On generalisation of Brown’s conjecture

Ishfaq Nazir\textsuperscript{a,} \*, Mohammad Ibrahim Mir\textsuperscript{a}, Irfan Ahmad Wani\textsuperscript{a}

\textsuperscript{a}Department of Mathematics, University of Kashmir, South Campus, Anantnag-192101, Jammu and Kashmir, India

(Communicated by Madjid Eshaghi Gordji)

Abstract

Let $P$ be the complex polynomial of the form $P(z) = z \prod_{j=1}^{n-1} (z - z_j)$, with $|z_j| \geq 1$, $1 \leq j \leq n - 1$. Then according to famous Brown’s Conjecture $p'(z) \neq 0$, for $|z| < \frac{1}{n}$. This conjecture was proved by Aziz and Zarger\textsuperscript{1}. In this paper, we present some interesting generalisations of this conjecture and the results of several authors related to this conjecture.

Keywords: polynomial, disk, zeros, derivative, conjecture.

2010 MSC: Primary 30C15; Secondary 12D10.

1. Introduction

If all the zeros of a polynomial lie in the disk $D = \{ z : |z - c| \leq r \}$, then according to Gauss-Lucas Theorem, every critical point of $p(z)$ lie in $D$. B. Sendov conjectured that, if all the zeros of $p(z)$ lie in $|z| \leq 1$, then for any zero $z_0$ of $p(z)$, the disk $|z - z_0| \leq 1$ contains at least one zero of $p'(z)$\textsuperscript{5}. This conjecture has attracted much attention for the last six decades and a broad overview can be found in\textsuperscript{7}. In connection with this conjecture, Brown\textsuperscript{3} posed the following problem.

Let $Q_n$ denote the set of all complex polynomials of the form $p(z) = z \prod_{j=1}^{n-1} (z - z_j)$ where $|z_j| \geq 1$ for $1 \leq j \leq n - 1$. Find the best constant $C_n$ such that $p'(z)$ does not vanish in $|z| < C_n$.

Brown himself conjectured that $C_n = \frac{1}{n}$. However, conjecture was verified by Aziz and Zarger\textsuperscript{1} by proving the following more general result.

\textbf{Theorem 1.1.} If all the zeros of polynomial $Q(z) = \prod_{j=1}^{n-1} (z - z_j)$ lie in $|z| \geq 1$ and $P(z) = z^m Q(z)$, then $P'(z)$ has $(m - 1)$ fold zero at origin and remaining $(n - m)$ zeros lie in $|z| \geq \frac{m}{n}$.

\textbf{Remark 1.2.} For $m = 1$, it reduces to conjecture of Brown.

\*Corresponding author

\textsuperscript{1}Email addresses: ishfaqnazir02@gmail.com (Ishfaq Nazir), ibrahimmath80@gmail.com (Mohammad Ibrahim Mir), irfanmushtaq62@gmail.com (Irfan Ahmad Wani)

Received: December 2020 Accepted: January 2021
Zarger and Manzoor [2] have proved following result for \( s \)th derivative of \( P(z) = z^m \prod_{j=1}^{n-m} (z - z_j) \) with \(|z_j| \geq 1, 1 \leq j \leq n - m\).

**Theorem 1.3.** Let \( P(z) = z^m \prod_{j=1}^{n-m} (z - z_j) \) be a polynomial of degree \( n \) with \(|z_j| \geq 1, 1 \leq j \leq n - m\), then for \( 1 \leq s \leq m \) the polynomial \( P^{(s)}(z) \), the \( s \)th derivative of \( P(z) \) does not vanish in

\[
0 < |z| < \frac{m(m-1)...(m-s+1)}{n(n-1)...(n-s+1)}
\]

N. A Rather and F. Ahmad [6] have proved following results:

**Theorem 1.4.** Let \( p(z) = (z - a) \prod_{k=1}^{n-1} (z - z_k) \) with \(|a| \leq 1\), be a polynomial of degree \( n \) with \(|a| \leq 1\), and \(|z_k| \geq 1\) for \( 1 \leq k \leq n - 1\), then \( p'(z) \) does not vanish in the region

\[
|z - \left( \frac{n-1}{n} \right) a| < \frac{1}{n}.
\]

The result is best possible as is shown by the polynomial

\[
p(z) = (z - a)(z - e^{i\alpha})^{n-1}, 0 \leq \alpha < 2\pi.
\]

**Theorem 1.5.** Let \( p(z) = (z - a)^m \prod_{k=1}^{n-m} (z - z_k) \) be a polynomial of degree \( n \) with \(|a| \leq 1\), and \(|z_k| \geq 1\) for \( 1 \leq k \leq n - m\), then \( p'(z) \) has \( (m-1) \) fold zero at \( z = a \) and remaining \( (n - m) \) zeros of \( p'(z) \) lie in the region

\[
|z - \left( \frac{n-m}{n} \right) a| \geq \frac{m}{n}.
\]

The result is best possible as is shown by the polynomial

\[
p(z) = (z - a)^m(z - e^{i\alpha})^{n-m}, 0 \leq \alpha < 2\pi.
\]

In this paper, we first prove a result which generalises the result of Theorem 1.1, Theorem 1.5 and includes Brown’s Conjecture as a special case.

For the proof of our results, we shall use the following result, which is Walsh’s Conincidence Theorem [5].

**Theorem 1.6.** (Walsh’s Conincidence Theorem): Let \( G(z_1, z_2, ... z_n) \) is a symmetric \( n \) - linear form of total degree \( n \) in \((z_1, z_2, ..., z_n)\) and let \( C \) be a circular region containing the \( n \) points \( \alpha_1, \alpha_2, ..., \alpha_n \), then there exists at least one point \( \alpha \in C \) such that

\[
G(\alpha_1, \alpha_2, ... \alpha_n) = G(\alpha, \alpha, ..., \alpha).
\]
2. Main Results

**Theorem 2.1.** If all the zeros of a polynomial $P(z)$ of degree $n$ lie in $|z| \geq 1$, then for every real $\lambda > 0$, the polynomial $\lambda P(z) + (z - a)P'(z)$, $|a| \leq 1$ has no zero in

$$\left| z - \frac{na}{n + \lambda} \right| < \frac{\lambda}{n + \lambda}.$$

**Proof.** Let $z_1, z_2, \ldots, z_n$ be the zeros of $P(z)$, so that $|z_j| \geq 1$, $1 \leq j \leq n$. Let $w$ be any zero of $\lambda P(z) + (z - a)P'(z)$, then

$$\lambda P(w) + (z - a)P'(w) = 0. \quad (1)$$

This is an equation which is linear and symmetric in $z_1, z_2, \ldots, z_n$. Hence by Theorem 1.6, for the circular region $C = \{ z : |z| \geq 1 \}$, there exists $\alpha \in C$ such that

$$P(z) = (z - \alpha)^n.$$

Thus equation (1) reduces to

$$\lambda(w - \alpha)^n + n(w - a)(w - \alpha)^{n-1} = 0.$$

Equivalently

$$(w - \alpha)^{n-1} (\lambda(w - \alpha) + n(w - a)) = 0.$$

This gives $w = \alpha$ and $w = \frac{\lambda \alpha}{n + \lambda} + \frac{na}{n + \lambda}$.

Now, if $w = \alpha$, then using the fact that $|a| \leq 1$, we have

$$\left| w - \frac{na}{n + \lambda} \right| = \left| \alpha - \frac{na}{n + \lambda} \right|$$

$$\geq |\alpha| - \frac{n}{n + \lambda}|a|$$

$$\geq 1 - \frac{n}{n + \lambda}$$

$$= \frac{\lambda}{n + \lambda}.$$

That is

$$\left| w - \frac{na}{n + \lambda} \right| \geq \frac{\lambda}{n + \lambda}.$$ 

Again, if $w = \frac{\lambda \alpha}{n + \lambda} + \frac{na}{n + \lambda}$, then

$$\left| w - \frac{na}{n + \lambda} \right| = \frac{|\lambda||\alpha|}{|n + \lambda|} \geq \frac{\lambda}{n + \lambda}.$$

This gives

$$\left| w - \frac{na}{n + \lambda} \right| \geq \frac{\lambda}{n + \lambda}.$$

Since $w$ is any zero of $\lambda P(z) + (z - a)P'(z)$, it follows that every zero of $\lambda P(z) + (z - a)P'(z)$ lies in

$$\left| z - \frac{na}{n + \lambda} \right| \geq \frac{\lambda}{n + \lambda}.$$

This completes the proof of the Theorem. □
**Remark 2.2.** If all the zeros of polynomial \( Q(z) = \prod_{j=1}^{n-m}(z - z_j) \) lie in \( |z| \geq 1 \) and \( P(z) = (z - a)^m Q(z) \), \( |a| \leq 1 \), then

\[
P'(z) = (z - a)^{m-1} \left( mQ(z) + (z - a)Q'(z) \right).
\]

Applying Theorem 2.1 with \( \lambda = m \) to the polynomial \( Q(z) \), which is of degree \( n - m \), we immediately obtain Theorem 1.5.

**Remark 2.3.** For \( a = 0 \) and \( \lambda = m \) in Remark 2, we obtain Theorem 1.1.

**Remark 2.4.** Theorem 2.1 also includes validity of the Brown’s Conjecture as a special case when \( a = 0, \lambda = 1 \) and \( P(z) \) is polynomial of degree \( n - 1 \).

Next, we shall prove following result which describes the regions which contains the zeros of higher derivatives of the polynomial \( P(z) = (z - a)^m \prod_{j=1}^{n-m}(z - z_j) \) with \( |a| \leq 1 \) and \( |z_j| \geq 1 \).

**Theorem 2.5.** Let \( P(z) = (z - a)^m \prod_{j=1}^{n-m}(z - z_j) \) be a polynomial of degree \( n \), with \( |a| \leq 1 \) and \( |z_j| \geq 1 \), \( 1 \leq j \leq n - m \). Then for \( 1 \leq s \leq m \), the polynomial \( P^{(s)}(z) \), \( s \)th derivative of \( P(z) \) has \( (m - s) \) fold zero at \( z = a \) and remaining \((n - m)\) zeros lie in

\[
|z - \left( 1 - \frac{m(m-1)...(m-s+1)}{n(n-1)...(n-s+1)} \right) a| \geq \frac{m(m-1)...(m-s+1)}{n(n-1)...(n-s+1)}.
\]

**Proof.** Let \( P(z) = (z - a)^m Q(z) \), where \( Q(z) = \prod_{j=1}^{n-m}(z - z_j) \). Then

\[
P'(z) = (z - a)^{m-1} \left( mQ(z) + (z - a)Q'(z) \right).
\]

Applying Theorem 2.1 with \( \lambda = m \) to the polynomial \( Q(z) \) which is of degree \( (n - m) \), it follows that \( P'(z) \) has \((m - 1)\) fold zero at \( z = a \) and remaining \((n - m)\) zeros lie in

\[
|z - \left( \frac{n-m}{n} \right) a| \geq \frac{m}{n}.
\]

That is \( P'(z) = (z - a)^{m-1} R(z) \), where \( R(z) = (z - a)Q'(z) + mQ(z) \) has \((m - 1)\) fold zero at \( z = a \) and remaining \((n - m)\) zeros lie in

\[
|z - \left( \frac{n-m}{n} \right) a| \geq \frac{m}{n}.
\]

We can write

\[
P''(z) = (z - a)^{m-2} T(z),
\]

where \( T(z) = (z - a)R'(z) + (m - 1)R(z) \).

Consider the polynomial

\[
S(z) = p' \left( \frac{m}{n} z + \frac{n-m}{n} a \right)
\]

or

\[
S(z) = \left( \frac{m}{n} \right)^{m-1} (z - a)^{m-1} R \left( \frac{m}{n} z + \frac{n-m}{n} a \right),
\]
then $S(z)$ is a polynomial of degree $n - 1$ with $(m - 1)$ fold zero at $z = a$ and remaining $(n - m)$ zeros lie in $|z| \geq 1$. Thus,  
\[ S'(z) = \left( \frac{m}{n} \right)^{m-2} (z-a)^{m-2} \left[ (z-a) \frac{m}{n} R \left( \frac{m}{n} z + \frac{n-m}{n} a \right) + (m-1) R \left( \frac{m}{n} z + \frac{n-m}{n} a \right) \right]. \]

Again applying Theorem 2.1 with $\lambda = m - 1$ to the polynomial $R \left( \frac{m}{n} z + \frac{n-m}{n} a \right)$ which is of degree $(n-m)$, it follows that $S'(z)$ has $(m-2)$ fold zero at $z = a$ and remaining $(n-m)$ zero lie in 
\[ \left| z - \left( \frac{n-m}{n-1} \right) a \right| \geq \frac{m-1}{n-1}. \]

Replacing $z$ by $\frac{m}{n} z + (\frac{m-n}{m}) a$ in (2) we obtain $P''(z)$ has $(m-2)$ fold zero at $z = a$ and remaining $(n-m)$ zeros lie in 
\[ \left| z - \left( 1 - \frac{m(m-1)}{n(n-1)} \right) a \right| \geq \frac{m(m-1)}{n(n-1)}. \]

Proceeding similarly as above and by repeated application of Theorem 2.1, we get for any positive integer $s$ such that $1 \leq s \leq m$, the polynomial $P^{(s)}(z)$, $s^{th}$ derivative has $(m-s)$ zeros at $z = a$ and remaining zeros lie in 
\[ \left| z - \left( 1 - \frac{m(m-1)...(m-s+1)}{n(n-1)...(n-s+1)} \right) a \right| \geq \frac{m(m-1)...(m-s+1)}{n(n-1)...(n-s+1)}. \]

\[ \square \]

**Remark 2.6.** For $s = 1$, $a = 0$ and $m = 1$, we get Brown’s conjecture.

**Remark 2.7.** For $a = 0$, we get Theorem 1.3 which is the result due to Zarger and Manzoor.

For $a = 0$ and $s = m$, following result proved in [8] immediately follows:

**Corollary 2.8.** If $p(z) = z^m \prod_{k=1}^{n-m} (z-z_k)$ be a polynomial of degree $n$ with $|z_k| \geq 1$ for $1 \leq k \leq n-m$, then the polynomial $p^{(m)}(z)$, $m \geq 1$ does not vanish in $|z| < \frac{m!}{n(n-1)...(n-m+1)}$.

For $s = m$, we obtain following result proved in [4].

**Corollary 2.9.** If $p(z) = (z-a)^m \prod_{k=1}^{n-m} (z-z_k)$ be a polynomial of degree $n$ with $|a| \leq 1$ and $|z_k| \geq 1$ for $1 \leq k \leq n-m$, then the polynomial $p^{(m)}(z)$, $m \geq 1$ has all its zeros in the region 
\[ \left| z - \left( 1 - \frac{m!}{n(n-1)...(n-m+1)} \right) a \right| \geq \frac{m!}{n(n-1)...(n-m+1)}. \]