).$$

Theorem 3.23. *The IVIFFGWA operator and IVIFFGOWA operator are the special case of the IVIFFGHWA operator. It is easy to prove that when $W = (\frac{1}{n}, \frac{1}{n}, \cdots, \frac{1}{n})$, the IVIFFGHWA operator will reduce to IVIFFGOWA operator, and when $\omega = (\frac{1}{n}, \frac{1}{n}, \cdots, \frac{1}{n})$ the IVIFFGHWA will reduce to IVIFFGWA operator.*

4. Multiple attribute group decision-making methods based on frank aggregation operators

In this section, we will use these Frank aggregation operators to the $MAGDM$ problems in which the attribute weights take the form of real numbers and attributes values take the form of $IVIFNs$, and we suggest two tools to solve such problems as follows.

4.1. Destiption the decision-making problems

For a $MAGDM$ problem, let $E = \{e_1, e_2, \cdots, e_q\}$ be the collection of decision makers, $A = \{A_1, A_2, \cdots, A_m\}$ be the collection of alternatives, and $C = \{C_1, C_2, \cdots, C_n\}$ be the collection of attributes. Suppose that $\tilde{a}_{ij}^k = ([a_{ij}^k, b_{ij}^k], [c_{ij}^k, d_{ij}^k])$ is an attribute value given by the decision marker e_k, which is expressed by an $IVIFN$ for the alternative A_i with respect to the attribute C_j, $W = (w_1, w_2, \cdots, w_n)$ is the wight vector of attribute set $C = \{C_1, C_2, \cdots, C_n\}$, and $w_j \in [0, 1]$, $\sum_{j=1}^{n} w_j = 1$. Let $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_q)$ be the weight vector of decision-markers $\{e_1, e_2, \cdots, e_q\}$, and $\lambda_k \in [0, 1]$, $\sum_{k=1}^{q} \lambda_k = 1$. Then we use the attribute weights, the decision makers'weights, and the attribute values to rank the order of the alternatives.

4.2. First method: The method based on the Frank hybrid weighted averaging operator

The decision-making steps of this method are shown as follows.

Step 1: Normalize the decision-making information:
In general, for attribute values, there are benefit attributes (I_1) (the bigger the attribute values the better) and cost attributes (I_2) (the smaller the attribute values the better). In order to eliminate the impact of different type attribute values, we need to normalize the decision-making information. Of couurse, if all the attributes are of the same type, then they do not need normalization.
We may transform the attribute values from cost type to benefit type; in such a case, decision matrices $A^k = [\tilde{a}_{ij}^k]_{m\times n}$ $(k = 1, 2, \cdots, q)$ can be transformed into matrices $R^k = [\tilde{r}_{ij}^k]_{m\times n}$ $(k = 1, 2, \cdots, q)$ where

$$\begin{aligned} \tilde{r}_{ij}^k &= \left([\underline{t}_{ij}^k, \overline{t}_{ij}^k], [\underline{f}_{ij}^k, \overline{f}_{ij}^k]\right) \\ &= \begin{cases} ([a_{ij}^k, b_{ij}^k], [c_{ij}^k, d_{ij}^k]) & C_j \in I_1 \\ ([c_{ij}^k, d_{ij}^k], [a_{ij}^k, b_{ij}^k]) & C_j \in I_2 \end{cases} \end{aligned} \tag{4.1}$$

$i = 1, 2, \cdots, m, \quad j = 1, 2, \cdots, n.$

Step 2: Utilize the $IVIFFHWA$ operator:

$$\begin{aligned} \tilde{r}_{ij} &= \left([\underline{t}_{ij}, \overline{t}_{ij}], [\underline{f}_{ij}, \overline{f}_{ij}]\right) \\ &= IVIFFHWA(\tilde{r}_{ij}^1, \tilde{r}_{ij}^2, \cdots, \tilde{r}_{ij}^q). \end{aligned} \tag{4.2}$$

to aggregate all the individual interval-valued intuitionistic fuzzy decision matrixes $R^k = [\tilde{r}_{ij}^k]_{m\times n}$ $(k = 1, 2, \cdots, q)$ into the collective interval-valued intuitionistic fuzzy decision matrixes $R = [\tilde{r}_{ij}]_{m\times n}$.

Step 3: Utilize the $IVIFFHWA$ operator:

$$\begin{aligned} \tilde{r}_{i} &= \left([\underline{t}_{i}, \overline{t}_{i}], [\underline{f}_{i}, \overline{f}_{i}]\right) \\ &= IVIFFHWA(\tilde{r}_{i1}, \tilde{r}_{i2}, \cdots, \tilde{r}_{in}) \end{aligned} \tag{4.3}$$

to derive the collective overall preference values $\tilde{r}_i (i = 1, 2, \cdots, m)$.

Step 4: Calculate the score function $S(\tilde{r}_i) (i = 1, 2, \cdots, m)$ of the collective overal values $\tilde{r}_i (i = 1, 2, \cdots, m)$, and then, rank all the alternatives $\{A_1, A_2, \cdots, A_m\}$. When two score functions $S(\tilde{r}_i)$ and $S(\tilde{r}_j)$ are equal, we need to calculate their accuracy functions $L(\tilde{r}_i)$ and $L(\tilde{r}_j)$, and then, we can rank them by accuracy functions.

Step 5: Rank the alternatives:
Rank all the alternatives $\{A_1, A_2, \cdots, A_m\}$ and select the best one(s) by score function $S(\tilde{r}_i)$ and accuracy function $L(\tilde{r}_i)$.

Step 6: End.

4.3. Second method: The method based on the Frank geometric hybrid weighted averaging operator

The decision-making steps of this method are shown as follows.

step 1: Normalize the decision-making information:
It is same as step 1 in the first method.

step 2: Utilize the $IVIFFGHWA$ operator:

$$\begin{aligned}\tilde{r}_{ij} =& \Big([\underline{t}_{ij}, \overline{t}_{ij}], [\underline{f}_{ij}, \overline{f}_{ij}]\Big) \\ =& IVIFFGHWA(\tilde{r}^1_{ij}, \tilde{r}^2_{ij}, \cdots, \tilde{r}^q_{ij})\end{aligned} \tag{4.4}$$

to aggregate all the individual interval-valued intuitionistic fuzzy decision matrixes $R^k = [\tilde{r}^k_{ij}]_{m\times n}$ $(k = 1, 2, \cdots, q)$ into the collective interval-valued intuitionistic fuzzy decision matrixes $R = [\tilde{r}_{ij}]_{m\times n}$.

Step 3: Utilize the $IVIFFGHWA$ operator:

$$\begin{aligned}\tilde{r}_{i} =& \Big([\underline{t}_{i}, \overline{t}_{i}], [\underline{f}_{i}, \overline{f}_{i}]\Big) \\ =& IVIFFGHWA(\tilde{r}_{i1}, \tilde{r}_{i2}, \cdots, \tilde{r}_{in})\end{aligned} \tag{4.5}$$

to derive the collective overall preference values $\tilde{r}_i (i = 1, 2, \cdots, m)$.

Step 4-5: It is same as steps 4-5 in the first method.

Step 6: End.

5. Applications and examples

Now, we give an example and solve it with the first method.

Example 5.1. *In order to demonstrate the application of proposed methods, we will cite an example about the air quality evaluation that is used in, [24]. To evaluate the air quality of Guangzhou for the 16th Asian Olympic Games which would be held during November 12-27, 2010, the air quality in Guangzhou for the November of 2006, 2007, 2008 and 2009 were collected in order to find out the trends and to forecast the situation in 2010. There are three air-quality monitoring stations* (e_1, e_2, e_3) *which can be regared as decision makers, and their weight is* $\lambda = (0.314, 0.355, 0.331)^T$. *There are three measured indexes, namely,* $SO_2(C_1)$, $NO_2(C_2)$, *and* $PM_{10}(C_3)$, *and their weight is* $W = (0.40, 0.20, 0.40)$. *The measured values from air-quality monitoring stations under these indexes are shown in Table 1-3 and they can be expressed by* $IVIFNs$. *Let* $\{A_1, A_2, A_3, A_4\}$ =\{*November of 2006, November of 2007, November of 2008, November of 2009*\} *be the set of alternatives, please give the rank of air quality from 2006 to 2009. We adopt two proposed methods to rank the alternatives.*

To get the best alternative(s), the following steps are involved.

step 1: Normalize the decision-making information: Because all the measured values of the same type, they do not need normalization.

Table 1: AIR QUALITY DATA FROM STATION e_1

	C_1	C_2	C_3
A_1	([0.22, 0.31], [0.23, 0.54])	([0.13, 0.53], [0.20, 0.36])	([0.12, 0.37], [0.40, 0.56])
A_2	([0.28, 0.41], [0.33, 0.49])	([0.33, 0.53], [0.20, 0.36])	([0.12, 0.37], [0.30, 0.46])
A_3	([0.32, 0.41], [0.23, 0.44])	([0.43, 0.53], [0.16, 0.25])	([0.23, 0.45], [0.21, 0.37])
A_4	([0.39, 0.47], [0.18, 0.36])	([0.39, 0.53], [0.27, 0.32])	([0.28, 0.34], [0.11, 0.23])

Table 2: AIR QUALITY DATA FROM STATION e_2

	C_1	C_2	C_3
A_1	([0.40, 0.21], [0.35, 0.46])	([0.10, 0.34], [0.27, 0.45])	([0.32, 0.37], [0.13, 0.20])
A_2	([0.32, 0.39], [0.27, 0.39])	([0.03, 0.57], [0.30, 0.36])	([0.16, 0.25], [0.14, 0.19])
A_3	([0.26, 0.37], [0.21, 0.40])	([0.23, 0.43], [0.06, 0.15])	([0.21, 0.35], [0.11, 0.29])
A_4	([0.30, 0.43], [0.19, 0.35])	([0.28, 0.43], [0.31, 0.34])	([0.39, 0.46], [0.01, 0.17])

Table 3: AIR QUALITY DATA FROM STATION e_3

	C_1	C_2	C_3
A_1	([0.25, 0.27], [0.23, 0.40])	([0.17, 0.27], [0.26, 0.40])	([0.21, 0.30], [0.17, 0.32])
A_2	([0.25, 0.29], [0.33, 0.39])	([0.18, 0.46], [0.43, 0.50])	([0.06, 0.21], [0.28, 0.30])
A_3	([0.22, 0.27], [0.27, 0.31])	([0.13, 0.37], [0.16, 0.20])	([0.11, 0.24], [0.14, 0.19])
A_4	([0.30, 0.48], [0.09, 0.45])	([0.08, 0.53], [0.20, 0.24])	([0.32, 0.61], [0.01, 0.09])

step 2: *Utilize the $IVIFFHWA$ operator expressed by (4.2) to agregate all the individual interval-valued intuitionistic fuzzy decision matrixes $R^k = [\tilde{r}^k_{ij}]_{4\times3}(\tilde{r}^k_{ij} = ([\underline{t}^k_{ij}, \overline{t}^k_{ij}], [\underline{f}^k_{ij}, \overline{f}^k_{ij}]), k = 1, 2, 3)$ into the collective interval-valued intuitionistic fuzzy decision matrixes $R = [\tilde{r}_{ij}]_{4\times3}$; suppose $(\gamma = 2, \omega = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3}))$.*
For example, $\tilde{r}_{11}$ can be calculated in Example 3.16 ($\tilde{r}_{11} = IVIFFHWA(\tilde{r}^1_{11}, \tilde{r}^2_{11}, \tilde{r}^3_{11})$, $\tilde{r}^1_{11}, \tilde{r}^2_{11}, \tilde{r}^3_{11}$ can be expressed by $\tilde{a}_1, \tilde{a}_2$ and $\tilde{a}_3$, and the weight vector λ is expressed by W in Example 3.16). Therefor can we get
R=

$$\begin{bmatrix} ([0.170, 0.261], [0.268, 0.462]) & ([0.133, 0.385], [0.243, 0.44]) & ([0.224, 0.347], [0.204, 0.326]) \\ ([0.285, 0.365], [0.307, 0.419]) & ([0.181, 0.523], [0.299, 0.402]) & ([0.115, 0.277], [0.225, 0.294]) \\ ([0.266, 0.351], [0.225, 0.379]) & ([0.268, 0.402], [0.113, 0.194]) & ([0.184, 0.349], [0.146, 0.273]) \\ ([0.329, 0.459], [0.146, 0.384]) & ([0.256, 0.496], [0.257, 0.298]) & ([0.333, 0.482], [0.021, 0.152]) \end{bmatrix}$$

step 3: *Utilize the $IVIFFHWA$ operator expressed by (4.3) to drive the collective overal prefernce values (suppose $\lambda = 2, \omega = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$); they are*

$$\begin{aligned} \tilde{r_1} &= ([0.175, 0.335], [0.237, 0.393]) \\ \tilde{r_2} &= ([0.194, 0.401], [0.275, 0.368]) \\ \tilde{r_3} &= ([0.240, 0.369], [0155, 0.269]) \\ \tilde{r_4} &= ([0.305, 0.480], [0.095, 0.260]). \end{aligned}$$

Step 4: *Calculating the score function $S(\tilde{r}_i), (i = 1, 2, 3, 4)$ of the collective overal values $\tilde{r}_i, (i = 1, 2, 3, 4)$ we can get $S(\tilde{r_1}) = -0.06, S(\tilde{r_2}) = -0.024, S(\tilde{r_3}) = 0.0925, S(\tilde{r_4}) = 0.215$.*

step 5: Rank the alternatives:
According to the score function $S(\tilde{r}_i)(i = 1, 2, 3, 4)$, we can get the ranking of alternatives $A_1, A_2, A_3, A_4 : A_4 \succ A_3 \succ A_2 \succ A_1$.
Thererfore, the best alternative is A_4, i.e., the best air quality in Guangzhou is November of 2009 among the Novembers of 2006, 2007, 2008, 2009.

It shoud be noted that, using the second method, we obtain the same result as the first method. That's why stopped writing it.

Here, we present another example that we solve with the second method.

Example 5.2. *Let us consider a high-tech company, which aims to select a supplier of USB connectors (adapted from [30]).There are four potential suppliers A_i $(i = 1, 2, 3, 4)$ that have been designated for further evaluation, and assume that there are four attributes to be considered in the evaluation process: C1: financial; C2: performance; C3: technical capacity; and C4: organizational culture and strategy, and $\omega = (0.35, 0.25, 0.25, 0.15)^T$ is the weight vector of them. The decision committee contains three decision makers e_1, e_2, e_3 including engineering expert, financial expert, and quality control expert, whose weight vector is $W = (0.35, 0.35, 0.3)^T$. The decision makers $e_i(i = 1, 2, 3)$ express the attribute values of the potential supplier $A_i(i = 1, 2, 3, 4)$ with respect to $C_i(i = 1, 2, 3, 4)$ by IVIFNs, respectively, which are listed in Tables 3, 4, and 5. (suppose $\lambda = 2, \omega = (\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}))$. With these assumptions, and with the help Theorem 3.23, the IVIFFGWA operator can be used instead of the IVIFFGHWA operator. Based on the IVIFFGWA operator, the decision-making steps are shown as follows;*

Table 4: ATTRIBUTE VALUES OF ENGINEERING EXPERT e_1

	C_1	C_2	C_3	C_4
A_1	([0.4, 0.5], [0.7, 0.8])	([0.5, 0.6], [0.8, 0.9])	([0.6, 0.7], [0.9, 1])	([0.6, 0.7], [0.9, 1])
A_2	([0.5, 0.6], [0.8, 0.9])	([0.5, 0.6], [0.8, 0.9])	([0.4, 0.5], [0.7, 0.8])	([0.6, 0.7], [0.9, 1])
A_3	([0.3, 0.4], [0.6, 0.7])	([0.4, 0.5], [0.7, 0.8])	([0.2, 0.3], [0.5, 0.6])	([0.3, 0.4], [0.6, 0.7])
A_4	([0.2, 0.3], [0.5, 0.6])	([0.6, 0.7], [0.9, 1])	([0.3, 0.4], [0.6, 0.7])	([0.4, 0.5], [0.7, 0.8])

Table 5: ATTRIBUTE VALUES OF FINANCIAL EXPERT e_2

	C_1	C_2	C_3	C_4
A_1	([0.5, 0.6], [0.8, 0.9])	([0.4, 0.5], [0.7, 0.8])	([0.5, 0.6], [0.8, 0.9])	([0.4, 0.5], [0.7, 0.8])
A_2	([0.6, 0.7], [0.9, 1])	([0.6, 0.7], [0.9, 1])	([0.5, 0.6], [0.8, 0.9])	([0.5, 0.6], [0.8, 0.9])
A_3	([0.2, 0.3], [0.5, 0.6])	([0.5, 0.6], [0.8, 0.9])	([0.1, 0.2], [0.4, 0.5])	([0.2, 0.3], [0.5, 0.6])
A_4	([0.3, 0.4], [0.6, 0.7])	([0.4, 0.5], [0.7, 0.8])	([0.2, 0.3], [0.5, 0.6])	([0.1, 0.2], [0.4, 0.5])

Table 6: ATTRIBUTE VALUES OF QUALITY EXPERT e_3

	C_1	C_2	C_3	C_4
A_1	([0.4, 0.5], [0.7, 0.8])	([0.5, 0.6], [0.8, 0.9])	([0.4, 0.5], [0.7, 0.8])	([0.5, 0.6], [0.8, 0.9])
A_2	([0.3, 0.4], [0.6, 0.7])	([0.6, 0.7], [0.9, 1])	([0.4, 0.5], [0.7, 0.8])	([0.6, 0.7], [0.9, 1])
A_3	([0.1, 0.2], [0.4, 0.5])	([0.6, 0.7], [0.9, 1])	([0.3, 0.4], [0.6, 0.7])	([0.4, 0.5], [0.7, 0.8])
A_4	([0.2, 0.3], [0.5, 0.6])	([0.2, 0.3], [0.5, 0.6])	([0.1, 0.2], [0.4, 0.5])	([0.3, 0.4], [0.6, 0.7])

To get the best alternative(s), the following steps are involved.

step 1: *Consider that all the attributes are benefit type; therefore, the decision matrices do not need normalization.*

step 2: *Utilize the* $IVIFFGWA$ *operator expressed by Definition 3.17 to agregate all the individual interval-valued intuitionistic fuzzy decision matrixes* $R^k = [\tilde{r}^k_{ij}]_{4\times 4}(\tilde{r}^k_{ij} = ([\underline{t}^k_{ij}, \overline{t}^k_{ij}], [\underline{f}^k_{ij}, \overline{f}^k_{ij}]), k = 1,2,3,4)$ *into the collective interval-valued intuitionistic fuzzy decision matrixes* $R = [\tilde{r}_{ij}]_{4\times 4}$.

Therefor can we get,
$R =$

$$\begin{bmatrix} ([0.4367, 0.5372], [0.7393, 0.8427]) & ([0.4667, 0.5671], [0.7692, 0.8722]) & ([0.5105, 0.6122], [0.8217, 1]) & ([0.5059, 0.6078], [0.8180, 1]) \\ ([0.4862, 0.5893], [0.8047, 1]) & ([0.5671, 0.6679], [0.8722, 1]) & ([0.4367, 0.5372], [0.7393, 0.8427]) & ([0.5671, 0.6679], [0.8722, 1]) \\ ([0.2081, 0.3086], [0.5105, 0.6122]) & ([0.5005, 0.6022], [0.8116, 1]) & ([0.1980, 0.2986], [0.5005, 0.6022]) & ([0.2986, 0.3994], [0.6022, 0.7051]) \\ ([0.2362, 0.3364], [0.5372, 0.6379]) & ([0.4283, 0.5333], [0.7577, 1]) & ([0.2081, 0.3086], [0.5105, 0.6122]) & ([0.2733, 0.3750], [0.5807, 0.6862]) \end{bmatrix}$$

step 3: *Utilize the* $IVIFFGWA$ *operator expressed by (4.5) to drive the collective overal prefernce values (suppose* $\lambda = 2, \omega = (\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4})$*); they are*

$$\begin{aligned} \tilde{r_1} &= ([0.4655, 0.5663], [0.7674, 0.8677]) \\ \tilde{r_2} &= ([0.4918, 0.5938], [0.7961, 0.8969]) \\ \tilde{r_3} &= ([0.2531, 0.3645], [0.5741, 0.6766]) \\ \tilde{r_4} &= ([0.2488, 0.3571], [0.5644, 0.6663]). \end{aligned}$$

Step 4: *Calculating the score function* $S(\tilde{r_i}), (i = 1,2,3,4)$ *of the collective overal values* $\tilde{r}_i, (i = 1,2,3,4)$ *we can get* $S(\tilde{r_1}) = 1.2027, S(\tilde{r_2}) = 1.2719, S(\tilde{r_3}) = 0.9022, S(\tilde{r_4}) = 0.8821.$

step 5: *Rank the alternatives:*
According to the score function $S(\tilde{r_i})(i = 1,2,3,4)$*, we can get the ranking of alternatives* $A_1, A_2, A_3, A_4 : A_2 \succ A_1 \succ A_3 \succ A_4$.
Thererfore, the best alternative is A_2.

Conclusion

The existing aggregation operators for IFS or IVIFNs are based on algebraic t-norm and t-conorm or Einstein t-norm and t-conorm. In this paper, some new aggregation operators for IVIFNs based on Frank t-norm and t-conorm, such as IVIFFWA, IVIFFOWA, IVIFFHWA, IVIFFGWA, IVIFFGOWA, and IVIFFGHWA operators, have been proposed, and various properties of these operators have been investigated. Then, they are applied to solve the MAGDM problems in which attribute values take the form of IVIFNs.
A practical suggestion that can be used in this paper is to follow the construction of Frank aggregation operators using nonlinear integrals such as Sugeno and choquet integral, which are introduced in [20]. Because, as mentioned earlier, nonlinear integrals are more practical in decision making science.

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