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On boundary value problems of higher order abstract fractional integro-differential equations

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Abstract

The aim of this paper is to establish the existence of solutions of boundary value problems of nonlinear fractional integro-differential equations involving Caputo fractional derivative by using the techniques such as fractional calculus, Hölder inequality, Krasnoselskii's fixed point theorem and nonlinear alternative of Leray-Schauder type. Examples are exhibited to illustrate the main results.

Keywords: Fractional integro-differential equations; boundary value problem; fixed point theorems.
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1. Introduction

Fractional differential equations have been recently used as effective tools in study the modelling of many real phenomena. Scientists and engineers have become more conscious of the fact that the description of natural phenomena in physics, chemistry, biophysics, biology, blood flow problems, control theory, aerodynamics, nonlinear oscillation of earthquake, the fluid-dynamic traffic model, etc. be more precise by fractional derivatives, see for examples [8, 11]. For more details on fractional calculus and fractional differential equations theory, see the monographs of Kilbas et al. [11], Miller and Ross [14], Podlubny [15] Samko et al. [16], and the references given therein. Agarwal et al. [2] studied the existence and uniqueness of solutions for various classes of fractional differential equations involving the Caputo fractional derivative with initial and boundary value conditions in finite dimensional spaces. Chalishajar and Karthikeyan [3] and Yang et al. [20] have extended the

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work in [1] from real line $\mathbb{R}$ to the abstract Banach space X by using more general assumptions on the nonlinear function f.

We are motivated by the works in [1, 20] and influenced by Chalishajar and Karthikeyan [3]. By applying different techniques and strong conditions, the purpose of the present paper is to study the following more general class of boundary value problem for fractional integro-differential equation

$$\begin{cases} {}^cD^\alpha x(t) = f\big(t, x(t), (Sx)(t)\big), t \in J = [0,T], \alpha \in (n-1, n], \\ x(0) = x_0, x'(0) = x_0^1, x''(0) = x_0^2, \ldots, x^{(n-2)}(0) = x_0^{n-2}, \\ x^{(n-1)}(T) = x_T, \end{cases} \tag{1.1}$$

where ${}^cD^\alpha$ is the Caputo fractional derivative of order $\alpha, f : J \times X \times X \to X$ and $x_0, x_0^i(i = 1, 2, \ldots, n-2, n \geq 3, n$ is an integer), x_T are elements of X and S is a nonlinear integral operator given by $(Sx)(t) = \int_0^t k\big(t, s, x(s)\big)ds$, where $k \in C(J \times J \times X, X)$. Here, X be a Banach space with the norm $\|\cdot\|$ and $C(J, X)$ denotes the Banach space of X-valued continuous functions on J with the supremum norm $\|x\|_\infty := \sup\{\|x(t)\| : t \in J\}$. For measurable functions $m : J \to \mathbb{R}$, define the norm $\|m\|_{L^p(J,\mathbb{R})} = \big(\int_J |m(t)|^p dt\big)^{\frac{1}{p}}, 1 \leq p < \infty$, where $L^p(J, \mathbb{R})$ the Banach space of all Lebesgue measurable functions m with $\|m\|_{L^p(J,\mathbb{R})} < \infty$.

Many authors have investigated the special forms of equation (1.1) with different boundary conditions by using various techniques, see for examples [1, 2, 4, 5, 6, 7, 10, 12, 13, 18, 20]. Here, we establish existence results for the fractional boundary value problem (BVP for short), (1.1) by applying Krasnoselskii's fixed point theorem, nonlinear alternative of Leray-Schauder type, Hölder inequality and fractional calculus.

This paper is organized as follows. In Section 2, we present some preliminaries. In Section 3, we give the proof of our main results by applying fixed point theorems. Finally, in Section 4, applications of the main results are provided.

2. Preliminaries

In this section, we set forth some preliminaries from [11, 21]. Throughout this paper, we denote $\mathbb{R}_+ = [0, \infty)$ and $\mathbb{R}^+ = (0, \infty)$.

Definition 2.1. The Riemann-Liouville fractional integral of order $\alpha \in \mathbb{R}^+$, of a suitable function h, is defined by

$$I_{a+}^\alpha h(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} h(s) ds,$$

where $a \in \mathbb{R}$ and Γ is the Gamma function.

Definition 2.2. For a suitable function h given on the interval $[a, b]$, the Riemann-Liouville fractional derivative of order $\alpha > 0$ of h, is defined by

$$(D_{a+}^\alpha h)(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^n \int_a^t (t-s)^{n-\alpha-1} h(s) ds,$$

where $n = [\alpha] + 1, [\alpha]$ denotes the integer part of α.

Definition 2.3. For a suitable function h given on the interval $[a, b]$, the Caputo fractional order derivative of order $\alpha > 0$ of h, is defined by

$$({}^cD^{\alpha}_{a+}h)(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t (t-s)^{n-\alpha-1} h^{(n)}(s)ds,$$

where $n = [\alpha] + 1, [\alpha]$ denotes the integer part of α.

Remark 2.4. We remark that
(i) The Caputo derivative of a constant is equal to zero.
(ii) ${}^cD^{\alpha}\ I^{\alpha}h(t) = h(t)$
(iii) For $\alpha, \beta > 0$ and $n = [\alpha] + 1$, we have
${}^cD^{\alpha}t^{\beta} = \frac{\Gamma(1+\beta)}{\Gamma(1+\beta-\alpha)}t^{\beta-\alpha}, \beta > n$ and ${}^cD^{\alpha}t^k = 0,\ k = 0, 1, 2, \ldots, n-1.$
(iv) If h is an abstract function with values in X, then integrals which appeared in Definitions 2.1, 2.2 and 2.3 are taken in Bochner's sense.

Lemma 2.5. ([21]) Let $\alpha > 0$; then the differential equation ${}^cD^{\alpha}h(t) = 0$, has the following general solution $h(t) = c_0 + c_1t + c_2t^2 + \cdots + c_{n-1}t^{n-1}$, where $c_i \in \mathbb{R}$, $i = 0, 1, 2, \ldots, n-1$, where $n = [\alpha] + 1$.

Lemma 2.6. ([21]) Let $\alpha > 0$; then

$$I^{\alpha}({}^cD^{\alpha}h)(t) = h(t) + c_0 + c_1t + c_2t^2 + \cdots + c_{n-1}t^{n-1},$$

for some $c_i \in \mathbb{R}$, $i = 0, 1, 2, \ldots, n-1$, where $n = [\alpha] + 1$.

For more details, see [11].

Definition 2.7. A function $x \in C(J, X)$ with it's α derivative existing on J is said to be a solution of the fractional BVP (1.1) if x satisfies the equation ${}^cD^{\alpha}x(t) = f\big(t, x(t), (Sx)(t)\big)$ a.e. on J, and the conditions $x(0) = x_0, x'(0) = x_0^1, x''(0) = x_0^2, \ldots,$
$x^{(n-2)}(0) = x_0^{n-2}, x^{(n-1)}(T) = x_T$.

We now prove the following auxiliary lemma.

Lemma 2.8. *Let $\overline{f} : J \to X$ be continuous. A function $x \in C(J, X)$ is solution of the fractional integral equation*

$$\begin{aligned} x(t) = & \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}\overline{f}(s)ds \\ & - \frac{t^{n-1}}{(n-1)!\Gamma(\alpha-n+1)} \int_0^T (T-s)^{\alpha-n}\overline{f}(s)ds \\ & + x_0 + x_0^1 t + \frac{x_0^2}{2!}t^2 + \cdots + \frac{x_0^{n-2}}{(n-2)!}t^{n-2} + \frac{x_T}{(n-1)!}t^{n-1}, \end{aligned} \tag{2.1}$$

if and only if x is a solution of the following fractional BVP

$${}^cD^{\alpha}x(t) = \overline{f}(t), t \in J = [0, T], \alpha \in (n-1, n], \tag{2.2}$$

$$x(0) = x_0, x'(0) = x_0^1, x''(0) = x_0^2, \ldots, x^{(n-2)}(0) = x_0^{n-2}, x^{(n-1)}(T) = x_T. \tag{2.3}$$

Proof . Suppose that x satisfies fractional BVP (2.2)-(2.3); then by Lemma 2.6 and Def. 2.1, we get

$$x(t)+c_0+c_1t+c_2t^2+\cdots+c_{n-1}t^{n-1}=\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}\overline{f}(s)ds,$$

where $c_i\in\mathbb{R}$, $i=0,1,2,\ldots,n-1$. That is:

$$x(t)=\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}\overline{f}(s)ds-c_0-c_1t-c_2t^2-\cdots-c_{n-2}t^{n-2}-c_{n-1}t^{n-1}. \tag{2.4}$$

By applying the condition $x(0)=x_0$, we get

$$x_0=-c_0\Rightarrow c_0=-x_0.$$

Now,

$$x'(t) = \frac{1}{\Gamma(\alpha-1)}\int_0^t(t-s)^{\alpha-2}\overline{f}(s)ds-c_1-2c_2t-\cdots-(n-2)c_{n-2}t^{n-3}-(n-1)c_{n-1}t^{n-2},$$

since $x'(0)=x_0^1$, we have

$$x_0^1=-c_1\Rightarrow c_1=-x_0^1.$$

And,

$$\begin{aligned}x''(t) &= \frac{1}{\Gamma(\alpha-2)}\int_0^t(t-s)^{\alpha-3}\overline{f}(s)ds-2c_2-\ldots\\ &\quad -(n-2)(n-3)c_{n-2}t^{n-4}-(n-1)(n-2)c_{n-1}t^{n-3},\end{aligned}$$

by using $x''(0)=x_0^2$, we have

$$x_0^2=-2c_2\Rightarrow c_2=-\frac{x_0^2}{2!}.$$

By continuing this process, we get

$$\begin{aligned}x^{(n-2)}(t) &= \frac{1}{\Gamma(\alpha-n+2)}\int_0^t(t-s)^{\alpha-n+1}\overline{f}(s)ds\\ &\quad -(n-2)(n-3)(n-4)\ldots(2)(1)c_{n-2}-(n-1)(n-2)(n-3)\ldots(3)(2)c_{n-1}t,\end{aligned}$$

applying $x^{(n-2)}(0)=x_0^{n-2}$, we have

$$x_0^{n-2}=-(n-2)!c_{n-2}\Rightarrow c_{n-2}=-\frac{x_0^{n-2}}{(n-2)!}.$$

Finally,

$$\begin{aligned}x^{(n-1)}(t) &= \frac{1}{\Gamma(\alpha-n+1)}\int_0^t(t-s)^{\alpha-n}\overline{f}(s)ds\\ &\quad -(n-1)(n-2)(n-3)\ldots(3)(2)(1)c_{n-1}\\ &= \frac{1}{\Gamma(\alpha-n+1)}\int_0^t(t-s)^{\alpha-n}\overline{f}(s)ds-(n-1)!c_{n-1},\end{aligned}$$

by using $x^{(n-1)}(T) = x_T$, we obtain

$$x_T = \frac{1}{\Gamma(\alpha-n+1)}\int_0^T (T-s)^{\alpha-n}\overline{f}(s)ds - (n-1)!c_{n-1},$$

and so that

$$c_{n-1} = -\frac{x_T}{(n-1)!} + \frac{1}{(n-1)!\Gamma(\alpha-n+1)}\int_0^T (T-s)^{\alpha-n}\overline{f}(s)ds.$$

By putting the values of $c_i(i=0,1,2,\ldots,n-1)$ in (2.4), we get

$$\begin{aligned} x(t) = & \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\overline{f}(s)ds \\ & - \frac{t^{n-1}}{(n-1)!\Gamma(\alpha-n+1)}\int_0^T (T-s)^{\alpha-n}\overline{f}(s)ds \\ & + x_0 + x_0^1 t + \frac{x_0^2}{2!}t^2 + \cdots + \frac{x_0^{n-2}}{(n-2)!}t^{n-2} + \frac{x_T}{(n-1)!}t^{n-1}. \end{aligned}$$

Conversely, assume that if x satisfies fractional integral equation (2.1), if $t \in [0,T]$ then $x(0) = x_0, x'(0) = x_0^1, x''(0) = x_0^2, \ldots, x^{(n-2)}(0) = x_0^{n-2}, x^{(n-1)}(T) = x_T$ and applying Remark 2.4 (i)-(iii), we get (2.2) is also satisfied. □

In view of Lemma 2.8, we have the following result which is useful in what follows.

Lemma 2.9. Let $f : J \times X \times X \to X$be continuous function. Then, $x \in C(J,X)$is a solution of the fractional integral equation

$$\begin{aligned} x(t) = & \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f\big(s, x(s), (Sx)(s)\big)ds \\ & -\frac{t^{n-1}}{(n-1)!\Gamma(\alpha-n+1)}\int_0^T (T-s)^{\alpha-n} f\big(s, x(s), (Sx)(s)\big)ds \\ & +x_0 + x_0^1 t + \frac{x_0^2}{2!}t^2 + \cdots + \frac{x_0^{n-2}}{(n-2)!}t^{n-2} + \frac{x_T}{(n-1)!}t^{n-1}, \end{aligned}$$

if and only if x is solution of the fractional BVP (1.1).

Lemma 2.10. (Mazur theorem, [17]) Let X be a Banach space. If $U \subset X$ is relatively compact, then $conv(U)$ is relatively compact and $\overline{conv}(U)$ is compact.

Lemma 2.11. (Ascoli-Arzela theorem) Let $S = \{s(t)\}$ is a function family of continuous mappings $s : [a,b] \to X$. If S is uniformly bounded and equicontinuous, and for any $t^* \in [a,b]$, the set $\{s(t^*)\}$ is relatively compact, then, there exists a uniformly convergent function sequence $\{s_n(t)\}(n = 1,2,\ldots, t \in [a,b])$ in S.

Lemma 2.12. (Krasnoselskii theorem) Let B be a closed convex and nonempty subset of X. Suppose that L and N are in general nonlinear operators which maps B into X such that:
(i) $Lu + Nv \in B$ whenever $u, v \in B$;
(ii) L is a contraction mapping;
(iii) N is compact and continuous. Then there exists $w \in B$ such that $w = Lw + Nw$.

Lemma 2.13. (Nonlinear alternative of Leray-Schauder type) Let C be a nonempty convex subset of a Banach space X. Let U be a nonempty open subset of C with $0 \in U$ and $F : \overline{U} \to C$ be a compact and continuous operators. Then either
(i)F has fixed points in $\overline{U}$, or
(ii) there exist $x \in \partial U$ and $\eta \in [0, 1]\}$ with $x = \eta F(x)$.

3. Main Results

First, we list hypotheses that will be used in our further discussion.

• (H1) The function $f : J \times X \times X \to X$ is measurable with respect to t on J and is continuous with respect to x on X.
• (H2) There exists a constant $\alpha_1 \in (0, \alpha - n + 1)$ and real-valued function $h(t) \in L^{\frac{1}{\alpha_1}}(J, \mathbb{R})$, such that $\|f\big(t, x(t), (Sx)(t)\big)\| \le h(t)$, for each $t \in J$, and all $x \in X$. For brevity, let $H = \|h\|_{L^{\frac{1}{\alpha_1}}(J,\mathbb{R})}$.
• (H3) There exists a constant $\alpha_2 \in (0, \alpha - n + 1)$, real-valued functions $\varphi_1(t), \varphi_2(t) \in L^{\frac{1}{\alpha_2}}(J, \mathbb{R}^+)$ and there exist a L^1-integrable and nondecreasing functions $\psi_1, \psi_2 : [0, \infty) \to (0, \infty)$ such that

$$\|f\big(t, x(t), (Sx)(t)\big)\| \le \varphi_1(t)\, \psi_1\big(\|x(t)\|\big) + \|Sx(t)\|,$$

$$\|k\big(t, s, x(s)\big)\| \le \varphi_2(t)\, \psi_2\big(\|x(s)\|\big),$$

for each $s \in [0, t]$, $t \in J$ and all $x, y \in X$.
• (H4) For every $t \in J$, the sets
$K_1 = \Big\{(t - s)^{\alpha-1} f\big(s, x(s), (Sx)(s)\big) : x \in C(J, X), s \in [0, t]\Big\}$ and
$K_2 = \Big\{(t - s)^{\alpha-n} f\big(s, x(s), (Sx)(s)\big) : x \in C(J, X), s \in [0, t]\Big\}$ are relatively compact.
• (H5) There exists a constant $Z > 0$ such that

$$\begin{aligned} Z \Bigg(&\frac{\Phi_1\psi_1\big(Z\big)}{\Gamma(\alpha)} \frac{T^{\alpha-\alpha_2}\,(1-\alpha_2)^{1-\alpha_2}}{(\alpha-\alpha_2)^{1-\alpha_2}} + \frac{\Phi_2\psi_2\big(Z\big)}{\Gamma(\alpha)} \frac{T^{\alpha-\alpha_2+1}\,(1-\alpha_2)^{1-\alpha_2}}{(\alpha-\alpha_2)^{1-\alpha_2}} \\ &+ \frac{\Phi_1\psi_1\big(Z\big)}{(n-1)!\Gamma(\alpha-n+1)} \frac{T^{\alpha-\alpha_2}\,(1-\alpha_2)^{1-\alpha_2}}{(\alpha-\alpha_2-n+1)^{1-\alpha_2}} \\ &+ \frac{\Phi_2\psi_2\big(Z\big)}{(n-1)!\Gamma(\alpha-n+1)} \frac{T^{\alpha-\alpha_2+1}\,(1-\alpha_2)^{1-\alpha_2}}{(\alpha-\alpha_2-n+1)^{1-\alpha_2}} + \chi \Bigg)^{-1} > 1, \end{aligned} \tag{3.1}$$

where $\Phi_1 = \Big(\int_J \big(\varphi_1(s)\big)^{\frac{1}{\alpha_2}} \Big)^{\alpha_2}$, $\Phi_2 = \Big(\int_J \big(\varphi_2(s)\big)^{\frac{1}{\alpha_2}} \Big)^{\alpha_2}$ and

$$\chi = \|x_0\| + \|x_0^1\|T + \frac{\|x_0^2\|}{2!}T^2 + \cdots + \frac{\|x_0^{n-2}\|}{(n-2)!}T^{n-2} + \frac{\|x_T\|}{(n-1)!}T^{n-1}.$$

Now, before dealing with main results, let us define the operator $F : C(J,X) \to C(J,X)$ as follows:

$$\begin{aligned}(F(x))(t) = &\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f\big(s, x(s), (Sx)(s)\big)ds \\ &- \frac{t^{n-1}}{(n-1)!\Gamma(\alpha-n+1)}\int_0^T (T-s)^{\alpha-n} f\big(s, x(s), (Sx)(s)\big)ds \\ &+ x_0 + x_0^1 t + \frac{x_0^2}{2!}t^2 + \cdots + \frac{x_0^{n-2}}{(n-2)!}t^{n-2} + \frac{x_T}{(n-1)!}t^{n-1}, t \in J.\end{aligned} \tag{3.2}$$

Theorem 3.1. *Assume that* (H1), (H2) *and* (H4) *hold. Then, the fractional BVP* (1.1) *has at least one solution on J.*

Proof . Let $B_r = \{x \in C(J,X) : \|x\|_\infty \le r\}$, where

$$\begin{aligned} r \ge &\frac{HT^{\alpha-\alpha_1}}{\Gamma(\alpha)(\frac{\alpha-\alpha_1}{1-\alpha_1})^{1-\alpha_1}} + \frac{HT^{\alpha-\alpha_1}}{(n-1)!\Gamma(\alpha-n+1)}\left(\frac{\alpha-\alpha_1-n+1}{1-\alpha_1}\right)^{-(1-\alpha_1)} \\ &+ \|x_0\| + \|x_0^1\|T + \frac{\|x_0^2\|}{2!}T^2 + \cdots + \frac{\|x_0^{n-2}\|}{(n-2)!}T^{n-2} + \frac{\|x_T\|}{(n-1)!}T^{n-1}.\end{aligned} \tag{3.3}$$

We subdivide the operators F defined by (3.2) into two operators $\mathcal{A}$ and $\mathcal{B}$ on B_r as

$$\begin{aligned}(\mathcal{A}x)(t) = &\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f\big(s, x(s), (Sx)(s)\big)ds \\ &- \frac{t^{n-1}}{(n-1)!\Gamma(\alpha-n+1)}\int_0^T (T-s)^{\alpha-n} f\big(s, x(s), (Sx)(s)\big)ds, \\ (\mathcal{B}x)(t) = &\, x_0 + x_0^1 t + \frac{x_0^2}{2!}t^2 + \cdots + \frac{x_0^{n-2}}{(n-2)!}t^{n-2} + \frac{x_T}{(n-1)!}t^{n-1}.\end{aligned}$$

The proof is divided into several steps.

Step 1. $\mathcal{A}x + \mathcal{B}y \in B_r$.

For any $x, y \in B_r$ and $t \in J$, by (H2), Hölder inequality and (3.3), we have

$$\begin{aligned} &\|(\mathcal{A}x)(t) + (\mathcal{B}y)(t)\| \\ &\le \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} \|f\big(s, x(s), (Sx)(s)\big)\|ds \\ &\quad + \frac{t^{n-1}}{(n-1)!\Gamma(\alpha-n+1)}\int_0^T (T-s)^{\alpha-n} \|f\big(s, x(s), (Sx)(s)\big)\|ds \\ &\quad + \|x_0\| + \|x_0^1\|t + \frac{\|x_0^2\|}{2!}t^2 + \cdots + \frac{\|x_0^{n-2}\|}{(n-2)!}t^{n-2} + \frac{\|x_T\|}{(n-1)!}t^{n-1} \\ &\le \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} h(s)ds \\ &\quad + \frac{t^{n-1}}{(n-1)!\Gamma(\alpha-n+1)}\int_0^T (T-s)^{\alpha-n} h(s)ds \\ &\quad + \|x_0\| + \|x_0^1\|t + \frac{\|x_0^2\|}{2!}t^2 + \cdots + \frac{\|x_0^{n-2}\|}{(n-2)!}t^{n-2} + \frac{\|x_T\|}{(n-1)!}t^{n-1}\end{aligned}$$

and so

$$
\begin{aligned}
&\|(\mathcal{A}x)(t)+(\mathcal{B}y)(t)\| \\
&\le \frac{1}{\Gamma(\alpha)}\left(\int_0^t (t-s)^{\frac{\alpha-1}{1-\alpha_1}}ds\right)^{1-\alpha_1}\left(\int_0^t (h(s))^{\frac{1}{\alpha_1}}ds\right)^{\alpha_1} \\
&\quad+\frac{t^{n-1}}{(n-1)!\Gamma(\alpha-n+1)}\left(\int_0^T (T-s)^{\frac{\alpha-n}{1-\alpha_1}}ds\right)^{1-\alpha_1}\left(\int_0^T (h(s))^{\frac{1}{\alpha_1}}ds\right)^{\alpha_1} \\
&\quad+\|x_0\|+\|x_0^1\|t+\frac{\|x_0^2\|}{2!}t^2+\cdots+\frac{\|x_0^{n-2}\|}{(n-2)!}t^{n-2}+\frac{\|x_T\|}{(n-1)!}t^{n-1} \\
&\le \frac{HT^{\alpha-\alpha_1}}{\Gamma(\alpha)(\frac{\alpha-\alpha_1}{1-\alpha_1})^{1-\alpha_1}}+\frac{HT^{\alpha-\alpha_1}}{(n-1)!\Gamma(\alpha-n+1)}\left(\frac{\alpha-\alpha_1-n+1}{1-\alpha_1}\right)^{-(1-\alpha_1)} \\
&\quad+\|x_0\|+\|x_0^1\|T+\frac{\|x_0^2\|}{2!}T^2+\cdots+\frac{\|x_0^{n-2}\|}{(n-2)!}T^{n-2}+\frac{\|x_T\|}{(n-1)!}T^{n-1} \\
&\le r.
\end{aligned}
$$

Thus, $\|\mathcal{A}x+\mathcal{B}y\|_\infty \le r$ and we conclude that for all $x, y \in B_r$, $\mathcal{A}x+\mathcal{B}y \in B_r$.

Step 2. $\mathcal{B}$ is a contraction mapping.
It is obvious that $\mathcal{B}$ is a contraction with the constant zero.

Step 3. $\mathcal{A}$ is continuous operator.
Let $\{x_n\}$ be a sequence such that $x_n \to x$ in C(J,X). Then for each $t \in J$, we have

$$
\begin{aligned}
&\|(\mathcal{A}(x_n))(t)-(\mathcal{A}(x))(t)\| \\
&\le \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\left\|f\big(s,x_n(s),(Sx_n)(s)\big)-f\big(s,x(s),(Sx)(s)\big)\right\|ds \\
&\quad+\frac{t^{n-1}}{(n-1)!\Gamma(\alpha-n+1)}\int_0^T (T-s)^{\alpha-n}\left\|f\big(s,x_n(s),(Sx_n)(s)\big)-f\big(s,x(s),(Sx)(s)\big)\right\|ds \\
&\le \|f(.,x_n(.),(Sx_n)(.))-f(.,x(.),(Sx)(.))\|_\infty \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}ds \\
&\quad+\frac{\left\|f\big(.,x_n(.),(Sx_n)(.)\big)-f\big(.,x(.),(Sx)(.)\big)\right\|_\infty t^{n-1}}{(n-1)!\Gamma(\alpha-n+1)}\int_0^T (T-s)^{\alpha-n}ds \\
&\le \left\|f\big(.,x_n(.),(Sx_n)(.)\big)-f\big(.,x(.),(Sx)(.)\big)\right\|_\infty \frac{t^\alpha}{\alpha\Gamma(\alpha)} \\
&\quad+\frac{\left\|f\big(.,x_n(.),(Sx_n)(.)\big)-f\big(.,x(.),(Sx)(.)\big)\right\|_\infty t^{n-1}}{(n-1)!\Gamma(\alpha-n+1)}\frac{T^{\alpha-n+1}}{(\alpha-n+1)} \\
&\le \left(\frac{T^\alpha}{\Gamma(\alpha+1)}+\frac{T^\alpha}{(n-1)!\Gamma(\alpha-n+2)}\right)\left\|f\big(.,x_n(.),(Sx_n)(.)\big)-f\big(.,x(.),(Sx)(.)\big)\right\|_\infty .
\end{aligned}
$$

Taking supremum, we get

$$\begin{aligned}&\|\mathcal{A}x_n-\mathcal{A}x\|_\infty\\ &\le\left(\frac{T^\alpha}{\Gamma(\alpha+1)}+\frac{T^\alpha}{(n-1)!\Gamma(\alpha-n+2)}\right)\left\|f\big(.,x_n(.),(Sx_n)(.)\big)-f\big(.,x(.),(Sx)(.)\big)\right\|_\infty,\end{aligned}$$

since f is continuous, we have

$$\|\mathcal{A}x_n-\mathcal{A}x\|_\infty\to 0 \text{ as } n\to\infty.$$

Therefore, $\mathcal{A}$ is continuous operator.

Step 4. $\mathcal{A}$ is compact operator.
For $x\in B_r$ and all $t\in J$, by using (H2) and Hölder inequality, we have

$$\begin{aligned}\|(\mathcal{A}(x))(t)\|&\le\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}\|f\big(s,x(s),(Sx)(s)\big)\|ds\\ &\quad+\frac{t^{n-1}}{(n-1)!\Gamma(\alpha-n+1)}\int_0^T(T-s)^{\alpha-n}\|f\big(s,x(s),(Sx)(s)\big)\|ds\\ &\le\frac{1}{\Gamma(\alpha)}\left(\int_0^t(t-s)^{\frac{\alpha-1}{1-\alpha_1}}ds\right)^{1-\alpha_1}\left(\int_0^t(h(s))^{\frac{1}{\alpha_1}}ds\right)^{\alpha_1}\\ &\quad+\frac{t^{n-1}}{(n-1)!\Gamma(\alpha-n+1)}\left(\int_0^T(T-s)^{\frac{\alpha-n}{1-\alpha_1}}ds\right)^{1-\alpha_1}\left(\int_0^T(h(s))^{\frac{1}{\alpha_2}}ds\right)^{\alpha_1}\\ &\le\frac{HT^{\alpha-\alpha_1}}{\Gamma(\alpha)(\frac{\alpha-\alpha_1}{1-\alpha_1})^{1-\alpha_1}}+\frac{HT^{\alpha-\alpha_1}}{(n-1)!\Gamma(\alpha-n+1)}\left(\frac{\alpha-\alpha_1-n+1}{1-\alpha_1}\right)^{-(1-\alpha_1)}\\ &\le l,\end{aligned}$$

where

$$l:=\frac{HT^{\alpha-\alpha_1}}{\Gamma(\alpha)(\frac{\alpha-\alpha_1}{1-\alpha_1})^{1-\alpha_1}}+\frac{HT^{\alpha-\alpha_1}}{(n-1)!\Gamma(\alpha-n+1)}\left(\frac{\alpha-\alpha_1-n+1}{1-\alpha_1}\right)^{-(1-\alpha_1)}.$$

Thus, we have
$\|\mathcal{A}x\|_\infty\le l$, and hence $\mathcal{A}$ is bounded.

Also, let $0\le t_1\le t_2\le T, x\in B_r$. Using (H2) and Hölder inequality, again we have

$$
\begin{aligned}
&\|(\mathcal{A}(x))(t_2)-(\mathcal{A}(x))(t_1)\| \\
&\leq \frac{1}{\Gamma(\alpha)}\int_0^{t_1}\left[(t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}\right]\left\|f\big(s,x(s),(Sx)(s)\big)\right\|ds \\
&\quad+\frac{1}{\Gamma(\alpha)}\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}\left\|f\big(s,x(s),(Sx)(s)\big)\right\|ds \\
&\quad+\frac{t_2^{n-1}-t_1^{n-1}}{(n-1)!\Gamma(\alpha-n+1)}\int_0^T(T-s)^{\alpha-n}\left\|f\big(s,x(s),(Sx)(s)\big)\right\|ds \\
&\leq \frac{1}{\Gamma(\alpha)}\int_0^{t_1}\left[(t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}\right]h(s)ds+\frac{1}{\Gamma(\alpha)}\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}h(s)ds \\
&\quad+\frac{t_2^{n-1}-t_1^{n-1}}{(n-1)!\Gamma(\alpha-n+1)}\int_0^T(T-s)^{\alpha-n}h(s)ds \\
&\leq \frac{1}{\Gamma(\alpha)}\Big(\int_0^{t_1}(t_2-s)^{\frac{\alpha-1}{1-\alpha_1}}ds\Big)^{1-\alpha_1}\Big(\int_0^{t}(h(s))^{\frac{1}{\alpha_1}}ds\Big)^{\alpha_1} \\
&\quad-\frac{1}{\Gamma(\alpha)}\Big(\int_0^{t_1}(t_1-s)^{\frac{\alpha-1}{1-\alpha_1}}ds\Big)^{1-\alpha_1}\Big(\int_0^{t}(h(s))^{\frac{1}{\alpha_1}}ds\Big)^{\alpha_1} \\
&\quad+\frac{1}{\Gamma(\alpha)}\Big(\int_{t_1}^{t_2}(t_2-s)^{\frac{\alpha-1}{1-\alpha_1}}ds\Big)^{1-\alpha_1}\Big(\int_0^{t}(h(s))^{\frac{1}{\alpha_1}}ds\Big)^{\alpha_1} \\
&\quad+\frac{t_2^{n-1}-t_1^{n-1}}{(n-1)!\Gamma(\alpha-n+1)}\Big(\int_0^{T}(T-s)^{\frac{\alpha-n}{1-\alpha_1}}ds\Big)^{1-\alpha_1}\Big(\int_0^{t}(h(s))^{\frac{1}{\alpha_1}}ds\Big)^{\alpha_1} \\
&\leq \frac{H}{\Gamma(\alpha)}\left(\frac{(t_2-t_1)^{\frac{\alpha-\alpha_1}{1-\alpha_1}}}{\frac{\alpha-\alpha_1}{1-\alpha_1}}-\frac{t_2^{\frac{\alpha-\alpha_1}{1-\alpha_1}}}{\frac{\alpha-\alpha_1}{1-\alpha_1}}\right)^{1-\alpha_1}-\frac{H}{\Gamma(\alpha)}\left(\frac{-t_1^{\frac{\alpha-\alpha_1}{1-\alpha_1}}}{\frac{\alpha-\alpha_1}{1-\alpha_1}}\right)^{1-\alpha_1} \\
&\quad+\frac{H}{\Gamma(\alpha)}\left(\frac{-(t_2-t_1)^{\frac{\alpha-\alpha_1}{1-\alpha_1}}}{\frac{\alpha-\alpha_1}{1-\alpha_1}}\right)^{1-\alpha_1}+\frac{H(t_2^{n-1}-t_1^{n-1})}{(n-1)!\Gamma(\alpha-n+1)}\left(\frac{-T^{\frac{\alpha-\alpha_1-n+1}{1-\alpha_1}}}{\frac{\alpha-\alpha_1-n+1}{1-\alpha_1}}\right)^{1-\alpha_1}.
\end{aligned}
$$

As $t_2 \to t_1$, the right-hand side of the above inequality tends to zero and since x is an arbitrary in B_r, $\mathcal{A}$ is equicontinuous.

Now, let $\{x_n\}, n = 1, 2, \dots$ be a sequence on B_r, and

$$(\mathcal{A}x_n)(t) = (\mathcal{A}_1x_n)(t) + (\mathcal{A}_2x_n)(t), t \in J,$$

where

$$
\begin{aligned}
(\mathcal{A}_1x_n)(t) &= \frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}f\big(s,x_n(s),(Sx_n)(s)\big)ds, t\in J, \\
(\mathcal{A}_2x_n)(t) &= -\frac{t^{n-1}}{(n-1)!\Gamma(\alpha-n+1)}\int_0^T(T-s)^{\alpha-n}f\big(s,x_n(s),(Sx_n)(s)\big)ds, t\in J.
\end{aligned}
$$

In view of hypothesis (H4) and lemma 2.10, the set $\overline{conv}K_1$ is compact. For any $t^* \in J$,

$$\begin{aligned}(\mathcal{A}_1x_n)(t^*) &= \frac{1}{\Gamma(\alpha)}\int_0^{t^*}(t^*-s)^{\alpha-1}f\big(s,x_n(s),(Sx_n)(s)\big)ds\\ &= \frac{1}{\Gamma(\alpha)}\lim_{k\to\infty}\sum_{i=1}^{k}\frac{t^*}{k}\left(t^*-\frac{it^*}{k}\right)^{\alpha-1}f\left(\frac{it^*}{k},x_n(\frac{it^*}{k}),(Sx_n)(\frac{it^*}{k})\right)\\ &= \frac{t^*}{\Gamma(\alpha)}\zeta_{n1},\end{aligned}$$

where

$$\zeta_{n1} = \lim_{k\to\infty}\sum_{i=1}^{k}\frac{1}{k}\left(t^*-\frac{it^*}{k}\right)^{\alpha-1}f\left(\frac{it^*}{k},x_n(\frac{it^*}{k}),(Sx_n)(\frac{it^*}{k})\right).$$

Now, we have $\{(\mathcal{A}_1x_n)(t)\}$ is a function family of continuous mappings $\mathcal{A}_1x_n : J \to X$, which is uniformly bounded and equicontinuous. As $\overline{conv}K_1$ is convex and compact, we know $\zeta_{n1} \in \overline{conv}K_1$. Hence, for any $t^* \in J = [0,T]$, the set $\{(\mathcal{A}_1x_n)(t^*)\}$, is relatively compact. Therefore by lemma 2.11, every $\{(\mathcal{A}_1x_n)(t)\}$ contains a uniformly convergent subsequence $\{(\mathcal{A}_1x_{n_k})(t)\}, k = 1,2,\ldots,$ on J. Thus, $\{\mathcal{A}_1x : x \in B_r\}$ is relatively compact.

Set

$$(\overline{\mathcal{A}}_2x_n)(t) = -\frac{t^{n-1}}{(n-1)!\Gamma(\alpha-n+1)}\int_0^{t}(t-s)^{\alpha-n}f\big(s,x_n(s),(Sx_n)(s)\big)ds, t \in J.$$

For any $t^* \in J$,

$$\begin{aligned}(\overline{\mathcal{A}}_2x_n)(t^*) &= -\frac{(t^*)^{n-1}}{(n-1)!\Gamma(\alpha-n+1)}\int_0^{t^*}(t^*-s)^{\alpha-n}f\big(s,x_n(s),(Sx_n)(s)\big)ds\\ &= -\frac{(t^*)^{n-1}}{(n-1)!\Gamma(\alpha-n+1)}\\ &\quad\times\lim_{k\to\infty}\sum_{i=1}^{k}\frac{t^*}{k}\left(t^*-\frac{it^*}{k}\right)^{\alpha-n}f\left(\frac{it^*}{k},x_n(\frac{it^*}{k}),(Sx_n)(\frac{it^*}{k})\right)\\ &= -\frac{(t^*)^{n}}{(n-1)!\Gamma(\alpha-n+1)}\zeta_{n2},\end{aligned}$$

where

$$\zeta_{n2} = \lim_{k\to\infty}\sum_{i=1}^{k}\frac{1}{k}\left(t^*-\frac{it^*}{k}\right)^{\alpha-n}f\left(\frac{it^*}{k},x_n(\frac{it^*}{k}),(Sx_n)(\frac{it^*}{k})\right).$$

Now, we have $\{(\overline{\mathcal{A}}_2x_n)(t)\}$ is a function family of continuous mappings $\overline{\mathcal{A}}_2x_n : J \to X$, which is uniformly bounded and equicontinuous. As $\overline{conv}K_2$ is convex and compact, we know $\zeta_{n2} \in \overline{conv}K_2$. Hence, for any $t^* \in J = [0,T]$, the set $\{(\overline{\mathcal{A}}_2x_n)(t^*)\}$, is relatively compact. Therefore by lemma

2.11, every $\{(\overline{\mathcal{A}}_2 x_n)(t)\}$ contains a uniformly convergent subsequence $\{(\overline{\mathcal{A}}_2 x_{n_k})(t)\}, k = 1, 2, \dots,$ on J. Particularly, $\{(\mathcal{A}_2 x_n)(t)\}$ contains a uniformly convergent subsequence $\{(\mathcal{A}_2 x_{n_k})(t)\}, k = 1, 2, \dots,$ on J. Thus, $\{\mathcal{A}_2 x : x \in B_r\}$ is relatively compact. As a result, the set $\{\mathcal{A}x : x \in B_r\}$ is relatively compact.

Therefore, the continuity of $\mathcal{A}$ and relatively compactness of the set $\{\mathcal{A}x : x \in B_r\}$ imply that $\mathcal{A}$ is compact operator. By Krasnoselskii's fixed point theorem given in Lemma 2.12, we deduce that $\mathcal{A} + \mathcal{B}$ has a fixed point that is the solution of fractional BVP (1.1). □

Theorem 3.2. *Assume that* (H1) *and* (H3)-(H5) *hold. Then the fractional BVP* (1.1) *has at least one solution on J.*

Proof . Clearly, the fixed points of the operator F defined by (3.2) are solutions of fractional BVP (1.1). We subdivide the proof into several steps.

Firstly, we prove that the operator F is continuous.
Since f is continuous, as in the proof of Theorem 3.1 (step 3), we can show that the operator F is continuous.

Secondly, we show that the operator F maps bounded sets into bounded sets in $C(J, X)$.
We have to show that for any $r^* > 0$, there exists a $l^* > 0$ such that for each $x \in B_{r^*} = \{x \in C(J, X) : \|x\|_\infty \le r^*\}$, we have $\|Fx\|_\infty \le l^*$.
For each $t \in J$, by (H3) and Hölder inequality, we get

$$\begin{aligned}
\|(F(x))(t)\| &\le \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \|f\big(s, x(s), (Sx)(s)\big)\| ds \\
&\quad + \frac{t^{n-1}}{(n-1)!\Gamma(\alpha-n+1)} \int_0^T (T-s)^{\alpha-n} \|f\big(s, x(s), (Sx)(s)\big)\| ds \\
&\quad + \|x_0\| + \|x_0^1\| t + \frac{\|x_0^2\|}{2!} t^2 + \dots + \frac{\|x_0^{n-2}\|}{(n-2)!} t^{n-2} + \frac{\|x_T\|}{(n-1)!} t^{n-1} \\
&\le \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \Big(\varphi_1(s)\, \psi_1\big(\|x(s)\|\big) + \|(Sx)(s)\|\Big) ds \\
&\quad + \frac{t^{n-1}}{(n-1)!\Gamma(\alpha-n+1)} \int_0^T (T-s)^{\alpha-n} \Big(\varphi_1(s)\, \psi_1\big(\|x(s)\|\big) + \|(Sx)(s)\|\Big) ds \\
&\quad + \|x_0\| + \|x_0^1\| t + \frac{\|x_0^2\|}{2!} t^2 + \dots + \frac{\|x_0^{n-2}\|}{(n-2)!} t^{n-2} + \frac{\|x_T\|}{(n-1)!} t^{n-1} \\
&\le \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \Big(\varphi_1(s)\, \psi_1\big(\|x(s)\|\big) + \int_0^s \|k\big(s, \tau, x(\tau)\big)\| d\tau\Big) ds \\
&\quad + \frac{t^{n-1}}{(n-1)!\Gamma(\alpha-n+1)} \int_0^T (T-s)^{\alpha-n} \\
&\quad \times \Big(\varphi_1(s)\, \psi_1\big(\|x(s)\|\big) + \int_0^s \|k\big(s, \tau, x(\tau)\big)\| d\tau\Big) ds \\
&\quad + \|x_0\| + \|x_0^1\| t + \frac{\|x_0^2\|}{2!} t^2 + \dots + \frac{\|x_0^{n-2}\|}{(n-2)!} t^{n-2} + \frac{\|x_T\|}{(n-1)!} t^{n-1}
\end{aligned}$$

and so

$$\begin{aligned}
\|(F(x))(t)\| &\le \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\Big(\varphi_1(s)\,\psi_1\big(\|x\|_\infty\big)+T\varphi_2(s)\,\psi_2\big(\|x\|_\infty\big)\Big)ds \\
&\quad + \frac{t^{n-1}}{(n-1)!\Gamma(\alpha-n+1)}\int_0^T (T-s)^{\alpha-n} \\
&\quad \times \Big(\varphi_1(s)\,\psi_1\big(\|x\|_\infty\big)+T\varphi_2(s)\,\psi_2\big(\|x\|_\infty\big)\Big)ds \\
&\quad + \|x_0\| + \|x_0^1\|t + \frac{\|x_0^2\|}{2!}t^2 + \cdots + \frac{\|x_0^{n-2}\|}{(n-2)!}t^{n-2} + \frac{\|x_T\|}{(n-1)!}t^{n-1} \\
&\le \frac{\psi_1\big(r^*\big)}{\Gamma(\alpha)}\left(\int_0^t (t-s)^{\frac{\alpha-1}{1-\alpha_2}}ds\right)^{1-\alpha_2}\left(\int_0^t \big(\varphi_1(s)\big)^{\frac{1}{\alpha_2}}ds\right)^{\alpha_2} \\
&\quad + \frac{T\psi_2\big(r^*\big)}{\Gamma(\alpha)}\left(\int_0^t (t-s)^{\frac{\alpha-1}{1-\alpha_2}}ds\right)^{1-\alpha_2}\left(\int_0^t \big(\varphi_2(s)\big)^{\frac{1}{\alpha_2}}ds\right)^{\alpha_2} \\
&\quad + \frac{\psi_1\big(r^*\big)t^{n-1}}{(n-1)!\Gamma(\alpha-n+1)}\left(\int_0^T (T-s)^{\frac{\alpha-n}{1-\alpha_2}}ds\right)^{1-\alpha_2}\left(\int_0^T \big(\varphi_1(s)\big)^{\frac{1}{\alpha_2}}ds\right)^{\alpha_2} \\
&\quad + \frac{T\psi_2\big(r^*\big)t^{n-1}}{(n-1)!\Gamma(\alpha-n+1)}\left(\int_0^T (T-s)^{\frac{\alpha-n}{1-\alpha_2}}ds\right)^{1-\alpha_2}\left(\int_0^T \big(\varphi_1(s)\big)^{\frac{1}{\alpha_2}}ds\right)^{\alpha_2} \\
&\quad + \|x_0\| + \|x_0^1\|t + \frac{\|x_0^2\|}{2!}t^2 + \cdots + \frac{\|x_0^{n-2}\|}{(n-2)!}t^{n-2} + \frac{\|x_T\|}{(n-1)!}t^{n-1} \\
&\le \frac{\Phi_1\psi_1\big(r^*\big)}{\Gamma(\alpha)}\frac{T^{\alpha-\alpha_2}}{\left(\frac{\alpha-\alpha_2}{1-\alpha_2}\right)^{1-\alpha_2}} + \frac{\Phi_2\psi_2\big(r^*\big)}{\Gamma(\alpha)}\frac{T^{\alpha-\alpha_2+1}}{\left(\frac{\alpha-\alpha_2}{1-\alpha_2}\right)^{1-\alpha_2}} \\
&\quad + \frac{\Phi_1\psi_1\big(r^*\big)}{(n-1)!\Gamma(\alpha-n+1)}\frac{T^{\alpha-\alpha_2}}{\left(\frac{\alpha-\alpha_2-n+1}{1-\alpha_2}\right)^{1-\alpha_2}} + \frac{\Phi_2\psi_2\big(r^*\big)}{(n-1)!\Gamma(\alpha-n+1)}\frac{T^{\alpha-\alpha_2+1}}{\left(\frac{\alpha-\alpha_2-n+1}{1-\alpha_2}\right)^{1-\alpha_2}} \\
&\quad + \|x_0\| + \|x_0^1\|T + \frac{\|x_0^2\|}{2!}T^2 + \cdots + \frac{\|x_0^{n-2}\|}{(n-2)!}T^{n-2} + \frac{\|x_T\|}{(n-1)!}T^{n-1} \\
&\le l^*,
\end{aligned}$$

where

$$\begin{aligned}
l^* &:= \frac{\Phi_1\psi_1\big(r^*\big)}{\Gamma(\alpha)}\frac{T^{\alpha-\alpha_2}}{\left(\frac{\alpha-\alpha_2}{1-\alpha_2}\right)^{1-\alpha_2}} + \frac{\Phi_2\psi_2\big(r^*\big)}{\Gamma(\alpha)}\frac{T^{\alpha-\alpha_2+1}}{\left(\frac{\alpha-\alpha_2}{1-\alpha_2}\right)^{1-\alpha_2}} \\
&\quad + \frac{\Phi_1\psi_1\big(r^*\big)}{(n-1)!\Gamma(\alpha-n+1)}\frac{T^{\alpha-\alpha_2}}{\left(\frac{\alpha-\alpha_2-n+1}{1-\alpha_2}\right)^{1-\alpha_2}} + \frac{\Phi_2\psi_2\big(r^*\big)}{(n-1)!\Gamma(\alpha-n+1)}\frac{T^{\alpha-\alpha_2+1}}{\left(\frac{\alpha-\alpha_2-n+1}{1-\alpha_2}\right)^{1-\alpha_2}} \\
&\quad + \|x_0\| + \|x_0^1\|T + \frac{\|x_0^2\|}{2!}T^2 + \cdots + \frac{\|x_0^{n-2}\|}{(n-2)!}T^{n-2} + \frac{\|x_T\|}{(n-1)!}T^{n-1}.
\end{aligned}$$

Thus, we have
$\|(F(x))(t)\| \le l^*$ for every $t \in J = [0,T]$ and hence $\|Fx\|_\infty \le l^*$.

Thirdly, we show that the operator F maps bounded sets into equicontinuous sets of $C(J,X)$. Let $0 \le t_1 \le t_2 \le T, x \in B_{r^*}$. Using (H3) and Hölder inequality, again we have

$$
\begin{aligned}
&\|(F(x))(t_2) - (F(x))(t_1)\| \\
&\le \frac{1}{\Gamma(\alpha)} \int_0^{t_1} \left[(t_2-s)^{\alpha-1} - (t_1-s)^{\alpha-1}\right] \left\|f\big(s, x(s), (Sx)(s)\big)\right\| ds \\
&\quad + \frac{1}{\Gamma(\alpha)} \int_{t_1}^{t_2} (t_2-s)^{\alpha-1} \left\|f\big(s, x(s), (Sx)(s)\big)\right\| ds \\
&\quad + \frac{t_2^{n-1} - t_1^{n-1}}{(n-1)!\Gamma(\alpha-n+1)} \int_0^T (T-s)^{\alpha-n} \left\|f\big(s, x(s), (Sx)(s)\big)\right\| ds \\
&\quad + \|x_0^1\|(t_2-t_1) + \frac{\|x_0^2\|}{2!}(t_2^2 - t_1^2) + \cdots + \frac{\|x_0^{n-2}\|}{(n-2)!}(t_2^{n-2} - t_1^{n-2}) + \frac{\|x_T\|}{(n-1)!}(t_2^{n-1} - t_1^{n-1}) \\
&\le \frac{1}{\Gamma(\alpha)} \int_0^{t_1} \left[(t_2-s)^{\alpha-1} - (t_1-s)^{\alpha-1}\right] \Big(\varphi_1(s)\, \psi_1\big(\|x\|_\infty\big) + T\varphi_2(s)\, \psi_2\big(\|x\|_\infty\big)\Big) ds \\
&\quad + \frac{1}{\Gamma(\alpha)} \int_{t_1}^{t_2} (t_2-s)^{\alpha-1} \Big(\varphi_1(s)\, \psi_1\big(\|x\|_\infty\big) + T\varphi_2(s)\, \psi_2\big(\|x\|_\infty\big)\Big) ds \\
&\quad + \frac{t_2^{n-1} - t_1^{n-1}}{(n-1)!\Gamma(\alpha-n+1)} \int_0^T (T-s)^{\alpha-n} \Big(\varphi_1(s)\, \psi_1\big(\|x\|_\infty\big) + T\varphi_2(s)\, \psi_2\big(\|x\|_\infty\big)\Big) ds \\
&\quad + \|x_0^1\|(t_2-t_1) + \frac{\|x_0^2\|}{2!}(t_2^2 - t_1^2) + \cdots + \frac{\|x_0^{n-2}\|}{(n-2)!}(t_2^{n-2} - t_1^{n-2}) + \frac{\|x_T\|}{(n-1)!}(t_2^{n-1} - t_1^{n-1}) \\
&\le \frac{\psi_1\big(r^*\big)}{\Gamma(\alpha)} \Big(\int_0^{t_1} (t_2-s)^{\frac{\alpha-1}{1-\alpha_2}} ds\Big)^{1-\alpha_2} \Big(\int_0^{t_1} \big(\varphi_1(s)\big)^{\frac{1}{\alpha_2}} ds\Big)^{\alpha_2} \\
&\quad - \frac{\psi_1\big(r^*\big)}{\Gamma(\alpha)} \Big(\int_0^{t_1} (t_1-s)^{\frac{\alpha-1}{1-\alpha_2}} ds\Big)^{1-\alpha_2} \Big(\int_0^{t_1} \big(\varphi_1(s)\big)^{\frac{1}{\alpha_2}} ds\Big)^{\alpha_2} \\
&\quad + \frac{T\psi_2\big(r^*\big)}{\Gamma(\alpha)} \Big(\int_0^{t_1} (t_2-s)^{\frac{\alpha-1}{1-\alpha_2}} ds\Big)^{1-\alpha_2} \Big(\int_0^{t_1} \big(\varphi_2(s)\big)^{\frac{1}{\alpha_2}} ds\Big)^{\alpha_2} \\
&\quad - \frac{T\psi_2\big(r^*\big)}{\Gamma(\alpha)} \Big(\int_0^{t_1} (t_1-s)^{\frac{\alpha-1}{1-\alpha_2}} ds\Big)^{1-\alpha_2} \Big(\int_0^{t_1} \big(\varphi_2(s)\big)^{\frac{1}{\alpha_2}} ds\Big)^{\alpha_2} \\
&\quad + \frac{\psi_1\big(r^*\big)}{\Gamma(\alpha)} \Big(\int_{t_1}^{t_2} (t_2-s)^{\frac{\alpha-1}{1-\alpha_2}} ds\Big)^{1-\alpha_2} \Big(\int_{t_1}^{t_2} \big(\varphi_1(s)\big)^{\frac{1}{\alpha_2}} ds\Big)^{\alpha_2} \\
&\quad + \frac{T\psi_2\big(r^*\big)}{\Gamma(\alpha)} \Big(\int_{t_1}^{t_2} (t_2-s)^{\frac{\alpha-1}{1-\alpha_2}} ds\Big)^{1-\alpha_2} \Big(\int_{t_1}^{t_2} \big(\varphi_2(s)\big)^{\frac{1}{\alpha_2}} ds\Big)^{\alpha_2} \\
&\quad + \frac{\psi_1\big(r^*\big)\big(t_2^{n-1} - t_1^{n-1}\big)}{(n-1)!\Gamma(\alpha-n+1)} \Big(\int_0^T (T-s)^{\frac{\alpha-n}{1-\alpha_2}} ds\Big)^{1-\alpha_2} \Big(\int_{t_1}^{t_2} \big(\varphi_1(s)\big)^{\frac{1}{\alpha_2}} ds\Big)^{\alpha_2} \\
&\quad + \frac{T\psi_2\big(r^*\big)\big(t_2^{n-1} - t_1^{n-1}\big)}{(n-1)!\Gamma(\alpha-n+1)} \Big(\int_0^T (T-s)^{\frac{\alpha-n}{1-\alpha_2}} ds\Big)^{1-\alpha_2} \Big(\int_{t_1}^{t_2} \big(\varphi_2(s)\big)^{\frac{1}{\alpha_2}} ds\Big)^{\alpha_2} \\
&\quad + \|x_0^1\|(t_2-t_1) + \frac{\|x_0^2\|}{2!}(t_2^2 - t_1^2) + \cdots + \frac{\|x_0^{n-2}\|}{(n-2)!}(t_2^{n-2} - t_1^{n-2}) + \frac{\|x_T\|}{(n-1)!}(t_2^{n-1} - t_1^{n-1})
\end{aligned}
$$

and so

$$
\begin{aligned}
&\|(F(x))(t_2)-(F(x))(t_1)\| \\
&\leq \frac{\Phi_1\psi_1(r^*)}{\Gamma(\alpha)}\left(\frac{(t_2-t_1)^{\frac{\alpha-\alpha_2}{1-\alpha_2}}}{\frac{\alpha-\alpha_2}{1-\alpha_2}}-\frac{t_2^{\frac{\alpha-\alpha_2}{1-\alpha_2}}}{\frac{\alpha-\alpha_2}{1-\alpha_2}}\right)^{1-\alpha_2}-\frac{\Phi_1\psi_1(r^*)}{\Gamma(\alpha)}\left(\frac{-t_1^{\frac{\alpha-\alpha_2}{1-\alpha_2}}}{\frac{\alpha-\alpha_2}{1-\alpha_2}}\right)^{1-\alpha_2} \\
&+\frac{T\Phi_2\psi_2(r^*)}{\Gamma(\alpha)}\left(\frac{(t_2-t_1)^{\frac{\alpha-\alpha_2}{1-\alpha_2}}}{\frac{\alpha-\alpha_2}{1-\alpha_2}}-\frac{t_2^{\frac{\alpha-\alpha_2}{1-\alpha_2}}}{\frac{\alpha-\alpha_2}{1-\alpha_2}}\right)^{1-\alpha_2}-\frac{T\Phi_2\psi_2(r^*)}{\Gamma(\alpha)}\left(\frac{-t_1^{\frac{\alpha-\alpha_2}{1-\alpha_2}}}{\frac{\alpha-\alpha_2}{1-\alpha_2}}\right)^{1-\alpha_2} \\
&+\frac{\Phi_1\psi_1(r^*)}{\Gamma(\alpha)}\left(\frac{-(t_2-t_1)^{\frac{\alpha-\alpha_2}{1-\alpha_2}}}{\frac{\alpha-\alpha_2}{1-\alpha_2}}\right)^{1-\alpha_2}+\frac{T\Phi_2\psi_2(r^*)}{\Gamma(\alpha)}\left(\frac{-(t_2-t_1)^{\frac{\alpha-\alpha_2}{1-\alpha_2}}}{\frac{\alpha-\alpha_2}{1-\alpha_2}}\right)^{1-\alpha_2} \\
&+\frac{\Phi_1\psi_1(r^*)(t_2^{n-1}-t_1^{n-1})}{(n-1)!\Gamma(\alpha-n+1)}\left(\frac{-T^{\frac{\alpha-\alpha_2-n+1}{1-\alpha_2}}}{\frac{\alpha-\alpha_2-n+1}{1-\alpha_2}}\right)^{1-\alpha_2}+\frac{T\Phi_2\psi_2(r^*)(t_2^{n-1}-t_1^{n-1})}{(n-1)!\Gamma(\alpha-n+1)}\left(\frac{-T^{\frac{\alpha-\alpha_2-n+1}{1-\alpha_2}}}{\frac{\alpha-\alpha_2-n+1}{1-\alpha_2}}\right)^{1-\alpha_2} \\
&+\|x_0^1\|(t_2-t_1)+\frac{\|x_0^2\|}{2!}(t_2^2-t_1^2)+\cdots+\frac{\|x_0^{n-2}\|}{(n-2)!}(t_2^{n-2}-t_1^{n-2})+\frac{\|x_T\|}{(n-1)!}(t_2^{n-1}-t_1^{n-1}).
\end{aligned}
$$

As $t_2 \to t_1$, the right-hand side of the above inequality tends to zero and since x is an arbitrary in B_{r^*}, F is equicontinuous.

Now, let $\{x_n\}, n = 1, 2, \dots$ be a sequence on B_{r^*}, and

$$(Fx_n)(t) = (F_1x_n)(t) + (F_2x_n)(t) + (F_3x_n)(t), t \in J,$$

where

$$
\begin{aligned}
(F_1x_n)(t) &= \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}f\big(s,x_n(s),(Sx_n)(s)\big)ds, t\in J,\\
(F_2x_n)(t) &= -\frac{t^{n-1}}{(n-1)!\Gamma(\alpha-n+1)}\int_0^T (T-s)^{\alpha-n}f\big(s,x_n(s),(Sx_n)(s)\big)ds, t\in J,\\
(F_3x_n)(t) &= x_0+x_0^1t+\frac{x_0^2}{2!}t^2+\cdots+\frac{x_0^{n-2}}{(n-2)!}t^{n-2}+\frac{x_T}{(n-1)!}t^{(n-1)}, t\in J.
\end{aligned}
$$

As in the proof of previous theorem, we can show that the sets $\{F_1x : x \in B_{r^*}\}$ and $\{F_2x : x \in B_{r^*}\}$ are relatively compact. Obviously, the set $\{F_3x : x \in B_{r^*}\}$ is relatively compact, and hence, the set $\{Fx : x \in B_{r^*}\}$ is relatively compact.
Consequently, we can conclude that F is continuous and completely continuous.

Finally, we suppose that for some $\eta \in [0, 1]$, let $x(t) = \eta(Fx)(t)$, by using (H3), (H5), Hölder inequality and for each $t \in J$, we have

$$
\begin{aligned}
\|x(t)\| &\leq \|(F(x))(t)\| \\
&\leq \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\Big(\varphi_1(s)\,\psi_1\big(\|x\|_\infty\big)+T\varphi_2(s)\,\psi_2\big(\|x\|_\infty\big)\Big)ds \\
&\quad + \frac{t^{n-1}}{(n-1)!\Gamma(\alpha-n+1)}\int_0^T (T-s)^{\alpha-n} \\
&\quad \times \Big(\varphi_1(s)\,\psi_1\big(\|x\|_\infty\big)+T\varphi_2(s)\,\psi_2\big(\|x\|_\infty\big)\Big)ds \\
&\quad + \|x_0\| + \|x_0^1\|t + \frac{\|x_0^2\|}{2!}t^2 + \cdots + \frac{\|x_0^{n-2}\|}{(n-2)!}t^{n-2} + \frac{\|x_T\|}{(n-1)!}t^{n-1} \\
&\leq \frac{\Phi_1\psi_1\big(\|x\|_\infty\big)}{\Gamma(\alpha)}\frac{T^{\alpha-\alpha_2}}{\Big(\frac{\alpha-\alpha_2}{1-\alpha_2}\Big)^{1-\alpha_2}} + \frac{\Phi_2\psi_2\big(\|x\|_\infty\big)}{\Gamma(\alpha)}\frac{T^{\alpha-\alpha_2+1}}{\Big(\frac{\alpha-\alpha_2}{1-\alpha_2}\Big)^{1-\alpha_2}} \\
&\quad + \frac{\Phi_1\psi_1\big(\|x\|_\infty\big)}{(n-1)!\Gamma(\alpha-n+1)}\frac{T^{\alpha-\alpha_2}}{\Big(\frac{\alpha-\alpha_2-n+1}{1-\alpha_2}\Big)^{1-\alpha_2}} \\
&\quad + \frac{\Phi_2\psi_2\big(\|x\|_\infty\big)}{(n-1)!\Gamma(\alpha-n+1)}\frac{T^{\alpha-\alpha_2+1}}{\Big(\frac{\alpha-\alpha_2-n+1}{1-\alpha_2}\Big)^{1-\alpha_2}} \\
&\quad + \|x_0\| + \|x_0^1\|T + \frac{\|x_0^2\|}{2!}T^2 + \cdots + \frac{\|x_0^{n-2}\|}{(n-2)!}T^{n-2} + \frac{\|x_T\|}{(n-1)!}T^{n-1} \\
&\leq \frac{\Phi_1\psi_1\big(\|x\|_\infty\big)}{\Gamma(\alpha)}\frac{T^{\alpha-\alpha_2}\,(1-\alpha_2)^{1-\alpha_2}}{(\alpha-\alpha_2)^{1-\alpha_2}} + \frac{\Phi_2\psi_2\big(\|x\|_\infty\big)}{\Gamma(\alpha)}\frac{T^{\alpha-\alpha_2+1}\,(1-\alpha_2)^{1-\alpha_2}}{(\alpha-\alpha_2)^{1-\alpha_2}} \\
&\quad + \frac{\Phi_1\psi_1\big(\|x\|_\infty\big)}{(n-1)!\Gamma(\alpha-n+1)}\frac{T^{\alpha-\alpha_2}\,(1-\alpha_2)^{1-\alpha_2}}{(\alpha-\alpha_2-n+1)^{1-\alpha_2}} \\
&\quad + \frac{\Phi_2\psi_2\big(\|x\|_\infty\big)}{(n-1)!\Gamma(\alpha-n+1)}\frac{T^{\alpha-\alpha_2+1}\,(1-\alpha_2)^{1-\alpha_2}}{(\alpha-\alpha_2-n+1)^{1-\alpha_2}} + \chi.
\end{aligned}
$$

Thus

$$
\begin{aligned}
\Big\|x\Big\|_\infty \Bigg(&\frac{\Phi_1\psi_1\big(\|x\|_\infty\big)}{\Gamma(\alpha)}\frac{T^{\alpha-\alpha_2}\,(1-\alpha_2)^{1-\alpha_2}}{(\alpha-\alpha_2)^{1-\alpha_2}} + \frac{\Phi_2\psi_2\big(\|x\|_\infty\big)}{\Gamma(\alpha)}\frac{T^{\alpha-\alpha_2+1}\,(1-\alpha_2)^{1-\alpha_2}}{(\alpha-\alpha_2)^{1-\alpha_2}} \\
&+ \frac{\Phi_1\psi_1\big(\|x\|_\infty\big)}{(n-1)!\Gamma(\alpha-n+1)}\frac{T^{\alpha-\alpha_2}\,(1-\alpha_2)^{1-\alpha_2}}{(\alpha-\alpha_2-n+1)^{1-\alpha_2}} \\
&+ \frac{\Phi_2\psi_2\big(\|x\|_\infty\big)}{(n-1)!\Gamma(\alpha-n+1)}\frac{T^{\alpha-\alpha_2+1}\,(1-\alpha_2)^{1-\alpha_2}}{(\alpha-\alpha_2-n+1)^{1-\alpha_2}} + \chi\Bigg)^{-1} \leq 1.
\end{aligned}
$$

But by (H5), there exists a $Z > 0$ such that $\|x\|_\infty \neq Z$ and $\|x\|_\infty < Z$.
Let $U = \{x \in C(J,X) : \|x\|_\infty < Z < r^*\}$. The operator $F : \overline{U} \to C(J,X)$ is continuous and completely continuous. From the choice of U, there is no $x \in \partial U$ such that $x = \eta F x, \eta \in [0,1]$. As

a consequence of the nonlinear alternative of Leray-Schuder type given in Lemma 2.13, we deduce that F has a fixed point $x \in \overline{U}$, which implies that the fractional BVP (1.1) has at least one solution $x \in C(J, X)$. □

4. Examples

In order to illustrate the applications of our results, we give the following an examples.

Example 4.1.

$$\begin{cases} {}^cD^\alpha x(t) = \frac{t}{3+\sin(t)}\cos(x(t)) + \int_0^t \frac{t^2(s+|x(s)|)}{1+s+|x(s)|}ds, t \in J_1, \alpha \in (2,3], \\ x(0) = 0, x'(0) = 1, x''(1) = 1, \end{cases} \tag{4.1}$$

Take $J_1 = [0, 1]$ and so $T = 1$.

Set

$$f_1\big(t, x(t), Sx(t)\big) = \frac{t}{3+\sin(t)}\cos(x(t)) + \int_0^t \frac{t^2(s+|x(s)|)}{1+s+|x(s)|}ds.$$

For all $x \in C(J_1, X)$ and each $t \in J_1 = [0, 1]$, we have

$$\begin{aligned} \big\|f_1\big(t, x(t), Sx(t)\big)\big\| &= \left\|\frac{t}{3+\sin(t)}\cos(x(t)) + \int_0^t \frac{t^2(s+|x(s)|)}{1+s+|x(s)|}ds\right\| \\ &\leq \frac{t}{2} + t^2 \in L^{\frac{1}{\alpha_1}}(J, \mathbb{R}). \end{aligned}$$

Now

$$\begin{aligned} \int_0^t (t-s)^{\alpha-1} f_1\big(s, x(s), Sx(s)\big)ds &\leq \int_0^t (t-s)^{\alpha-1}\left(\frac{s}{2} + s^2\right)ds \\ &\leq \Gamma(\alpha) I^\alpha \left(\frac{t}{2} + t^2\right) \\ &\leq \frac{\Gamma(\alpha)\Gamma(2)}{2\Gamma(2+\alpha)} t^{1+\alpha} + \frac{\Gamma(\alpha)\Gamma(1+2)}{\Gamma(1+2+\alpha)} t^{2+\alpha} \\ &\leq \frac{\Gamma(\alpha)\Gamma(2)}{2\Gamma(2+\alpha)} + \frac{\Gamma(\alpha)\Gamma(3)}{\Gamma(3+\alpha)}, \end{aligned}$$

also

$$\begin{aligned} \int_0^t (t-s)^{\alpha-3} f_1\big(s, x(s), Sx(s)\big)ds &\leq \int_0^t (t-s)^{\alpha-3}\left(\frac{s}{2} + s^2\right)ds \\ &\leq \Gamma(\alpha-2) I^{\alpha-2}\left(\frac{t}{2} + t^2\right) \\ &\leq \frac{\Gamma(\alpha-2)\Gamma(2)}{2\Gamma(\alpha)} t^{\alpha-1} + \frac{\Gamma(\alpha-2)\Gamma(3)}{\Gamma(\alpha+1)} t^\alpha \\ &\leq \frac{\Gamma(\alpha-2)\Gamma(2)}{2\Gamma(\alpha)} + \frac{\Gamma(\alpha-2)\Gamma(3)}{\Gamma(\alpha+1)}. \end{aligned}$$

As a result, the sets

$$K_{11} = \Big\{(t-s)^{\alpha-1} f_1\big(s, x(s), Sx(s)\big) : x \in C(J_1, X), s \in [0,t]\Big\},$$
$$K_{12} = \Big\{(t-s)^{\alpha-3} f_1\big(s, x(s), Sx(s)\big) : x \in C(J_1, X), s \in [0,t]\Big\},$$

are bounded which implies that K_{11}, K_{12} are relatively compact. Thus, all the assumptions in Theorem 3.1 satisfied, and, hence, the fractional BVP 4.1 has at least one solution on J_1.

Example 4.2.

$$\begin{cases} {}^cD^{\frac{7}{2}}x(t) = \frac{t^3}{1+e^t}\frac{|x(t)|^3}{1+|x(t)|^3} + \int_0^t \frac{t^2|x(s)|^2}{2(1+|x(s)|^2)}ds, t \in J_1, \alpha \in (3,4], \\ x(0) = 0, x'(0) = 0, x''(0) = 0, x'''(1) = 0. \end{cases} \tag{4.2}$$

Take $X_1 = \mathbb{R}_+, J_1 = [0,1]$ and so $T = 1$.
Set

$$f_2\big(t, x(t), (Sx)(t)\big) = \frac{t^3}{1+e^t}\frac{|x(t)|^3}{1+|x(t)|^3} + (Sx)(t),\ k_2(t,s,x(s)) = \frac{t^2|x(s)|^2}{2(1+|x(s)|^2)},$$

$\alpha = \frac{7}{2}, \alpha_2 = \frac{1}{4}$. For all $x \in C(J_1, X_1)$ and each $t \in J_1 = [0,1]$, we have

$$|k_2(t,s,x(s))| = \Big|\frac{t^2|x(s)|^2}{2(1+|x(s)|^2)}\Big| \le \frac{t^2}{2}|x(s)|^2,$$

and

$$\big|f_2\big(t, x(t), Sx(t)\big)\big| \le \frac{t^3}{2}|x(t)|^3 + \Big|(Sx)(t)\Big|.$$

Then, we have $\varphi_1(t) = \frac{t^3}{2} \in L^{\frac{1}{\alpha_2}}(J, \mathbb{R}^+), \varphi_2(t) = \frac{t^2}{2} \in L^{\frac{1}{\alpha_2}}(J, \mathbb{R}^+), \psi_1\big(|x(t)|\big) = |x(t)|^3$ and $\psi_2\big(|x(t)|\big) = |x(s)|^2$. Further, we can choose a real number $Z = 1$. Therefore $\Phi_1 = \Big(\int_0^1 \big(\frac{t^3}{2}\big)^4 dt\Big)^{\frac{1}{4}} = \big(\frac{1}{208}\big)^{\frac{1}{4}} = 0.263, \Phi_2 = \Big(\int_0^1 \big(\frac{t^2}{2}\big)^4 dt\Big)^{\frac{1}{4}} = \big(\frac{1}{144}\big)^{\frac{1}{4}} = 0.289$, $\psi_1(Z) = \psi_2(Z) = 1$, $\Gamma(0.5) = 1.77$, and $\Gamma(3.5) = 3.32$. Then, the inequality (3.1) is satisfies, that is ($5.75237080 > 1$.) Now,

$$\begin{aligned} \int_0^t (t-s)^{\alpha-1} f_2\big(s, x(s), Sx(s)\big)ds &\le \int_0^t (t-s)^{\frac{7}{2}-1}\left(s^3 + \frac{s^2}{2}\right)ds \\ &\le \Gamma(\frac{7}{2}) I^{\frac{7}{2}}\left(t^3 + \frac{t^2}{2}\right) \\ &\le \frac{\Gamma(\frac{7}{2})\Gamma(4)}{\Gamma(\frac{15}{2})} + \frac{\Gamma(\frac{7}{2})\Gamma(3)}{2\Gamma(\frac{13}{2})}, \end{aligned}$$

$$\begin{aligned} \int_0^t (t-s)^{\alpha-n} f_2\big(s, x(s), Sx(s)\big)ds &\le \int_0^t (t-s)^{\frac{7}{2}-4}\left(s^3 + \frac{s^2}{2}\right)ds \\ &\le \Gamma(\frac{1}{2}) I^{\frac{1}{2}}\left(t^3 + \frac{t^2}{2}\right) \\ &\le \frac{\Gamma(\frac{1}{2})\Gamma(4)}{\Gamma(\frac{9}{2})} + \frac{\Gamma(\frac{1}{2})\Gamma(3)}{2\Gamma(\frac{7}{2})}. \end{aligned}$$

As a result, the sets

$$K_{13} = \Big\{(t-s)^{\alpha-1} f_2\big(s, x(s), Sx(s)\big) : x \in C(J_1, X_1), s \in [0,t]\Big\},$$

$$K_{14} = \Big\{(t-s)^{\alpha-4} f_2\big(s, x(s), Sx(s)\big) : x \in C(J_1, X_1), s \in [0,t]\Big\},$$

are bounded which implies that K_{13}, K_{14} are relatively compact. Thus, all the assumptions in Theorem 3.2 satisfied, and, hence, the fractional BVP 4.2 has at least one solution on J_1.

5. Conclusions

In this paper, we have discussed the achievement of sufficient conditions for the existence of solutions of fractional BVP (1.1) which is more general than problems in literatures review [1, 3, 20], by applying Krasnoselskii's fixed point theorem, nonlinear alternative of Leray-Schauder type, Hölder inequality and fractional calculus. Examples provided to illustrate main results.

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