



# Dynamic investment portfolio optimization using a multivariate Merton model with correlated jump risk

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## Abstract

In this paper, we are concerned with the optimization of a dynamic investment portfolio when the securities which follow a multivariate Merton model with dependent jumps are periodically invested and proceed by approximating the Condition-Value-at-Risk (CVaR) by comonotonic bounds and maximize the expected terminal wealth. Numerical studies, as well as applications of our results to real datasets, are also provided.

*Keywords:* Risk analysis, Conditional tail expectation, Merton Model, Geometric Brownian motion, Comonotonicity.

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## 1. Introduction

One of the most important problems in investment strategies is choosing the portfolio that ideally leads to the maximum return but has minimum risk. This problem was first proposed by Markowitz in 1950 [11], who used a quadratic programming model for solving this optimization problem. Since then, the problem has been studied by several researchers using different criteria. For example, Cura [3] used the particle swarm optimization method, while Xu et al. [15] assumed a Black-Scholes market with stochastic drift and used the fractional Kelly-strategy developed by MacLean et al. [9]. Furthermore, Afhami et al. [1] used comonotonic approximation to solve the problem when the securities followed a multivariate Merton model with dependent jumps and the portfolio was

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constant. In this paper, we use the fractional Kelly-strategy to optimize wealth in a jump-diffusion Merton model with a dynamic control asset allocation (DCAA). The solution leads to a stochastic differential equation (SDE), which was solved for the constant mix strategy in Afhami et al. [1] and can be generalized to the DCAA strategy to obtain the terminal wealth of a periodic investment problem in a DCAA strategy framework. The optimization problem is solved based on a lower bound of the risk measure of terminal wealth using comonotonic theory.

The rest of the paper is organized as follows. Section 2 provides the definitions and existing results used throughout the paper. The main results are presented in Section 3, we first obtain the solution of the SDE corresponding to the terminal wealth process of a portfolio following a jump-diffusion model using the DCAA strategy, and then calculate the lower comonotonic bound for CVaR of the terminal wealth in order to solve the portfolio optimization problem. In Section 4, we implement our optimization method on data containing the daily prices of the stocks of Zoom Video Communication Inc. (ZM), and Tesla Inc. (TSLA). These data are readily available from the *S&P 500* market index.

## 2. Preliminaries

The Black-Scholes model is a mathematical model used for options pricing and estimating the variation of financial investments over time. The model assumes that price of assets follows a geometric Brownian motion with constant drift and volatility, and is widely used in the literature. Al-Zhour et al. [2] investigated pricing using a nonlinear volatility model. Jankova [6] focused on the methods of derivative contract pricing and derived the basic differential equation of the Black-Scholes model for option contract pricing. He also referred to the significant drawbacks and limitations of other option pricing models which are based on unrealistic assumptions. Merton [12] showed that the Black-Scholes model is generally not suitable for real market stock pricing, and introduced the jump-diffusion “Merton” model [13] which is commonly used in the literature and has abundant applications in the analysis of financial data. For example, Stübinger et al. [14] developed a pairs trading framework based on a mean-reverting jump-diffusion model and applied it to the *S&P 500* oil companies. Li et al. [8] introduced a dynamic model for a risky asset which is governed by its compensation process, a Brownian motion, and a stationary compound Poisson process. Maruddani et al. [10] applied jump-diffusion processes to derive bond parameters, equity, and default probability when the asset prices have extreme values. In this paper, we consider the return of a portfolio including one risk-free asset together with several risky assets which follow the jump-diffusion Merton model and optimize it using the DCAA strategy.

*VaR* and *CVaR* are the most common risk measures used in an optimization problem for maximizing expected terminal wealth.

**Definition 2.1.** Let  $X$  be a random variable with cdf  $F_X$  the risk measures *VaR*, *CVaR* and *Conditional Left-side Value at Risk (CLVaR)* for the set of real numbers  $\mathbb{R}$  and  $p \in (0, 1)$ , are defined as

$$\begin{aligned} VaR_p(X) &:= \inf\{x \in \mathbb{R} \mid F_X(x) \geq p\}, \\ CVaR_p(X) &:= \mathbb{E}(X \mid X > VaR_p(X)), \\ CLVaR_p(X) &:= \mathbb{E}(X \mid X < VaR_p(X)). \end{aligned}$$

where  $F_X(x) = Pr(X \leq x)$  and by convention,  $\inf\{\emptyset\} = +\infty$ .

It is obvious that

$$CVaR_{1-p}(X) = -CLVaR_p(-X).$$

The following definition characterizes comonotonicity (see Theorem 3 in Dhaene et al. [4]).

**Definition 2.2.** *The random vector  $\mathbf{X} = (X_1, X_2, \dots, X_n)$  is comonotonic if and only if one of the following equivalent conditions holds:*

- (1)  $\mathbf{X}$  has a comonotonic support;
- (2) For all  $\mathbf{x} = (x_1, x_2, \dots, x_n)$ , we have

$$F_{\mathbf{X}}(\mathbf{x}) = \min\{F_{X_1}(x_1), F_{X_2}(x_2), \dots, F_{X_n}(x_n)\};$$

- (3) For  $U \sim \text{Uniform}(0, 1)$ , we have

$$\mathbf{X} \stackrel{d}{=} (F_{X_1}^{-1}(U), F_{X_2}^{-1}(U), \dots, F_{X_n}^{-1}(U));$$

- (4) There exist a random variable  $Z$  and non-decreasing functions  $f_i$ , ( $i = 1, 2, \dots, n$ ), such that

$$\mathbf{X} \stackrel{d}{=} (f_1(Z), f_2(Z), \dots, f_n(Z)).$$

The following theorem presents the additivity property of risk measures for sums of comonotonic risks (Dhaene et al. [5]).

**Theorem 2.3.** *For the comonotonic random vector  $(X_1, X_2, \dots, X_n)$  we have*

$$\begin{aligned} VaR_p\left(\sum_{j=1}^n X_j\right) &= \sum_{j=1}^n VaR_p(X_j), \\ CVaR_p\left(\sum_{j=1}^n X_j\right) &= \sum_{j=1}^n CVaR_p(X_j), \end{aligned}$$

*if all marginal distributions  $F_k$ , ( $k = 1, 2, \dots, n$ ) are continuous.*

A random variable  $X$  is said to precede  $Y$  in the convex order sense ( $X \leq_{cx} Y$ ), if  $\mathbb{E}(X) = \mathbb{E}(Y)$  and  $\mathbb{E}[\max(X - d, 0)] \leq \mathbb{E}[\max(Y - d, 0)]$  for all real  $d$ . With respect to convex order, Kaas et al. [7] obtains bounds for sum of random variables.

**Theorem 2.4.** *For any random vector  $(X_1, \dots, X_n)$  and any random variable  $\Lambda$ ,*

$$\sum_{j=1}^n \mathbb{E}(X_j \mid \Lambda) \leq_{cx} \sum_{j=1}^n X_j \leq_{cx} \sum_{j=1}^n F_{X_j}^{-1}(U).$$

Table 1: Notations used throughout this paper

<i>Symbol</i>	<i>Definition</i>
$\mathcal{P}_j(t)$	Price of the $j^{th}$ risky asset at time $t$
$\mu_j$	Drift of geometric Brownian motion
$\sigma_j$	Volatility of the asset $\mathcal{P}_j, j = 1, 2, \dots, m$
$\mu$	$(r + \mu_1 - \lambda h_{1,0} - \lambda_1 h_{1,1}, \dots, r + \mu_m - \lambda h_{m,0} - \lambda_m h_{m,1})^\top$
$\mathbf{x}(l-1)$	$(x_1(l-1), \dots, x_m(l-1))^\top$
$r$	Interest rate of risk-free asset
$\mathbf{A}$	$\mu - r\mathbf{1}$
$\mu(\mathbf{x}(l-1))$	$\mathbf{A}^\top \mathbf{x}(l-1) + r$
$\mathfrak{B}'_j$	m-dimensional standard Brownian motions, $j = 1, 2, \dots, m$
$\rho_{i,j}$	$\frac{1}{t} Cov(\mathfrak{B}'_i(t), \mathfrak{B}'_j(t+s)), t, s \geq 0, i, j = 1, \dots, m$
$\Sigma_{m \times m}$	$(\sigma_i \sigma_j \rho_{i,j})_{i,j}, i, j \in \{1, 2, \dots, m\}$
$\sigma^2(\mathbf{x}(l-1))$	$(\mathbf{x}(l-1))^\top \Sigma \mathbf{x}(l-1)$
$\mathfrak{B}_{\mathbf{x}(l-1)}(t)$	$(\sigma^2(\mathbf{x}(l-1)))^{-1} \sum_{j=1}^m x_j(l-1) \sigma_j \mathfrak{B}'_j(t)$
$\mathcal{N}(t)$	Poisson processes with intensity rates $\lambda$
$\mathcal{N}_j(t)$	Poisson processes with intensity rates $\lambda_j$
$\mathcal{Z}_{k,j,0}$	The jump magnitude of the $k$ -th common jump for the asset $\mathcal{P}_j$ in $(0, t]$ , $k = 1, 2, \dots, \mathcal{N}(t)$
$\mathcal{Z}_{k,j,1}$	The jump magnitude of the $k$ -th individual jump of asset $\mathcal{P}_j$ , in $(0, t]$ , $k = 1, 2, \dots, \mathcal{N}_j(t)$
$h_{j,\ell}$	$E[e^{\mathcal{Z}_{i(t),j,\ell}}] - 1, j = 1, 2, \dots, m, \ell = 0, 1$
$\alpha_{l^*}$	The periodic endowments at predetermined points, $l^* = 0, 1, \dots, \tau - 1$
$\mu_{\mathcal{Z}_{1,j,\ell}}$	$\mathbb{E}(\mathcal{Z}_{k,j,\ell})$
$\sigma_{\mathcal{Z}_{1,j,\ell}}^2$	$Var(\mathcal{Z}_{k,j,\ell})$
$r_n$	$corr(V_n, \Lambda)$
$w_{l-1}$	The amount of wealth at the beginning of the period $l$

### 3. Main results

In this section, we consider  $(m + 1)$  securities consisting of one risk-free and  $m$  risky asset that is traded continuously. We also assume that the decision-maker invests a fraction  $x_j(l - 1)$ , ( $j = 1, 2, \dots, m$ ) of his/her wealth in the  $j$ -th risky asset at the beginning of the  $l$ -th period and  $1 - \sum_{j=1}^m x_j(l - 1)$  in the risk-free asset. In what follows, Theorem 3.1 gives the solution of the corresponding SDE (Eq. 3.3) and Theorem 3.4 characterizes the solution to the optimization problem. All notations are provided in Table 1 for simplicity.

In the constant mix strategy, the initial proportions of assets will remain fixed until the end of the terminal horizon. However, when using the DCAA strategy, the optimal portfolio strategy is obtained based on the current wealth to design a policy that remains fixed until the end of the terminal horizon [10]. If at the start of the next period the amount of wealth changes, the strategy is updated.

Assume that  $\mathcal{P}_0(t)$ , the price of the risk-free asset at time  $t$ , satisfies

$$\frac{d\mathcal{P}_0(t)}{\mathcal{P}_0(t-)} = r dt, \quad r > 0, \tag{3.1}$$

and the dynamics of risky assets prices for  $j = 1, 2, \dots, m$ , follow

$$\begin{aligned} \frac{d\mathcal{P}_j(t)}{\mathcal{P}_j(t-)} &= (r + \mu_j - \lambda h_{j,0} - \lambda_j h_{j,1}) dt + \sigma_j d\mathfrak{B}'_j(t) + (e^{\mathcal{Z}_{\iota(t-),j,0}} - 1) d\mathcal{N}(t) \\ &\quad + (e^{\mathcal{Z}_{\iota(t-),j,1}} - 1) d\mathcal{N}_j(t). \end{aligned} \tag{3.2}$$

Discounted changes in assets price are divided into two groups: individual changes corresponding to  $\mathcal{N}_j(t)$  and common changes corresponding to  $\mathcal{N}(t)$ . We assume that for all  $j, k$ , and  $\ell = 0, 1$  the random variables  $\mathcal{Z}_{k,j,\ell}$  are independent of  $\mathfrak{B}'_j(t)$  and for fixed  $j = 1, 2, \dots, m$ , the random variables  $\mathcal{Z}_{k,j,\ell}$  are *i.i.d* for all  $k$  and  $\ell$ . The variables  $\mathcal{Z}_{k,j,\ell}$  and  $\mathcal{Z}_{k,j',\ell}$  are also independent for  $j \neq j'$ , but they may not be identically distributed. We assume that  $\mathcal{Z}_{k,j,\ell}$ , ( $j = 1, \dots, m, \ell = 0, 1$ ) are continuous random variables.

Then, according to (3.2) and (3.1), the wealth process  $W(t)$  within the  $l$ -th period is

$$\begin{aligned} \frac{d\mathcal{P}(t)}{\mathcal{P}(t-)} &= \sum_{j=1}^m x_j(l - 1) \frac{d\mathcal{P}_j(t)}{\mathcal{P}_j(t-)} + (1 - \sum_{j=1}^m x_j(l - 1)) \frac{d\mathcal{P}_0(t)}{\mathcal{P}_0(t-)} \\ &= \mu(\mathbf{x}(l - 1)) dt + \sigma^2(\mathbf{x}(l - 1)) d\mathfrak{B}_{\mathbf{x}(l-1)}(t) + \sum_{j=1}^m x_j(l - 1) (e^{\mathcal{Z}_{\iota(t-),j,0}} - 1) d\mathcal{N}(t) \\ &\quad + \sum_{j=1}^m x_j(l - 1) (e^{\mathcal{Z}_{\iota(t-),j,1}} - 1) d\mathcal{N}_j(t). \end{aligned} \tag{3.3}$$

With a simple modification of Proposition 2.1 in Afhami et al. [1] we can obtain the solution of the Equation (3.3), and because of the similarity of proof, we do not provide the proof here.

**Theorem 3.1.** *If the random processes  $\mathcal{N}, \mathcal{N}_{j_1}$  and  $\mathcal{N}_{j_2}$  do not have common jumps for  $j_1 \neq j_2 = 1, 2, \dots, m$ , then the solution of (3.3) is as bellow*

$$\begin{aligned} \mathcal{P}(t) &= e^{\left(\mu(\mathbf{x}(l-1)) - \frac{\sigma^2(\mathbf{x}(l-1))}{2}\right)t + \sigma^2(\mathbf{x}(l-1)) \mathfrak{B}_{\mathbf{x}(l-1)}(t) + \sum_{j=1}^m \sum_{k=1}^{\mathcal{N}(t)} \mathcal{Z}_{k,j,0}^*(\mathbf{x}(l-1))} \\ &\quad \times e^{\sum_{j=1}^m \sum_{k=1}^{\mathcal{N}_j(t)} \mathcal{Z}_{k,j,1}^*(\mathbf{x}(l-1))}. \end{aligned} \tag{3.4}$$

where  $e^{\mathcal{Z}_{k,j,\ell}^*(\mathbf{x}(l-1))} - 1 = x_j(l-1)(e^{\mathcal{Z}_{k,j,\ell}} - 1)$ ,  $\ell = 0, 1, j = 1, \dots, m$ .

An investment of a unit amount of wealth at time  $t - 1$  will grow to amount  $e^{\mathcal{Y}_t(\mathbf{x}(l-1))}$  at time  $t$ , where

$$\begin{aligned} \mathcal{Y}_t(\mathbf{x}(l-1)) &= \mu(\mathbf{x}(l-1)) - \frac{\sigma^2(\mathbf{x}(l-1))}{2} + \sum_{j=1}^m \sum_{k=N(t-1)+1}^{\mathcal{N}(t)} \mathcal{Z}_{k,j,0}^*(\mathbf{x}(l-1)) \\ &+ \sum_{j=1}^m \sum_{k=N_j(t-1)+1}^{\mathcal{N}_j(t)} \mathcal{Z}_{k,j,1}^*(\mathbf{x}(l-1)) + \sigma^2(\mathbf{x}(l-1)) (\mathfrak{B}_{\mathbf{x}(l-1)}(t) - \mathfrak{B}_{\mathbf{x}(l-1)}(t-1)). \end{aligned} \tag{3.5}$$

In Afhami et al. [1], the terminal wealth was calculated from (3.4) assuming constant proportions and an optimal strategy was obtained to maximize the expectation of terminal wealth when the comonotonic approximation of *CLVaR*, was controlled. We now consider portfolio optimization under the DCAA strategy.

So the wealth  $\mathcal{W}_l$  at the end of the  $l$ -th period satisfy the recursion equation

$$\mathcal{W}_l = \mathcal{W}_{l-1} e^{\mathcal{Y}_l(\mathbf{x}(l-1))} + \alpha_l, \quad l = 1, 2, \dots, \tau,$$

with  $\mathcal{W}_0 = \alpha_0$ . Hence the terminal wealth can be written as

$$\mathcal{W}_\tau(\mathbf{x}(l-1)) = w_{l-1} e^{\sum_{k=l}^{\tau} \mathcal{Y}_k(\mathbf{x}(l-1))} + \sum_{j=l}^{\tau-1} \alpha_j e^{\sum_{k=j+1}^{\tau} \mathcal{Y}_k(\mathbf{x}(l-1))}. \tag{3.6}$$

Due to the limitation of borrowing from risky assets, we impose the upper bound  $c_0$  to  $\mu(\mathbf{x}(l-1))$ , so the portfolio optimization problem is given by:

$$\begin{aligned} &\max_{\mathbf{x}(l-1)} \mathbb{E}(\mathcal{W}_\tau(\mathbf{x}(l-1))), \\ &CLVaR_p(\mathcal{W}_\tau(\mathbf{x}(l-1))) \geq K, \\ &\mu(\mathbf{x}(l-1)) \leq c_0. \end{aligned} \tag{3.7}$$

Since we cannot obtain the exact distribution of  $\mathcal{W}_\tau$ , instead we approximate its comonotonic lower bound.

**Theorem 3.2.** Let  $\Lambda = \sigma^2(\mathbf{x}(l-1)) \sum_{k=l-1}^{\tau-1} \alpha_k (\mathfrak{B}_{\mathbf{x}(l-1)}(\tau) - \mathfrak{B}_{\mathbf{x}(l-1)}(k))$ , then the comonotonic lower bound is

$$\begin{aligned} \mathcal{W}_\tau^L(\mathbf{x}(l-1)) &= w_{l-1} e^{c_{1,l-1} + c_{2,l-1}(\mathbf{x}(l-1)) + c_{3,l-1} \sqrt{\sigma^2(\mathbf{x}(l-1))} \frac{\Lambda}{\sigma_\Lambda}} \\ &+ \sum_{i=l}^{\tau-1} \alpha_i e^{c_{1,i} + c_{2,i}(\mathbf{x}(l-1)) + c_{3,i} \sqrt{\sigma^2(\mathbf{x}(l-1))} \frac{\Lambda}{\sigma_\Lambda}}, \end{aligned} \tag{3.8}$$

where  $c_{1,n}$ ,  $c_{2,n}$ , and  $c_{3,n}$  are defined in (A.2), (A.3) and (A.4).

Calculating risk measures of (3.8) is difficult, to come up with this issue, we use the first-order Taylor expansion of the exponential function. Its result is given in following Lemma.

**Lemma 3.3.** *We can approximate  $\mathcal{W}_\tau^L(\mathbf{x}(l-1))$  using the first-order Taylor expansion as*

$$\mathcal{W}_\tau^L(\mathbf{x}(l-1)) = c_4 + c_5(\mathbf{x}(l-1)) + c_6 \sqrt{\sigma^2(\mathbf{x}(l-1))} \frac{\Lambda}{\sigma_\Lambda}, \tag{3.9}$$

where  $c_4$ ,  $c_5(\mathbf{x}(l-1))$  and  $c_6$  are given in (A.5), (A.6) and (A.7).

One can easily shown that  $\mathbb{E}(\mathcal{W}_\tau^L) = c_4 + c_5(\mathbf{x}(l-1))$ , and its risk measure is

$$CVaR_{1-p}(-\mathcal{W}_\tau^L) = -c_4 - c_5(\mathbf{x}(l-1)) + c_7 \sqrt{\sigma^2(\mathbf{x}(l-1))},$$

where  $c_7 = c_6 \frac{1}{p\sqrt{2\pi}} e^{-\frac{1}{2}(\Phi^{-1}(p))^2}$ .

So portfolio optimization problem reduces to

$$\begin{aligned} & \max_{\mathbf{x}(l-1)} (c_4 + c_5(\mathbf{x}(l-1))), \\ & -c_4 - c_5(\mathbf{x}(l-1)) + c_7 \sqrt{\sigma^2(\mathbf{x}(l-1))} \leq -K, \\ & r + \mathbf{A}^\top \mathbf{x}(l-1) \leq c_0, \end{aligned} \tag{3.10}$$

and its solution is given in the following result.

**Theorem 3.4.** *The solution of problem (2.9) for  $p < 0.5$  is of the form  $\mathbf{x}(l-1) = q(l-1) \mathbf{x}^*$ , where  $\mathbf{x}^* = \Sigma^{-1} \mathbf{A}$ ,  $q(l-1) = \min\{q_1, q_2, q_3\}$ , and*

$$q_1 = \frac{-B_2 - \sqrt{B_2^2 - 4 B_1 B_3}}{2 B_1}, \quad q_2 = \frac{c_8}{2 c_9}, \quad q_3 = \frac{c_0 - r}{\mathbf{A}^\top \Sigma^{-1} \mathbf{A}}, \tag{3.11}$$

with

$$B_1 = -c_9 (\mathbf{A}^\top \Sigma^{-1} \mathbf{A}), \quad B_2 = \frac{c_8}{-c_9} B_1 - c_7 \sqrt{\frac{B_1}{-c_9}}, \quad B_3 = c_4 - K,$$

and

$$\begin{aligned} c_8 &= w_{l-1} (\tau - l + 1) + \sum_{i=l}^{\tau-1} \alpha_i (\tau - i), \\ c_9 &= \frac{1}{2} w_{l-1} (\tau - l + 1) r_{l-1}^2 + \frac{1}{2} \sum_{i=l}^{\tau-1} \alpha_i r_i^2. \end{aligned}$$

### 4. Data Example

In Example 4.1, we compute the optimal portfolio for a DCAA strategy for two stocks listed in the S&P 500 stock market index.

**Example 4.1.** *In this example, we consider the daily prices of ZM and TSLA for time horizon  $\tau = 6$  months from August of 2020 to January of 2021. These prices are plotted in Figure 1.*

*For simplification, we set  $\alpha_0 = \alpha_1 = \dots = \alpha_5 = 1$ , and assume that the interest rate and the risk budget are  $r = 0.03$  and  $p = 0.05$ , respectively. In Equation (3.10) we set  $K = k^* \sum_{i=1}^{\tau} e^{\frac{(\tau-i+1)r}{\tau}}$  where  $k^*$  is the stop loss rate. From Theorem 3.4 we obtain the optimal proportions  $x_1(l)$  and  $x_2(l)$*

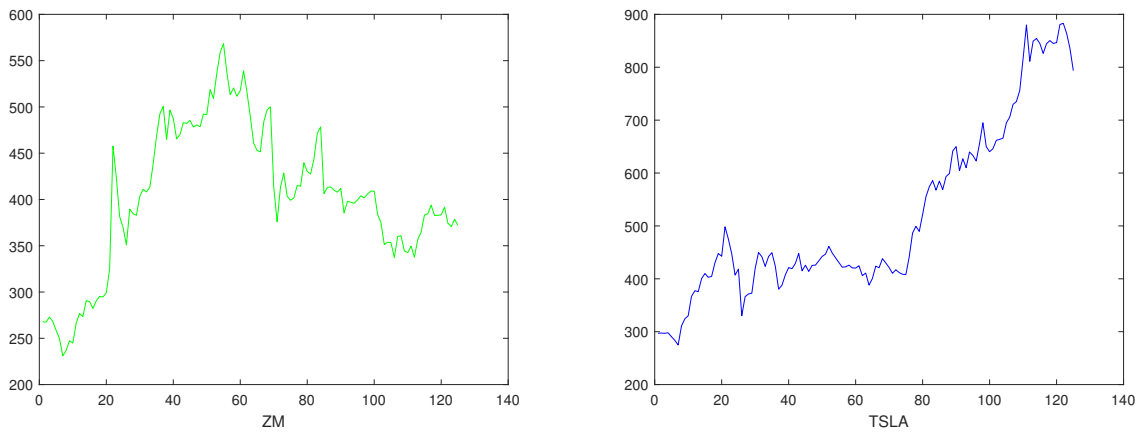


Figure 1: Diagram of daily prices of assets.

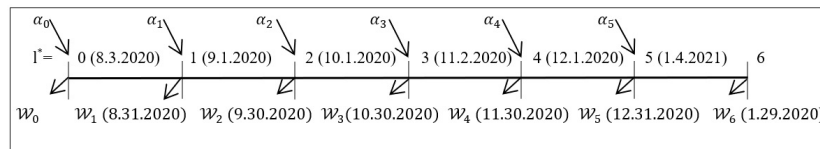


Figure 2: Illustration of the algorithm for six periods.

respectively of the stocks ZM and TSLA at the beginning of the  $l$ -th period i.e. at time  $l^* = l - 1$  and write  $\mathbf{x}(l^*) = (x_1(l^*), x_2(l^*))$ , ( $l^* = 0, 1, \dots, 5$ ).

Let  $\mathbf{S}_b(l) = (S_{b,1}(l), S_{b,2}(l))$  and  $\mathbf{S}_e(l) = (S_{e,1}(l), S_{e,2}(l))$  be vectors respectively containing the prices of ZM and TSLA at the beginning and end of the  $l$ -th period, respectively.

Consider  $\mathbf{f}_l = (\frac{\mathcal{W}_l x_1(l)}{S_{b,1}(l)}, \frac{\mathcal{W}_l x_2(l)}{S_{b,2}(l)})$  where  $\frac{\mathcal{W}_l x_1(l)}{S_{b,1}(l)}$  and  $\frac{\mathcal{W}_l x_2(l)}{S_{b,2}(l)}$  are respectively the stock volumes of ZM and TSLA which are bought at the beginning of the  $l$ -th period. We compute terminal wealth ( $\mathcal{W}_6$ ) as follows:

**First step:** At time  $l^* = 0$  (or  $l = 1$ ), with  $\mathcal{W}_0 = 1$  we obtain  $\mathbf{x}(0)$  using Theorem 3.4.

**Second step:** At time  $l^* = 1$  (or  $l = 2$ ), we calculate  $\mathcal{W}_1 = \mathbf{f}_0 \mathbf{S}_e(0)^\top + \alpha_1$ , and again using Theorem 3.4 obtain  $\mathbf{x}(1)$ .

**Third step:** We repeat second step for  $l^* = 2, 3, \dots, 5$  (or  $l = 3, 4, \dots, 6$ ), with  $\mathcal{W}_{l^*} = \mathbf{f}_{l^*-1} \mathbf{S}_e(l^* - 1)^\top + \alpha_{l^*}$ , and using Theorem 3.4 to obtain  $\mathbf{x}(l^*)$ .

This algorithm is illustrated in Figure 2.

The results for different values of  $k^*$  are given in Table 2. This table illustrates that wealth and return of portfolio decreases as  $k^*$  increases.

optimization of a dynamic investment portfolio when the securities which follow a multivariate Merton model with dependent jumps are periodically invested and proceed by approximating

Table 2: Optimal wealth for the first six periods.

$k^*$	$\mathcal{W}_1$	$\mathcal{W}_2$	$\mathcal{W}_3$	$\mathcal{W}_4$	$\mathcal{W}_5$	$\mathcal{W}_6$	Return
0.50	2.4640	3.3657	4.0509	6.0675	7.2707	7.7254	26.5194%
0.85	2.3898	3.2944	3.9863	5.9867	7.1871	7.6366	25.0651%
0.90	2.3118	3.0252	3.7422	5.6813	6.8715	7.3012	19.5722%
0.95	2.2326	2.7155	3.4355	5.2977	6.4751	6.8801	12.6758%



the Condition-Value-at-Risk (CVaR) by comonotonic bounds and maximize the expected terminal wealth.

### 5. Discussion

In this paper, we considered the problem of portfolio optimization of a dynamic investment portfolio when the securities which follow a multivariate Merton model with dependent jumps are periodically invested by controlling *CVaR*. The comonotonic bounds of CVaR were obtained since the solution to the optimization problem does not have a closed-form. Real data analysis is provided for more illustration. This work can be extended in several directions. For example, the compound Poisson process can be replaced by a multivariate Lévy process, or a Lévy copula can be implemented to model stock prices in a portfolio optimization problem.

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### Appendix A.

#### Proof of Theorem 3.2:

**Proof .** From (3.6) and (3.5) it can be easily shown

$$\mathcal{W}_\tau(\mathbf{x}(l-1)) = w_{l-1} e^{V_{l-1}+S_{l-1}} + \sum_{i=l}^{\tau-1} \alpha_i e^{V_i+S_i},$$

where for  $n = l - 1, l, \dots, \tau - 1$ ,

$$\begin{aligned} V_n &= \sigma^2(\mathbf{x}(l-1)) (\mathfrak{B}_{\mathbf{x}(l-1)}(\tau) - \mathfrak{B}_{\mathbf{x}(l-1)}(n)), \\ S_n &= (\tau - n) \left( \mu(\mathbf{x}(l-1)) - \frac{\sigma^2(\mathbf{x}(l-1))}{2} \right) + \sum_{j=1}^m \sum_{k=N(n)+1}^{N(\tau)} \mathcal{Z}_{k,j,0}^*(\mathbf{x}(l-1)) \\ &\quad + \sum_{j=1}^m \sum_{k=N_j(n)+1}^{N_j(\tau)} \mathcal{Z}_{k,j,1}^*(\mathbf{x}(l-1)), \end{aligned}$$

By using Theorem 2.4, we obtain lower bound for  $\mathcal{W}_\tau(\mathbf{x}(l-1))$ . First we set

$$\Lambda = \sigma^2(\mathbf{x}(l-1)) \sum_{k=l-1}^{\tau-1} \alpha_k (\mathfrak{B}_{\mathbf{x}(l-1)}(\tau) - \mathfrak{B}_{\mathbf{x}(l-1)}(k)), \text{ so}$$

$$\sigma_\Lambda^2 = \sigma^2(\mathbf{x}(l-1)) \sum_{k,w=l-1}^{\tau-1} \alpha_k \alpha_w \min\{\tau - k, \tau - w\},$$

$$\sigma_{V_n}^2 = (\tau - n) \sigma^2(\mathbf{x}(l-1)),$$

moreover

$$r_n = Corr(V_n, \Lambda) = \frac{\sum_{k=l-1}^{\tau-1} \alpha_k \min\{\tau - n, \tau - k\}}{\sqrt{(\tau - n) \sum_{k,w=l-1}^{\tau-1} \alpha_k \alpha_w \min\{\tau - k, \tau - w\}}}.$$

Therefore the lower bound is

$$\begin{aligned} \mathcal{W}_\tau^L(\mathbf{x}(l-1)) &= \mathbb{E} \left( w_{l-1} e^{V_{l-1}+S_{l-1}} + \sum_{i=l}^{\tau-1} \alpha_i e^{V_i+S_i} \mid \Lambda \right) \\ &= w_{l-1} \mathbb{E}(e^{V_{l-1}} \mid \Lambda) \mathbb{E}(e^{S_{l-1}}) + \sum_{i=l}^{\tau-1} \alpha_i \mathbb{E}(e^{V_i} \mid \Lambda) \mathbb{E}(e^{S_i}). \end{aligned} \tag{A.1}$$

Since  $V_n \mid \Lambda = \lambda$  has standard normal distribution with mean and variance, respectively  $r_n \sigma_{V_n} \frac{\lambda}{\sigma_\Lambda}$ , and  $\sigma_{V_n}^2 (1 - r_n^2)$ . We have

$$\mathbb{E}(e^{V_n} \mid \Lambda) = e^{r_n \sqrt{(\tau-n)\sigma^2(\mathbf{x}(l-1))} \frac{\Lambda}{\sigma_\Lambda} + \frac{1}{2}(\tau-n)\sigma^2(\mathbf{x}(l-1))(1-r_n^2)},$$

and

$$\begin{aligned} \mathbb{E}(e^{S_n}) &= e^{(\tau-n)\mu(\mathbf{x}(l-1)) - \frac{1}{2}(\tau-n)\sigma^2(\mathbf{x}(l-1)) + \lambda(\tau-n) \left( \prod_{j=1}^m M_{Z_{1,j,0}^*(\mathbf{x}(l-1))}(1) - 1 \right)} \\ &\quad \times e^{(\tau-n) \sum_{j=1}^m \lambda_j \left( M_{Z_{1,j,1}^*(\mathbf{x}(l-1))}(1) - 1 \right)}. \end{aligned}$$

Thus

$$\begin{aligned} \mathcal{W}_\tau^L(\mathbf{x}(l-1)) &= w_{l-1} e^{c_{1,l-1} + c_{2,l-1}(\mathbf{x}(l-1)) + c_{3,l-1} \sqrt{\sigma^2(\mathbf{x}(l-1))} \frac{\Lambda}{\sigma_\Lambda}} \\ &\quad + \sum_{i=l}^{\tau-1} \alpha_i e^{c_{1,i} + c_{2,i}(\mathbf{x}(l-1)) + c_{3,i} \sqrt{\sigma^2(\mathbf{x}(l-1))} \frac{\Lambda}{\sigma_\Lambda}}, \end{aligned}$$

where

$$\begin{aligned} c_{1,n} &= (\tau - n)r + \lambda(\tau - n) \left( \prod_{j=1}^m M_{Z_{1,j,0}^*(\mathbf{x}(l-1))}(1) - 1 \right) \\ &\quad + (\tau - n) \sum_{j=1}^m \lambda_j \left( M_{Z_{1,j,1}^*(\mathbf{x}(l-1))}(1) - 1 \right), \end{aligned} \tag{A.2}$$

$$c_{2,n}(\mathbf{x}(l-1)) = (\tau - n) (\mathbf{A}^\top \mathbf{x}(l-1)) - \frac{1}{2} (\tau - n) r_n^2 \sigma^2(\mathbf{x}(l-1)), \tag{A.3}$$

$$c_{3,n} = \sqrt{(\tau - n)} r_n, \tag{A.4}$$

which completes the proof of the result.  $\square$

**Proof of Lemma 3.3:** Here, we use the first-order Taylor expansion of the exponential function in (3.8). Therefore,

$$\begin{aligned} \mathcal{W}'_\tau^L(\mathbf{x}(l-1)) &= w_{l-1} \left( 1 + c_{1,l-1} + c_{2,l-1}(\mathbf{x}(l-1)) + c_{3,l-1} \sqrt{\sigma^2(\mathbf{x}(l-1))} \frac{\Lambda}{\sigma_\Lambda} \right) \\ &\quad + \sum_{i=l}^{\tau-1} \alpha_i \left( 1 + c_{1,i} + c_{2,i}(\mathbf{x}(l-1)) + c_{3,i} + \frac{\Lambda}{\sigma_\Lambda} \sqrt{\sigma^2(\mathbf{x}(l-1))} \right). \end{aligned}$$

By straightforward calculation, we have

$$\mathcal{W}'^L(\mathbf{x}(l-1)) = c_4 + c_5(\mathbf{x}(l-1)) + c_6 \sqrt{\sigma^2(\mathbf{x}(l-1))} \frac{\Lambda}{\sigma_\Lambda},$$

where

$$c_4 = w_{l-1} (1 + c_{1,l-1}) + \sum_{i=l}^{\tau-1} \alpha_i (1 + c_{1,i}), \quad (\text{A.5})$$

$$c_5(\mathbf{x}(l-1)) = w_{l-1} c_{2,l-1}(\mathbf{x}(l-1)) + \sum_{i=l}^{\tau-1} \alpha_i c_{2,i}(\mathbf{x}(l-1)), \quad (\text{A.6})$$

and

$$c_6 = w_{l-1} c_{3,l-1} + \sum_{i=l}^{\tau-1} \alpha_i c_{3,i}. \quad (\text{A.7})$$

This completes the proof of results.

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