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Existence results for common solution of equilibrium and vector equilibrium problems

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Abstract

In this paper, by using the notion of locally segment-dense subsets and sequentially sign property for bifunctions, we establish existence results for a common solution of a finite family of equilibrium problems in the setting of Hausdorff locally convex topological vector spaces. Also similar results obtain for vector equilibrium problems.

Keywords: Equilibrium problem, Common solution, Locally segment-dense, Sequentially sign property.

1. Introduction

Let X be a real Hausdorff, locally convex topological vector space and K be a nonempty subset of X. An equilibrium problem associated to f and K, or briefly $EP(f, K)$ in the sense of Blum and Oettli [7], is stated as follows:

$$find\ x^* \in K\ such\ that\ f(x^*, x) \geqslant 0\ \ \forall\ x \in K,$$

that $f : K \times K \to \mathbb{R}$ is a bifunction. We denote the set of solutions $EP(f, K)$, by $S(f, K)$. This problem is also called Ky Fan inequality due to his contribution to this field [11]. It is well known that some important problems such as convex programs, variational inequalities, fixed point, Nash equilibrium models and minimax problems can be formulated as an equilibrium problem (see e.g. [7, 10, 22] and the references therein).
In 2015 Laszla and Viorel [19] introduced a notion of a self-segment-dense set in order to establish

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some existence results for set-valued equilibrium problems, where the conditions are imposed on a self-segment-dense subset of the domain of the involved bifunction. Jafari et al. in [14], presented a new concept "locally segment-dense set" and study existence results for equilibrium problems where the conditions are imposed only on a locally segment-dense subset in the domain of the involved bifunction. Indeed, the locally segment-dense sets need not necessarily be dense in the whole convex subset under consideration. Using the mentioned approach [3, 5, 9, 12, 13, 14, 16, 17, 18], we study existence results for common solution of equilibrium problems where the conditions are imposed only on a locally segment-dense subset in the domain of the involved bifunction. For this purpose, we recall that notions of finding a common solution of a finite family of equilibrium problems, locally segment-dense, Minty solutions, sequentially sign property, and also present notions the common S-property of the considered bifunction.

2. Preliminaries

Let X be a real Hausdorff locally convex topological vector space, X^* its dual and $\langle\cdot,\cdot\rangle$ the duality pairing between X and X^*. Given a set $A \subseteq X$, $convA$ is the convex hull of A and $cl(A)$ is the closure of A. Suppose that $[x, y] := \{(1-t)x + ty : t \in [0, 1]\}$ is the closed segment joining x and y. Similarly, we can define semiopen segments $[x, y[$, $]x, y]$ and the open segment $]x, y[$.
In this paper, we consider the problem of finding a solution of a system of equilibrium problems in [14]. This problem, so-called the common solutions to equilibrium problems ($CSEP$), is stated as follows:
Let K be a nonempty subset of X and let for all $1 \leqslant i \leqslant N$, $f_i : K \times K \to \mathbb{R}$ be bifunctions. The common solutions to equilibrium problems ($CSEP$) is a problem of finding $\bar{x} \in K$ such that for every $1 \leqslant i \leqslant N$,

$$f_i(\bar{x}, y) \geqslant 0, \quad \forall\, y \in K, \qquad (CSEP)$$

The set of solutions of ($CSEP$) is denoted by $S(f_1, f_2, \cdots, f_N; K)$. Obviously, $S(f_1, f_2, \cdots, f_N; K) = \bigcap_{i=1}^{N} S(f_i; K)$.

Also, an element $\bar{x} \in K$ is a local Minty common solution (introduced in [5] by Bianchi and Pini), if there exists a neighbourhood U of $\bar{x}$ such that for every $1 \leqslant i \leqslant N$,

$$f_i(y, \bar{x}) \leqslant 0, \quad \forall\, y \in K \cap U.$$

The set of all local Minty common solutions is denoted by

$$M_L(f_1, f_2, \cdots, f_N; K).$$

Obviously, $M_L(f_1, f_2, \cdots, f_N; K) = \bigcap_{i=1}^{N} M_L(f_i; K)$. For example, if $X = K := \mathbb{R}$ and $f_1, f_2 : K \times K \to \mathbb{R}$ are defined by $f_1(x, y) = y^2 - x^2$ and $f_2(x, y) = 2y^2 - x^2$, then $\bar{x} = 0 \in M_L(f_1, f_2; K)$.
In 2016, Alleche and Radulescu, provide the following necessary and sufficient condition for the lower semi-continuity of functions. This property is useful for the main results of this paper.

Proposition 2.1. *[1] Let X be Hausdorff topological space, $g : X \to \mathbb{R}$ be a function and A be a subset of X. Then, the following conditions are equivalent:*

1. *g is lower semi-continuous on A;*
2. *for every $a \in \mathbb{R}$, $cl(\{x \in X : g(x) \leqslant a\}) \cap A = \{x \in A : g(x) \leqslant a\}$.*

In particular, if g is lower semi-continuous on A, then the intersection A with any lower level set of g is closed in A.

Let X be a real Hausdorff locally convex topological vector space and $x, y \in X$. The well-known segment-dense sets have been introduced by Luc [20]. Let $K \subseteq X$ be a convex set. We say that $U \subseteq K$ is segment-dense in K iff for each $x \in K$, there exists $y \in U$ such that x is a cluster point of the set $[x, y] \cap U$. In 2015, Laszlo and Viorel [19] introduced a notion of a self-segment-dense set, which is slightly different from the notion of the segment-dense set introduced by Luc [20]. Let K be a convex subset of X and $U \subseteq K \subseteq X$. The set U is called self-segment-dense in K iff U is dense in K and for every $x, y \in U$, $cl([x; y] \cap U) = [x, y]$.
Laszlo and Viorel [19] presented some examples and explained the difference between dense, segment-dense and self-segment-dense sets. Jafari et al. in [14], presented a concept of locally segment-dense sets. Let K be a convex subset of X and $D \subseteq K \subseteq X$. The set D is called locally segment dense in K, iff for every $x, y \in D$, $cl([x, y] \cap D) = [x, y]$; and for every $x \in D$ and $y \in K$, the set $]x, y] \cap D$ is nonempty. Notice that it can be concluded, $cl([x, z] \cap D) = [x, z]$ for every $z \in]x, y] \cap D$. As the next example shows, we can find locally segment-dense sets in K, which is neither segment-dense in K nor self-segment-dense in it.

Example 2.2. *Let $X = K := \mathbb{R}^2$, and let $D := \{(x, y) : x \in \mathbb{Q} \cap]-1, 1[, \; y \in]-1, 1[\}$, where $\mathbb{Q}$ denotes the set of all rational numbers. It is clear that D is locally segment-dense in K, but not dense in K.*

Jafari et al. in [14] noted that even in one dimension, the concept of a locally segment-dense is different the concept of a segment-dense set and a self-segment-dense set. Also, they provided an example for their claim.

Remark 2.3. *[14] It is worth mentioning that if U is a convex open neighbourhood of an element $x \in X$, then U is locally segment-dense in X. Indeed, every convex algebraically open subset $U \subseteq X$ is locally segment-dense in X. We recall that U is algebraically open (due to [15]) if $U = core(U)$, where*

$$core(U) := \{\bar{x} \in U : \forall \; x \in X \; \exists \; \bar{t} > 0 \text{ such that } \bar{x} + tx \in U, \; \forall \; t \in [0, \bar{t}]\}.$$

Remark 2.4. *Suppose D be a locally segment-dense set in K. If $x \in D$ and $y \in K$, then there can be found $\{z_n\} \subset]x, y] \cap D$ such that $z_n \to x$ as $n \to +\infty$. This is due to the definition of locally segment-dense set D in K, which allows us to find $z \in]x, y] \cap D$ such that $cl([x, z] \cap D) = [x, z]$.*

We need the following useful lemma for the main results of this paper.

Lemma 2.5. *Let X be a real Hausdorff locally convex topological vector space, K be a convex subset of X, and let $U \subseteq K$ be such that for every $x, y \in U$, it holds that $cl([x, y] \cap U) = [x, y]$. Then for all finite subsets $\{u_1, u_2, \cdots, u_n\} \subseteq U$, one has*

$$cl(conv\{u_1, u_2, \cdots, u_n\} \cap U) = conv\{u_1, u_2, \cdots, u_n\}.$$

Throughout this paper, if not otherwise specified, X stands for a real Hausdorff locally convex topological vector space and K denotes a convex subset of X.
For our purpose, we need the following notions of convexity of functions.

Definition 2.6. *Let D be a locally segment-dense set in K, and let $g : K \to \mathbb{R}$ be a function. We say that g is* *(i)* *quasiconvex on* D*, iff for all* $x, y \in D$ *and* $t \in [0,1]$ *such that* $(1-t)x + ty \in D$*, then* $g((1-t)x+ty) \leqslant \max\{g(x), g(y)\}$*;*

(ii) *semistrictly quasiconvex iff for all* $x, y \in K$ *such that* $g(x) \neq g(y)$ *it holds that* $g((1-t)x+ty) < \max\{g(x), g(y)\}$*, for all* $t \in]0,1[$*.*

Definition 2.7. *Let* D *be a locally segment-dense set in* K*, and let* $f : K \times K \to \mathbb{R}$ *be a bifunction. We say that* f *is quasimonotone on* D*, iff for* $x, y \in D$

$$f(x, y) > 0 \Rightarrow f(y, x) \leqslant 0.$$

Definition 2.8. *Let* D *be a locally segment-dense set in* K*, and let* $f : K \times K \to \mathbb{R}$ *be a bifunction. We say that* f *is properly quasimonotone on* D*, iff for every subset of finite elements* $x_1, x_2, \cdots, x_n \subseteq D$ *and every* $\bar{x} \in conv x_1, x_2, \cdots, x_n \cap D$*, there exists* $j \in 1, 2, \cdots, n$ *such that* $f(x_j, \bar{x}) \leqslant 0$*.*

Motivated by the notion of the strong upper sign property introduced in [13], we define a useful notion of sequentially sign property for bifunctions in this subsection.

Definition 2.9. *[14] Let* D *be a locally segment-dense set in* K*, and let* $f : K \times K \to \mathbb{R}$ *be a bifunction. We say that* f *has the sequentially sign property with respect to the first variable at* $x \in K$*, iff for every* $y \in K$ *the following implication holds:*

$$if\ \{z_n\} \subset]x, y] \cap D : z_n \to x \ and\ f(z_n, x) \leqslant 0, \forall\ n \in \mathbb{N}\ then\ f(x, y) \geqslant 0.$$

We say that f *has the sequentially sign property on* D*, iff* f *has this property at every* $x \in D$*.*

Also, Jafari et al. [14], provided a proposition and introduced a large class of bifunctions, that have the sequentially sign property.
In the following, we give a notion of locally segment-dense Minty common solution, that is needed to obtain existence result for $(CSEP)$.

Definition 2.10. *Let* D *be a locally segment-dense set in* K*, and let for every* $1 \leqslant i \leqslant N$*,* $f_i : K \times K \to \mathbb{R}$ *be bifunctions.We say that* $\bar{x} \in D$ *is a locally segment-dense Minty common solution, iff there exists a neighborhood* U *of* $\bar{x}$ *such that for every* $1 \leqslant i \leqslant N$*,*

$$f_i(y, \bar{x}) \leqslant 0,\ \forall\ y \in D \cap U.$$

$M_L^D(f_1, f_2, \cdots, f_N; K)$ *denotes the set of all locally segment-dense Minty common solutions. Obviously,* $M_L^D(f_1, f_2, \cdots, f_N; K) = \bigcap_{i=1}^{N} M_L^D(f_i; K)$*.*

It is notice that $M_L(f_1, f_2, \cdots, f_N; K) \subseteq M_L^D(f_1, f_2, \cdots, f_N; K)$ and the inclusion may be strict. Hence $M_L(f_1, f_2, \cdots, f_N; K)$ may be empty and $M_L^D(f_1, f_2, \cdots, f_N; K)$ may be nonempty.(See the following example)

Example 2.11. *Let* $X = K := \mathbb{R}$ *and* $D :=]-1, 1[\cap \mathbb{Q}$*. Consider two bifunctions* $f_1, f_2 : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ *defined by*

$$f_1(x, y) := \begin{cases} -1, & if\ x, y \in D, \\ 1, & otherwise, \end{cases}$$

and

$$f_2(x,y) := \begin{cases} y^2 - x^2, & if\ x, y \in D, \\ \\ x^2 + y^2, & otherwise. \end{cases}$$

Obviously, $M_L(f_1, f_2; K) = \varnothing$ *and* $\bar{x} = 0 \in M_L^D(f_1, f_2; K)$ *and hence* $M_L^D(f_1, f_2; K) \neq \varnothing$.

We have underlined that under a mild assumption of convexity, the sequentially sign property is a weak form of continuity, weaker than the upper hemicontinuity of f on D. In the following lemma, we show that for every $1 \leqslant i \leqslant N$, the rather large set $M_L^D(f_1, f_2, \cdots, f_N; K)$ is a subset of $S(f_1, f_2, \cdots, f_N; K)$ under the weak condition of the sequentially sign property of the involved bifunctions.

Lemma 2.12. *Let D be a locally segment-dense set in K, and let for every $1 \leqslant i \leqslant N$, $f_i : K \times K \to \mathbb{R}$ be a bifunction with the sequentially sign property. Then $M_L^D(f_1, f_2, \cdots, f_N; K) \subseteq S(f_1, f_2, \cdots, f_N; K)$.*

Proof . The proof follows immediately from intersection property and Lemma 2.2 in [14]. □

3. Existence results for common solution of equilibrium problems

By using the locally segment-dense set, we obtain some existence results for common solution equilibrium problems on non compact domains.
For real bifunctions $f_1, f_2, \cdots, f_N$ on $K \times K$, let $F_1 : K \rightrightarrows K$ be a set-valued mapping by

$$F_1(y) := \big\{x \in K : f_i(y, x) \leqslant 0, \quad \forall\ 1 \leqslant i \leqslant N\big\},$$

for all $y \in K$.

Definition 3.1. *Let D be a locally segment-dense set in K and let $f_1, f_2, \cdots, f_N : K \times K \to \mathbb{R}$ be bifunctions. We say that the bifunctions $f_1, f_2, \cdots, f_N$ have the common S-property on D, if the following condition holds:*
For every nonempty subset $A_1, A_2, \cdots, A_N$ of D if for all $1 \leqslant i \leqslant N$,

$$\exists\ \bar{x}_i \in conv A_i \cap D \ s.t.\ f_i(x, \bar{x}_i) > 0 \ \ \forall\ x \in A_i,$$

then there exists some $\bar{x} \in conv(A_1 \cup A_2 \cup \cdots \cup A_N) \cap D$ such that for all $1 \leqslant i \leqslant N$,

$$f_i(z, \bar{x}) > 0, \quad \forall\ z \in (A_1 \cup A_2 \cup \cdots \cup A_N) \cap D.$$

The following theorem is one of the main results of this paper.

Theorem 3.2. *Let D be a locally segment-dense set in K, and let for every $1 \leqslant i \leqslant N$, $f_i : K \times K \to \mathbb{R}$ be bifunctions satisfying the following conditions:*

(i) for every $1 \leqslant i \leqslant N$, f_i is quasimonotone on D, which is not properly quasimonotone on D;
(ii) $f_1, f_2, \cdots, f_N$ have the common s-property on D;
(iii) for every $y \in D$, $F_1(y)$ is closed in $K \setminus D$, i.e.,

$$cl(F_1(y)) \cap (K \setminus D) = F_1(y) \cap (K \setminus D) = \{x \in K \setminus D : f_i(y, x) \leqslant 0 \ \ 1 \leqslant i \leqslant N\};$$

(iv) for every $x_1, x_2 \in F_1(y) \cap D$ *and* $t \in [0,1]$ *such that* $\bar{x} = (1-t)x_1 + tx_2 \in D$*, then* $\bar{x} \in F_1(y)$*.*

Then $M_L^D(f_1, f_2, \cdots, f_N; K) \neq \varnothing$*.*

Proof . Since for every $1 \leqslant i \leqslant N$, f_i is not properly quasimonotone on D, there exist $x_{i1}, x_{i2}, \cdots, x_{in_i} \in D$ and $\bar{x}_i \in conv\{x_{i1}, x_{i2}, \cdots, x_{in_i}\} \cap D$ such that for every $j \in \{1, 2, \cdots, n_i\}$

$$f_i(x_{ij}, \bar{x}_i) > 0.$$

Thus, $\bar{x}_i \notin F_1(x_{ij}) \cap (K \setminus D)$. Hence for every $1 \leqslant i \leqslant N$ and every $j \in \{1, 2, \cdots, n_i\}$,

$$\bar{x}_i \notin cl\big(F_1(x_{ij})\big) \cap (K \setminus D).$$

For every $1 \leqslant i \leqslant N$, set $A_i = \{x_{i1}, x_{i2}, \cdots, x_{in_i}\}$. Since $f_1, f_2, \cdots, f_N$ have the common s-property on D, there exists $\bar{x} \in conv(A) \cap D$ $(A = A_1 \cup A_2 \cup \cdots \cup A_N)$ such that for every $z \in A \cap D$,

$$f_i(z, \bar{x}) > 0, \quad (1 \leqslant i \leqslant N).$$

Thus for each $z \in A \cap D$ that $A \cap D$ is finite there exists a neighborhood U_z of $\bar{x}$ such that

$$U_z \cap D \subseteq \Big(X \setminus \big(F_1(z) \cap D\big)\Big).$$

We set $U = \bigcap_{z \in A \cap D} U_z$. So for every $y \in U \cap D$ and $z \in A \cap D$, we get

$$f_i(z, y) > 0, \quad (1 \leqslant i \leqslant N).$$

Now, the quasimonotonicity of each f_i on D implies that for every $y \in U \cap D$ and $z \in A \cap D$:

$$f_i(y, z) \leqslant 0, \quad (1 \leqslant i \leqslant N).$$

Furthermore, for arbitrary and fixed $y \in U \cap D$, we have $z \in F_1(y)$. Using the convexity of $F_1(y)$ on D, we deduce that for all $y \in U \cap D$,

$$f_i(y, \bar{x}) \leqslant 0, \quad (1 \leqslant i \leqslant N).$$

Hence, $\bar{x} \in M_L^D(f_1, f_2, \cdots, f_N; K)$ and this completes the proof. □

Example 3.3. *Let* $X = K := \mathbb{R}$ *and* $D :=]0,1[\cap \mathbb{Q}$*. Consider two bifunctions* $f_1, f_2 : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ *defined by* $f_1(x, y) := y^2(y - x)$ *and*

$$f_2(x, y) := \begin{cases} x^2(y-x)^3, & if\ x, y \in D, \\ \\ 1, & otherwise. \end{cases}$$

Obviousely, all the conditions of Theorem 3.2 are satisfied, and hence $M_L^D(f_1, f_2; K) \neq \varnothing$*.*

In the following corollary, we provide some conditions on the bifunctions $f_1, f_2, \cdots, f_N$ to guarantee that the set-valued mapping F_1 satisfies the conditions (iii) and (iv) of Theorem 3.2.

Corollary 3.4. *Let* D *be a locally segment-dense set in* K*, and let for every* $1 \leqslant i \leqslant N$*,* $f_i : K \times K \to \mathbb{R}$ *be a bifunction satisfying the following conditions:*

(i) for every $1 \leqslant i \leqslant N$, f_i *is quasimonotone on* D*, which is not properly quasimonotone on* D*;*

(ii) $f_1, f_2, \cdots, f_N$ *have the common* S*-property on* D*;*

(iii) for every $1 \leqslant i \leqslant N$ *and* $x \in D$, $f_i(., x)$ *is lower semicontinuous on* $K \setminus D$*;*

(iv) for every $1 \leqslant i \leqslant N$ *and* $x \in D$, $f_i(., x)$ *is quasiconvex on* D*.*

Then $M_L^D(f_1, f_2, \cdots, f_N; K) \neq \varnothing$*.*

Proof . Suppose that $y \in D$. Since for every $1 \leqslant i \leqslant N$, $f_i(., x)$ is lower semicontinuous on $K \setminus D$, by using Proposition 2.1, for every $a \in \mathbb{R}$, we get

$$cl\big(\{y \in K : f_i(y, x) \leqslant a\}\big) \cap (K \setminus D) = \{y \in K \setminus D : f_i(y, x) \leqslant a\}.$$

Hence

$$\begin{aligned}
\bigcap_{i=1}^{N} cl\big(\{y \in K : f_i(y, x) \leqslant a\}\big) \cap (K \setminus D) &= \bigcap_{i=1}^{N} \{y \in K \setminus D : f_i(y, x) \leqslant a\} \\
&\subseteq \bigcap_{i=1}^{N} \{y \in K : f_i(y, x) \leqslant a\} \cap (K \setminus D) \\
&\subseteq cl\big(\bigcap_{i=1}^{N} \{y \in K : f_i(y, x) \leqslant a\}\big) \cap (K \setminus D) \\
&\subseteq \bigcap_{i=1}^{N} cl\big(\{y \in K : f_i(y, x) \leqslant a\}\big) \cap (K \setminus D),
\end{aligned}$$

The above inequality shows that

$$cl\big(\bigcap_{i=1}^{N} \{y \in K : f_i(y, x) \leqslant a\}\big) \cap (K \setminus D) = \bigcap_{i=1}^{N} \{y \in K \setminus D : f_i(y, x) \leqslant a\}.$$

Hence

$$cl\{y \in K : f_i(y, x) \leqslant a \quad 1 \leqslant i \leqslant N\} \cap (K \setminus D) = \{y \in K \setminus D : f_i(y, x) \leqslant a \quad 1 \leqslant i \leqslant N\}.$$

This shows that $F_1(y)$ is closed in $K \setminus D$. The convexity of $F_1(y)$ is obtained by the quasiconvexity of $f_1, f_2, \cdots, f_N$ on D. Therefore, by Theorem 3.2, we conclude that $M_L^D(f_1, f_2, \cdots, f_N; K) \neq \varnothing$ and this completes the proof. □

Now, the existence of solutions for $(CSEP)$ can be obtained.

Corollary 3.5. *Let* D *be a locally segment-dense set in* K*, and let for every* $1 \leqslant i \leqslant N$, $f_i : K \times K \to \mathbb{R}$ *be a bifunction satisfying all conditions of Theorem 3.2. If for each* $1 \leqslant i \leqslant N$, f_i *has the sequentially sign property on* D*, then* $S(f_1, f_2, \cdots, f_N; K) \neq \varnothing$*.*

Proof . The proof follows immediately from Theorem 3.2 and then Lemma 2.12. □

Jafari et al. [14] by an example shown that the requirement that f should not be properly quasimonotone is essential. Alike of [14], we show the nonemptiness of $S(f_1, f_2, \cdots, f_N; K)$, where $f_1, f_2, \cdots, f_N$ are quasimonotone on D and for every i, f_i is nonproperly quasimonotone on D.

Example 3.6. *Let $X := \mathbb{R}$, $K := [0, +\infty[$ and $D :=]0, 1[$. Consider two bifunctions $f_1, f_2 : K \times K \to \mathbb{R}$ defined by*

$$f_1(x, y) := \begin{cases} x - y, & if\ x, y \in [0, 1], \\ \\ 1, & otherwise, \end{cases}$$

and $f_2(x, y) := x^2 - y$. It is easy to check that all the other conditions of Corollary 3.5 are satisfied, while $S(f_1, f_2; K) = \varnothing$.

4. Existence results for common solution of vector equilibrium problems

In this section by the same method used in Section 2, we obtain a similar results for vector equilibrium problems.

Let X be a real Hausdorff, locally convex topological vector space. We say that $P \subseteq D$ is dense in D iff $D \subseteq clP$. Recall that a set $C \subseteq X$ is a cone iff $tc \in C$ for all $c \in C$ and $t > 0$. The cone C is called a convex cone iff $C + C = C$. The cone C is called a pointed cone iff $C \cap (-C) = \{0\}$. Note that a closed, convex and pointed cone C induces a partial ordering on X, that is, $z_1 \leqslant z_2 \Leftrightarrow z_2 - z_1 \in C$ and $z_1 < z_2 \Leftrightarrow z_2 - z_1 \in intC$. It is obvious that $C + C \setminus \{0\} = C \setminus \{0\}$ and $intC + C = intC$.

Let Z be a locally convex Hausdorff topological vector spaces, $K \subseteq X$ be a nonempty subset and $C \subseteq Z$ be a convex and pointed cone with nonempty interior. For all $1 \leqslant i \leqslant N$, $f_i : K \times K \to Z$, the common solution of vector equilibrium problem ($CSVEP$), consists in finding $\bar{x} \in K$, such that

$$f_i(\bar{x}, y) \notin -intC,\ \forall\ y \in K, 1 \leq i \leq N.$$

The set of common solutions of vector equilibrium problems ($CSVEP$) is denoted by

$$S(f_1, f_2, \cdots, f_N; K; C).$$

Obviously, $S(f_1, f_2, \cdots, f_N; K; C) = \bigcap_{i=1}^{N} S(f_i; K; C)$. We say that an element $\bar{x} \in K$ is a local Minty common solution for $f_1, f_2, ..., f_N$, if there exists a neighbourhood U of $\bar{x}$ such that

$$f_i(y, \bar{x}) \notin intC,\ \forall\ y \in K \cap U.$$

The set of all local Minty common solution of vector equilibrium problems is denoted by

$$M_L(f_1, f_2, \cdots, f_N; K; C).$$

Obviously, $M_L(f_1, f_2, \cdots, f_N; K; C) = \bigcap_{i=1}^{N} M_L(f_i; K; C)$.

Definition 4.1. *[23] A map $f : K \longrightarrow Z$ is said to be C-lower semi-continuous (C-upper semi-continuous) at $x \in K$, iff for any neighbourhood V of $f(x)$ there exists a neighbourhood U of x such that $f(u) \in V + C$ ($f(u) \in V - C$) for all $u \in U \cap K$.*

Obviously, if f is continuous at $x \in K$, then it is also C-lower semi-continuous at $x \in K$. Assume that C has nonempty interior. According to [25], f is C-lower semi-continuous at $x \in K$ iff for any $k \in intC$, there exists a neighbourhood U of x, such that $f(u) \in f(x) + k + intC$ for all $u \in U \cap K$.

Remark 4.2. *The map $f : K \longrightarrow Z$ is C-upper semi-continuous at $x \in K$ iff the map $-f$ is C-lower semi-continuous at $x \in K$.*

We say that f is C-lower semi-continuous, (C-upper semi-continuous) on K, if f is C-lower semi-continuous, (C-upper semi-continuous) at every $x \in K$. Obviously, if f is C-lower (resp. upper) semi-continuous on a subset A of X, then the restriction $f|_A : A \longrightarrow Z$ of f on A is C-lower (resp. upper) semi-continuous on A. The function f is said to be C-continuous on D, if it is C-lower semi-continuous and C-upper semi-continuous on D.
In the sequal, we suppose X and Z are real Hausdorff locally convex topological vector spaces, D is a locally segment-dense set in K (a nonempty subset of X) and $f : X \longrightarrow Z$ is a function. Assume also that $C \subseteq Z$ is a convex and pointed cone with $intC \neq \varnothing$ that C induces a partial ordering on Z.

Definition 4.3. *[21] The function f is C-convex on D, iff for all $x, y \in D$ and $t \in [0,1]$ such that $tx + (1-t)y \in D$, then*

$$tf(x) + (1-t)f(y) - f(tx + (1-t)y) \in C, \ \forall\ t \in [0;1].$$

f is said to be C-concave iff $-f$ is C-convex.

Definition 4.4. *[2, 12] The function f is C-quasimonotone on D, iff for $x, y \in D$,*

$$f(x,y) \in intC \Rightarrow f(y,x) \notin intC.$$

Definition 4.5. *[2] The function f is properly C-quasimonotone on D, iff for every subset of finite elements $\{x_1, x_2, \cdots, x_n\} \subseteq D$ and every $\bar{x} \in conv\{x_1, x_2, \cdots, x_n\} \cap D$, there exists $j \in \{1, 2, \cdots, n\}$ such that $f(x_j, \bar{x}) \notin intC$.*

Definition 4.6. *[24] Let K a convex subset of X and D be a locally segment-dense set in K. We say that f has the C-sequentially sign property with respect to the first variable at $x \in K \subseteq X$, iff for every $y \in K$ the following implication holds:*

$$if\ \{z_n\} \subset]x, y] \cap D : z_n \to x \ and\ f(z_n, x) \notin intC, \forall\ n \in \mathbb{N}\ then\ f(x, y) \notin -intC.$$

Also, Shokouhnia et al. [24], provided a proposition and introduced a large class of bifunctions, that have the C-sequentially sign property.
In the following, we give a notion of locally segment-dense Minty common solution to the vector case, that is needed to obtain existence result for $(CSVEP)$.

Definition 4.7. *Let K be a convex subset of X and D be a locally segment-dense set in K, and let for every $1 \leqslant i \leqslant N$, $f_i : K \times K \to \mathbb{R}$ be bifunctions. We say that $\bar{x} \in D$ is a locally segment-dense Minty common solution vector equilibrium problems, iff there exists a neighbourhood U of $\bar{x}$ such that for every $1 \leqslant i \leqslant N$,*

$$f_i(y, \bar{x}) \notin intC, \ \forall\ y \in D \cap U.$$

The set of all locally segment-dense Minty common solutions vector equilibrium problems is denoted by $M_L^D(f_1, f_2, \cdots, f_N; K; C)$. Obviously, $M_L^D(f_1, f_2, \cdots, f_N; K; C) = \bigcap_{i=1}^{N} M_L^D(f_i; K; C)$.

It is notice that if K be a subset of X, then $M_L(f_1, f_2, \cdots, f_N; K; C) \cap D \subseteq M_L^D(f_1, f_2, \cdots, f_N; K; C)$ and the inclusion may be strict. Hence $M_L(f_1, f_2, \cdots, f_N; K; C)$ may be empty and $M_L^D(f_1, f_2, \cdots, f_N; K; C)$ may be nonempty.
See the following example.

Example 4.8. *Let $X = K := \mathbb{R}$ and $D :=]-1, 1[\cap\mathbb{Q}$. Consider two bifunctions $f_1, f_2 : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ defined by*

$$f_1(x, y) := \begin{cases} -2, & if\ x, y \in D, \\ \\ 2, & otherwise, \end{cases}$$

and

$$f_2(x, y) := \begin{cases} y^2 - x^2, & if\ x, y \in D, \\ \\ x^2 + y^2, & otherwise. \end{cases}$$

Obviously, $M_L(f_1, f_2; K; C) = \varnothing$ while $M_L^D(f_1, f_2; K; C) \neq \varnothing$.

In the following lemma, we show that for every $1 \leqslant i \leqslant N$, the rather large set $M_L^D(f_1, f_2, \cdots, f_N; K; C)$ is a subset of $S(f_1, f_2, \cdots, f_N; K; C)$ under the weak condition of the C-sequentially sign property of the involved bifunctions.

Lemma 4.9. *Let K a convex subset of X and D be a locally segment-dense set in K, and let for $1 \leqslant i \leqslant N$, $f_i : K \times K \to \mathbb{R}$ be bifunctions with the C-sequentially sign property. Then $M_L^D(f_1, f_2, \cdots, f_N; K; C) \subseteq S(f_1, f_2, \cdots, f_N; K; C)$.*

Proof . The proof follows immediately from intersection property and Lemma 2.16 in [24]. □

By using the locally segment-dense set, we obtain some existence results for common solution vector equilibrium problems with unnecessarily compact domains.
For real bifunctions $f_1, f_2, \cdots, f_N$ on $K \times K$, let $F_1 : K \rightrightarrows K$ be a set-valued mapping by

$$F_2(y) := \big\{x \in K : f_i(y, x) \notin intC \quad \forall\ 1 \leqslant i \leqslant N\big\},$$

for all $y \in K$.

Definition 4.10. *Let D be a locally segment-dense set in K and let $f_1, f_2, \cdots, f_N : K \times K \to \mathbb{R}$ be bifunctions. we say that the bifunctions $f_1, f_2, \cdots, f_N$ have the common s^*-property on D, if the following condition holds:*
For every nonempty subset $A_1, A_2, \cdots, A_N$ of D if for all $1 \leqslant i \leqslant N$,

$$\exists\ \bar{x}_i \in convA_i \cap D \ s.t.\ f_i(x, \bar{x}_i) \in intC \quad \forall\ x \in A_i.$$

Then there exists some $\bar{x} \in conv(A_1 \cup A_2 \cup \cdots \cup A_N) \cap D$ such that for all $1 \leqslant i \leqslant N$,

$$f_i(z, \bar{x}) \in intC, \quad \forall\ z \in (A_1 \cup A_2 \cup \cdots \cup A_N) \cap D.$$

The following theorem is the vector form of Theorem 3.2.

Theorem 4.11. *Let K be a convex subset of X and D be a locally segment-dense set in K, and let for every $1 \leqslant i \leqslant N$, $f_i : K \times K \to Z$ be bifunctions satisfying the following conditions:*

(i) for every $1 \leqslant i \leqslant N$, f_i is C-quasimonotone on D, which is not properly C-quasimonotone on D;

(ii) $f_1, f_2, \cdots, f_N$ have the common S^*-property on D;

(iii) for every $y \in D$, $F_2(y)$ is closed in $K \setminus D$, i.e.,

$$cl(F_2(y)) \cap (K \setminus D) = F_2(y) \cap (K \setminus D) = \{x \in K \setminus D : f_i(y, x) \notin intC \quad 1 \leqslant i \leqslant N\};$$

(iv) for every $x_1, x_2 \in F_2(y) \cap D$ and $t \in [0, 1]$ such that $\bar{x} = (1 - t)x_1 + tx_2 \in D$, then $\bar{x} \in F_2(y)$.

Then $M_L^D(f_1, f_2, \cdots, f_N; K; C) \neq \varnothing$.

Proof . Since for every $1 \leqslant i \leqslant N$, f_i is not properly C-quasimonotone on D, there exist $x_{i1}, x_{i2}, \cdots, x_{in_i} \in D$ and $\bar{x}_i \in conv\{x_{i1}, x_{i2}, \cdots, x_{in_i}\} \cap D$ such that for every $j \in \{1, 2, \cdots, n_i\}$

$$f_i(x_{ij}, \bar{x}_i) \in intC.$$

Thus, $\bar{x}_i \notin F_2(x_{ij}) \cap (K \setminus D)$. Hence for every $1 \leqslant i \leqslant N$ and every $j \in \{1, 2, \cdots, n_i\}$

$$\bar{x}_i \notin cl\big(F_2(x_{ij})\big) \cap (K \setminus D).$$

For every $1 \leqslant i \leqslant N$, set $A_i = \{x_{i1}, x_{i2}, \cdots, x_{in_i}\}$. Since $f_1, f_2, \cdots, f_N$ have the common S^*-property on D, there exists $\bar{x} \in conv(A) \cap D \quad (A = A_1 \cup A_2 \cup \cdots \cup A_N)$ such that for every $z \in A \cap D$,

$$f_i(z, \bar{x}) \in intC, \quad (1 \leqslant i \leqslant N).$$

Then for each $z \in A \cap D$ that $A \cap D$ is finite there exists a neighbourhood U_z of $\bar{x}$ such that

$$U_z \cap D \subseteq \Big(X \setminus \big(F_2(z) \cap D\big)\Big).$$

We set $U = \bigcap_{z \in A \cap D} U_z$. So for every $y \in U \cap D$ and $z \in A \cap D$, we get

$$f_i(z, y) \in intC, \quad (1 \leqslant i \leqslant N).$$

Now, the C-quasimonotonicity of f_is on D implies that for every $y \in U \cap D$ and $z \in A \cap D$, we have

$$f_i(y, z) \notin intC, \quad (1 \leqslant i \leqslant N).$$

Furthermore, for arbitrary and fixed $y \in U \cap D$, we have $z \in F_2(y)$. Using the convexity of $F_2(y)$ on D, we deduce that for all $y \in U \cap D$,

$$f_i(y, \bar{x}) \notin intC, \quad (1 \leqslant i \leqslant N).$$

Hence, $\bar{x} \in M_L^D(f_1, f_2, \cdots, f_N; K; C)$ and this completes the proof. □

Corollary 4.12. *Let X be a real Hausdorff locally convex topological vector space, K be a convex subset of X and D be a locally segment-dense set in K, and let for every $1 \leqslant i \leqslant N$, f_i be bifunctions satisfying all conditions of Theorem 4.11. If for each $1 \leqslant i \leqslant N$, f_i has the C-sequentially sign property, then for every subset convex K of X, that $D \subseteq K$, $S(f_1, f_2, \cdots, f_N; K; C) \neq \varnothing$.*

5. Conclusions

In this paper, by using notion of locally segment-dense subsets, existence results for common solution of (vector) equilibrium problems are obtained, where the involved bifunction is quasimonotone just on a locally segment-dense subset of the domain. In fact, conditions are not imposed on thewhole domain, but rather on a locally segment-dense subset of it.

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