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Fixed points of a new class of compatible mappings satisfying an implicit relation via inverse C-class function

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Abstract

In this paper, we establish some common fixed point theorems of a new class of compatible mappings satisfying an implicit relation via inverse C-class function.

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1. Introduction and Preliminaries

In 1922, Stephen Banach [5] established a fixed point theorem using Picard iteration known as "Banach contraction principle", which is one of important result and it is widely used fixed point theorem not only in mathematical analysis but also in other related branch of applied Sciences. In 1976, Jungck [9] established existence theorems of common fixed point of two commuting mappings by extending Banach contraction principle. Sessa [20] introduced a weaker hypothesis than commutativity, called weak commutativity of two mappings and generalized the results of Jungck[9] and Daneš [6]. In 1986, Jungck [10] generalized the notion of commutativity and weak commutativity by introducing the notion of compatible mappings. Further in 1998, Jungck and Rhoades [12] introduced the weaker form of compatibility named weak compatibility. In 2008, Al-Thagafi et al. [2]

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introduced other weaker form of compatible mappings by introducing notion of occasionally weakly compatible (in short, *owc*) and proved that weakly compatible mappings are *owc*, but converse is not true in general. On the other hand, many authors also introduced different types of compatible and non compatible mappings to prove common fixed point theorems (for more details and their relationship, we refer to Jungck et al. [11], Pathak and Khan [17, 18], Pathak et al.[15], Pathak et al. [16], Singh and Singh[21, 22] and references therein).

In this paper, we establish the existence of common fixed point theorems for six mappings using a new class of compatible mappings satisfying with an implicit relation via inverse C-class function. Let $\mathcal{A}, \mathcal{B}$ and $\mathcal{T}$ be self mappings on a metric space (X, d).

Definition 1.1. *[10] A pair $(\mathcal{A}, \mathcal{B})$ on a metric space (X, d) is called compatible mappings if*

$$\lim_{n\to+\infty} d(\mathcal{A}\mathcal{B}x, \mathcal{B}\mathcal{A}x_n) = 0,$$

whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to+\infty} \mathcal{A}x_n = \lim_{n\to+\infty} \mathcal{B}x_n = w$, for some $w \in X$.

Definition 1.2. *[18] A pair $(\mathcal{A}, \mathcal{B})$ on a metric space (X, d) is called $\mathcal{A}$-compatible if*

$$\lim_{n\to+\infty} d(\mathcal{A}\mathcal{B}x, \mathcal{B}\mathcal{B}x_n) = 0,$$

whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to+\infty} \mathcal{A}x_n = \lim_{n\to+\infty} \mathcal{B}x_n = w$, for some $w \in X$.

Definition 1.3. *[18] A pair $(\mathcal{A}, \mathcal{B})$ on a metric space (X, d) is called $\mathcal{B}$-compatible if*

$$\lim_{n\to+\infty} d(\mathcal{B}\mathcal{A}x, \mathcal{A}\mathcal{A}x_n) = 0,$$

whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to+\infty} \mathcal{A}x_n = \lim_{n\to+\infty} \mathcal{B}x_n = w$, for some $w \in X$.

Definition 1.4. *[12] A pair $(\mathcal{A}, \mathcal{B})$ on a metric space (X, d) is called weakly compatible if they commute at their coincidence points i.e., $\mathcal{A}w = \mathcal{B}w$ implies $\mathcal{A}\mathcal{B}w = \mathcal{B}\mathcal{A}w$, for some $w \in X$.*

We define the notion of $\mathcal{S}_{\mathcal{T}}$-compatible mappings.

Definition 1.5. *[14] A pair $(\mathcal{A}, \mathcal{B})$ is said to be compatible w.r.t $\mathcal{T}$ (in short, $\mathcal{S}_{\mathcal{T}}$-compatible) if there exists a sequence $\{x_n\}$ in X such that*

$$\lim_{n\to+\infty} \mathcal{T}x_n = w \text{ and } \lim_{n\to+\infty} \mathcal{A}\mathcal{T}x_n = \lim_{n\to+\infty} \mathcal{B}\mathcal{T}x_n = \mathcal{T}w,$$

for some $w \in X$.

Example 1.6. *Let $X = \mathbb{R}$ with usual metric $d(x, y) = |x - y|$, for all $x, y \in X$. Define $\mathcal{A}, \mathcal{B}, \mathcal{T} : X \to X$ as $\mathcal{A}x = 2x$, $\mathcal{B}x = 3 - x$ and $\mathcal{T}x = x + 1$, for all $x \in X$. Consider a sequence $\{x_n\}$ in X, where $x_n = \frac{1}{3} + \epsilon_n, n \in \mathbb{N}$ and $\epsilon_n \to 0$ as $n \to +\infty$, then $\lim_{n\to+\infty} \mathcal{T}x_n = \frac{4}{3} = w$(say). Moreover, $\lim_{n\to+\infty} \mathcal{A}\mathcal{T}x_n = \lim_{n\to+\infty} \mathcal{B}\mathcal{T}x_n = \frac{8}{3} \neq \mathcal{T}w = \mathcal{T}\frac{4}{3} = \frac{7}{3}$. On the other hand, let $\{y_n\}$ be a sequence in X, defined by $y_n = \frac{1}{n}$, $n \in \mathbb{N}$, then $\lim_{n\to+\infty} \mathcal{T}y_n = 1 = w \in X$ and $\mathcal{T}w = 2$. Also we have, $\lim_{n\to+\infty} \mathcal{A}\mathcal{T}y_n = \lim_{n\to+\infty} \mathcal{B}\mathcal{T}y_n = 2 = \mathcal{T}w$. Therefore, the pair $(\mathcal{A}, \mathcal{B})$ is $\mathcal{S}_{\mathcal{T}}$-compatible for the sequence $\{y_n\}$ in X.*

Example 1.7. *Let $X = [0, \infty)$ with usual metric $d(x, y) = |x - y|$. Define $\mathcal{A}, \mathcal{B}, \mathcal{T} : X \to X$ as $\mathcal{A}x = x$, $\mathcal{B}x = x^2$, $\mathcal{T}x = 3x$, for all $x \in X$. Consider a sequence $\{x_n\}$ in X, where $x_n = \frac{1}{n}$, $n \in \mathbb{N}$. Then $\lim_{n\to+\infty} \mathcal{T}x_n = 0$ and also $\lim_{n\to+\infty} \mathcal{A}\mathcal{T}x_n = \lim_{n\to+\infty} \mathcal{B}\mathcal{T}x_n = T0$. Moreover, if $\{y_n\}$ be a sequence in X such that $y_n \to \frac{1}{3}$ as $n \to +\infty$. Then, $\lim_{n\to+\infty} \mathcal{T}y_n = 1 = w$(say) and $\lim_{n\to+\infty} \mathcal{A}\mathcal{T}y_n = \lim_{n\to+\infty} \mathcal{B}\mathcal{T}y_n = 1 \neq 3 = \mathcal{T}w$. Therefore, the pair $(\mathcal{A}, \mathcal{B})$ is $\mathcal{S}_{\mathcal{T}}$- compatible for the sequence $\{x_n\}$ in X.*

We define the above definition for a pair of mappings $(\mathcal{A}, \mathcal{B})$ on a metric space (X, d) as follows:

Definition 1.8. *A pair $(\mathcal{A}, \mathcal{B})$ is said to be $\mathcal{S}_{\mathcal{A}}$-compatible if there exists a sequence $\{x_n\}$ in X such that $\lim_{n\to+\infty} \mathcal{A}x_n = w$ and $\lim_{n\to+\infty} \mathcal{A}\mathcal{A}x_n = \lim_{n\to+\infty} \mathcal{B}\mathcal{A}x_n = \mathcal{A}w$, for some $w \in X$.*

Definition 1.9. *A pair $(\mathcal{A}, \mathcal{B})$ is said to be $\mathcal{S}_{\mathcal{B}}$-compatible if there exists a sequence $\{x_n\}$ in X such that $\lim_{n\to+\infty} \mathcal{B}x_n = w$ and $\lim_{n\to+\infty} \mathcal{A}\mathcal{B}x_n = \lim_{n\to+\infty} \mathcal{B}\mathcal{B}x_n = \mathcal{B}w$, for some $w \in X$.*

Example 1.10. *Let $X = [1, \infty)$ with usual metric d. Define self mappings $\mathcal{A}, \mathcal{B} : X \to X$ as:*

$$\mathcal{A}x = \begin{cases} 2x - 1, & x \in [1, 2) \\ 4, & x \in [2, \infty) \end{cases} ; \quad \mathcal{B}x = \begin{cases} 3x - 1, & x \in [1, 2) \\ x + 2, & x \in [2, \infty). \end{cases}$$

Choose $x_n = 1 + \epsilon_n$, $n \in \mathbb{N}$, where $\epsilon_n \to 0$ as $n \to +\infty$, then $\lim_{n\to+\infty} \mathcal{B}x_n = 2 = w$(say). Also $\lim_{n\to+\infty} \mathcal{A}\mathcal{B}x_n = \lim_{n\to+\infty} \mathcal{B}\mathcal{B}x_n = 4 = \mathcal{B}w$, then $(\mathcal{A}, \mathcal{B})$ is $\mathcal{S}_{\mathcal{B}}$–compatible.

Example 1.11. *Let $X = [1, \infty)$ with usual metric d. Define self mappings $\mathcal{A}, \mathcal{B} : X \to X$ as:*

$$\mathcal{A}x = \begin{cases} 1, & x \in [1, 2) \\ 3, & x \in [2, \infty) \end{cases} ; \quad \mathcal{B}x = \begin{cases} x + 1, & x \in [1, 2) \\ \frac{x+4}{2}, & x \in [2, \infty). \end{cases}$$

Choose $x_n = 1 + \epsilon_n$ in X such that $\epsilon_n \to 0$ as $n \to +\infty$, then $\lim_{n\to+\infty} \mathcal{B}x_n = 2 = w$(say). Moreover $\lim_{n\to+\infty} \mathcal{A}\mathcal{B}x_n = 3 = \mathcal{B}w$. Since $\lim_{n\to+\infty} \mathcal{B}\mathcal{B}x_n = \mathcal{B}w = 3$ as $\mathcal{B}$ is continuous. Then $(\mathcal{A}, \mathcal{B})$ is $\mathcal{S}_{\mathcal{B}}$-compatible.

Proposition 1.12. *Let $\mathcal{A}$ and $\mathcal{B}$ be self mappings of a metric space (X, d).*
(i) If $(\mathcal{A}, \mathcal{B})$ is compatible (resp. $\mathcal{B}$-compatible) and $\mathcal{A}$ is continuous, then the pair $(\mathcal{A}, \mathcal{B})$ is $\mathcal{S}_{\mathcal{A}}$-compatible;
(ii) If $(\mathcal{A}, \mathcal{B})$ is compatible (resp. $\mathcal{A}$-compatible) and $\mathcal{B}$ is continuous, then the pair $(\mathcal{A}, \mathcal{B})$ is $\mathcal{S}_{\mathcal{B}}$-compatible.

Proof . (i) Suppose$(\mathcal{A}, \mathcal{B})$ is compatible, then $\lim_{n\to\infty} d(\mathcal{A}\mathcal{B}x_n, \mathcal{B}\mathcal{A}x_n) = 0$ whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} \mathcal{A}x_n = \lim_{n\to\infty} \mathcal{B}x_n = w$, for some $w \in X$. By continuity of A, we have $\lim_{n\to\infty} \mathcal{A}\mathcal{A}x_n = \lim_{n\to\infty} \mathcal{A}\mathcal{B}x_n = \mathcal{A}w$ and consequently, we have $\lim_{n\to\infty} \mathcal{B}\mathcal{A}x_n = \mathcal{A}w$. Thus, $(\mathcal{A}, \mathcal{B})$ is S_A-compatible. Again suppose that $(\mathcal{A}, \mathcal{B})$ is $\mathcal{B}$-compatible, then $\lim_{n\to\infty} d(\mathcal{B}\mathcal{A}x_n, \mathcal{A}\mathcal{A}x_n) = 0$, whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} \mathcal{A}x_n = \lim_{n\to\infty} \mathcal{B}x_n = w$, for some $w \in X$. Since A is continuous, then $\lim_{n\to\infty} \mathcal{A}\mathcal{A}x_n = Aw$. Thus, $\lim_{n\to\infty} \mathcal{A}\mathcal{A}x_n = \lim_{n\to\infty} \mathcal{B}\mathcal{A}x_n = \mathcal{A}w$. Therefore, $(\mathcal{A}, \mathcal{B})$ is $\mathcal{S}_{\mathcal{A}}$-compatible.
(ii) Similarly, one can prove as in (i). □

Definition 1.13. *[3] A continuous function $K : [0, \infty) \times [0, \infty) \to \mathbb{R}$ is called a C-class function if for any $s, t \in [0, \infty)$, the following conditions holds:*
(i) $K(s, t) \leq s$;
(ii) $K(s, t) = s$ implies $s = 0$ or $t = 0$.

Definition 1.14. *[19] A continuous function $F : [0,\infty) \times [0,\infty) \to \mathbb{R}$ is called an inverse C-class function if for any $s, t \in [0,\infty)$, the following conditions holds:*
(i) $F(s,t) \geq s$;
(ii) $F(s,t) = s$ implies $s = 0$ or $t = 0$.

Denote C_{inv}, the set of all inverse C-class functions. For more examples on C-class and inverse C-class functions, we refer to Ansari [3] and Saleem et al.[19].

Definition 1.15. *[13] A function $\varphi : [0,\infty) \to [0,\infty)$ is called an altering distance if φ is continuous, non-decreasing and $\varphi(0) = 0$.*

Definition 1.16. *[3] A function $\varphi : [0,\infty) \to [0,\infty)$ is called an ultra altering distance if φ is continuous, $\varphi(t) > 0$, for all $t > 0$ and $\varphi(0) \geq 0$. We denote Φ_u, the set of all ultra altering distance functions.*

An implicit relation, which was used by Djoudi[7, 8] is as follows:
Let $\mathbb{R}_+$ be the set of all non-negative real numbers and g be the set of all continuous functions $g(t_1, t_2, ..., t_6) : \mathbb{R}_+^6 \to \mathbb{R}$ that satisfies the following conditions:
(g_1) : g is non decreasing in variables t_5 and t_6;
(g_2): There exists $h \in (1,\infty)$ such that for every $u, v \geq 0$ with
$(g_{2a}) : g(u, v, u, v, u + v, 0) \geq 0$, or $(g_{2b}) : g(u, v, v, u, 0, u + v) \geq 0$, we have $u \geq hv$;
$(g_3) : g(u, u, 0, 0, u, u) < 0$, for every $u > 0$.

Example 1.17. *Define $g(t_1, t_2, ..., t_6) = 1 - \frac{t_2(h+1)}{\max\{t_1,t_2,t_3,t_4,t_5+t_6\}}$, where $h \in (1,\infty)$.*
(g_1): It is obvious;
(g_2): Let $u, v \in \mathbb{R}_+$, for $h \in (1,\infty)$, we have
(g_{2a}): $g(u, v, u, v, u + v, 0) = 1 - \frac{v(h+1)}{\max\{u,v,u,v,u+v\}} = 1 - \frac{v(h+1)}{u+v} \geq 0$, then $u \geq hv$;
(g_{2b}): $g(u, v, v, u, 0, u + v) = 1 - \frac{v(h+1)}{\max\{u,v,v,u,0,u+v\}} = 1 - \frac{v(h+1)}{u+v} \geq 0$, then $u \geq vh$,
(g_3): $g(u, u, 0, 0, u, u) = 1 - \frac{u(h+1)}{\max\{u,u,0,0,2u\}} = 1 - \frac{u(h+1)}{2u} < 0$, for all $u > 0$.

Example 1.18. *Define $g(t_1, t_2, ..., t_6) = at_1 - bt_2 + c(t_3 + t_4) - \min\{t_3, t_5t_6\}$, where $0 < c < \frac{b-a}{2}$ and $h = \frac{b-c}{a+c} > 1$.*
(g_1): It is obvious;
(g_2): For $u, v \geq 0$ and $h \in (1,\infty)$, we have
(g_{2a}):$g(u, v, u, v, u + v, 0) = au - bv + c(u + v) \geq 0$, then $u \geq hv$;
(g_{2b}): $g(u, v, v, u, 0, u + v) = au - bv + c(u + v) \geq 0$, then $u \geq hv$;
(g_3): $g(u, u, 0, 0, u, u) = au - bu = -(b - a)u < 0$, for all $u > 0$.

For more examples and fixed point results, we refer to Akkouchi[1] and Djoudi [7, 8] and references therein. Motivated by Ansari et al.[4], we extend an implicit relation of Djoudi [7, 8] by using the inverse C-class function.
Let $\mathcal{G}_C$ be the set of all continuous functions $\mathcal{G}(t_1, t_2, ..., t_6) : \mathbb{R}_+^6 \to \mathbb{R}$ satisfying the conditions:
$(\mathcal{G}_1)$: $\mathcal{G}$ is non decreasing in variables t_5 and t_6;
$(\mathcal{G}_2)$: There exists $F \in C_{inv}$ such that for every $u, v \geq 0$ with
$(\mathcal{G}_{2a}) : \mathcal{G}(u, v, u, v, u + v, 0) \geq 0$, or $(\mathcal{G}_{2b}) : G(u, v, v, u, 0, u + v) \geq 0$, we have $u \geq F(v, \varphi(v))$;
$(\mathcal{G}_3) : \mathcal{G}(u, u, 0, 0, u, u) < 0$, for every $u > 0$.

2. Main Results

Theorem 2.1. *Let (X,d) be a complete metric space and $\mathcal{A},\mathcal{B},\mathcal{P},\mathcal{T},\mathcal{F},\mathcal{I}: X \to X$ be self mappings that satisfy the conditions:*
(i) $\mathcal{A}$ and $\mathcal{B}$ are surjective;
(ii) there exists $\mathcal{G} \in \mathcal{G}_C$ such that

$$\mathcal{G}\Big(d(\mathcal{A}x,\mathcal{B}y), d(\mathcal{P}\mathcal{F}x,\mathcal{T}\mathcal{I}y), d(\mathcal{A}x,\mathcal{P}\mathcal{F}x), d(\mathcal{B}y,\mathcal{T}\mathcal{I}y), d(\mathcal{A}x,\mathcal{T}\mathcal{I}y),\\ d(\mathcal{B}y,\mathcal{P}\mathcal{F}x)\Big) \geq 0$$

for all $x,y \in X$;
(iii) $(\mathcal{B},\mathcal{T}\mathcal{I})$ is $\mathcal{S}_{\mathcal{A}}$-compatible and $(\mathcal{A},\mathcal{P}\mathcal{F})$ is $\mathcal{S}_{\mathcal{B}}$-compatible.
Then $\mathcal{A}$, $\mathcal{B}$, $\mathcal{P}\mathcal{F}$ and $\mathcal{T}\mathcal{I}$ have a common fixed point in X.
Further, if the pairs $(\mathcal{A},\mathcal{F})$, $(\mathcal{P},\mathcal{F})$, $(\mathcal{B},\mathcal{I})$, and $(\mathcal{T},\mathcal{I})$ are commuting at common fixed points of $\mathcal{A}$, $\mathcal{B}$, $\mathcal{P}\mathcal{F}$ and $\mathcal{T}\mathcal{I}$. Then $\mathcal{A}$, $\mathcal{B}$, $\mathcal{P}$, $\mathcal{T}$, $\mathcal{F}$ and $\mathcal{I}$ have a unique common fixed point in X.

Proof . Let $x_0 \in X$ be an arbitrary. Since $\mathcal{A}$ is surjective, there exists $x_1 \in X$ such that $\mathcal{A}x_1 = \mathcal{T}\mathcal{I}x_0 = y_0$. For $x_1 \in X$ and $\mathcal{B}$ is surjective, there exists $x_2 \in X$ such that $\mathcal{B}x_2 = \mathcal{P}\mathcal{F}x_1 = y_1$. Similarly, we can choose $x_3, x_4 \in X$ such that $\mathcal{A}x_3 = \mathcal{T}\mathcal{I}x_2 = y_2$ and $\mathcal{B}x_4 = \mathcal{P}\mathcal{F}x_3 = y_3$. Inductively, we can construct two sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$$y_{2n} = \mathcal{A}x_{2n+1} = \mathcal{T}\mathcal{I}x_{2n};\ y_{2n+1} = \mathcal{B}x_{2n+2} = \mathcal{P}\mathcal{F}x_{2n+1},\ \text{where } n \in \mathbb{N}.$$

By condition (ii), we obtain

$$\mathcal{G}\Big(d(\mathcal{A}x_{2n+1},\mathcal{B}x_{2n+2}), d(\mathcal{P}\mathcal{F}x_{2n+1},\mathcal{T}\mathcal{I}x_{2n+2}), d(\mathcal{A}x_{2n+1},\mathcal{P}\mathcal{F}x_{2n+1}),\\ d(\mathcal{B}x_{2n+2},\mathcal{T}\mathcal{I}x_{2n+2}), d(\mathcal{A}x_{2n+1},\mathcal{T}\mathcal{I}x_{2n+2}), d(\mathcal{B}x_{2n+2},\mathcal{P}\mathcal{F}x_{2n+1})\Big) \geq 0$$
$$\mathcal{G}\Big(d(y_{2n},y_{2n+1}), d(y_{2n+1},y_{2n+2}), d(y_{2n},y_{2n+1}), d(y_{2n+1},y_{2n+2}), d(y_{2n},y_{2n+2}), d(y_{2n+1},y_{2n+1})\Big) \geq 0$$
$$\mathcal{G}\Big(d(y_{2n},y_{2n+1}), d(y_{2n+1},y_{2n+2}), d(y_{2n},y_{2n+1}), d(y_{2n+1},y_{2n+2}), d(y_{2n},y_{2n+2}), 0\Big) \geq 0.$$

By triangle inequality, we have $d(y_{2n},y_{2n+2}) \leq d(y_{2n},y_{2n+1}) + d(y_{2n+1},y_{2n+2})$. Then by $(\mathcal{G}_1)$, we obtain

$$\mathcal{G}\Big(d(y_{2n},y_{2n+1}), d(y_{2n+1},y_{2n+2}), d(y_{2n},y_{2n+1}), d(y_{2n+1},y_{2n+2}),\\ d(y_{2n},y_{2n+1}) + d(y_{2n+1},y_{2n+2}), 0\Big) \geq 0.$$

By using $(\mathcal{G}_{2a})$, we obtain

$$d(y_{2n},y_{2n+1}) \geq F\Big(d(y_{2n+1},y_{2n+2}), \varphi(d(y_{2n+1},y_{2n+2}))\Big). \tag{2.1}$$

Therefore,

$$d(y_{2n+1},y_{2n+2}) \leq d(y_{2n},y_{2n+1}).$$

Similarly, we can show that

$$d(y_{2n+2}, y_{2n+3}) \leq d(y_{2n+1}, y_{2n+2}).$$

Therefore, $\{d(y_n, y_{n+1})\}$, where $n \in \mathbb{N}$ is even or odd, is a monotone decreasing sequence and there exists $r \geq 0$ such that $\lim_{n\to+\infty} d(y_n, y_{n+1}) = r$. Now we claim that $\lim_{n\to+\infty} d(y_n, y_{n+1}) = 0$, otherwise by (2.1) letting with $n \to +\infty$, we obtain

$$r \geq F(r, \varphi(r)) \geq r \Rightarrow F(r, \varphi(r)) = r \Rightarrow r = 0, \text{ or } \varphi(r) = 0 \ \Rightarrow \ r = 0.$$

This is a contradiction. Therefore, $\lim_{n\to+\infty} d(y_n, y_{n+1}) = 0$. In order to show $\{y_n\}$ is a Cauchy sequence in X, it is sufficient to show that subsequence $\{y_{2n}\}$ is Cauchy. Suppose on contrary that $\{y_{2n}\}$ is not a Cauchy sequence, then there exists a number $\epsilon > 0$, and integers $(2m)_k$ and $(2n)_k$ such that $(2m)_k > (2n)_k \geq k$, for each even integer k satisfying $d(y_{(2m)_k}, y_{(2n)_k}) > \epsilon$. Further assume that $(2m)_k$ is the smallest even integer greater that $(2n)_k$, for each even interger k, then we obtain $d(y_{(2m)_k-2}, y_{(2n)_k}) < \epsilon$.
By triangular inequality, we obtain

$$\begin{aligned}\epsilon \leq d(y_{(2m)_k}, y_{(2n)_k}) &\leq d(y_{(2m)_k}, y_{(2m)_k-2}) + d(y_{(2m)_k-2}, y_{(2n)_k}) \\ &< \epsilon + d(y_{(2m)_k}, y_{(2m)_k-1}) + d(y_{(2m)_k-1}, y_{(2m)_k-2})\end{aligned}$$

Letting limit as $k \to +\infty$, we obtain

$$\lim_{k\to+\infty} d(y_{(2m)_k}, y_{(2n)_k}) = \epsilon. \tag{2.2}$$

Also by triangular inequality, we obtain

$$d(y_{(2m)_k}, y_{(2n)_k}) \leq d(y_{(2m)_k}, y_{(2m)_k+1}) + d(y_{(2m)_k+1}, y_{(2n)_k-1}) + d(y_{(2n)_k-1}, y_{(2n)_k}).$$

Letting limit as $k \to +\infty$, we obtain

$$\epsilon \leq \lim_{k\to+\infty} d(y_{(2m)_k+1}, y_{(2n)_k-1}) \tag{2.3}$$

On the other hand, we obtain

$$d(y_{(2m)_k+1}, y_{(2n)_k-1}) \leq d(y_{(2m)_k+1}, y_{(2m)_k}) + d(y_{(2m)_k}, y_{(2n)_k}) + d(y_{(2n)_k}, y_{(2n)_k-1})$$

Letting limit as $k \to +\infty$, we obtain

$$\lim_{k\to+\infty} d(y_{(2m)_k+1}, y_{(2n)_k-1}) \leq \epsilon. \tag{2.4}$$

From (2.3) and (2.4), we obtain

$$\lim_{k\to+\infty} d(y_{(2m)_k+1}, y_{(2n)_k-1}) = \epsilon. \tag{2.5}$$

Similarly, by triangular inequality, we obtain

$$\begin{aligned}d(y_{(2m)_k+1}, y_{(2n)_k-1}) &\leq d(y_{(2m)_k+1}, y_{(2n)_k}) + d(y_{(2n)_k}, y_{(2n)_k-1}) \\ &\leq d(y_{(2m)_k+1}, y_{(2m)_k}) + d(y_{(2m)_k}, y_{(2n)_k}) + d(y_{(2n)_k}, y_{(2n)_k-1}).\end{aligned}$$

Letting limit as $k \to +\infty$ with (2.2) and (2.5), we obtain

$$\epsilon \leq \lim_{k\to+\infty} d(y_{(2m)_k+1}, y_{(2n)_k}) \leq \epsilon \Rightarrow \lim_{k\to+\infty} d(y_{(2m)_k+1}, y_{(2n)_k}) = \epsilon \tag{2.6}$$

Also, we obtain

$$\begin{aligned} d(y_{(2m)_k+1}, y_{(2n)_k-1}) \leq & d(y_{(2m)_k+1}, y_{(2m)_k}) + d(y_{(2m)_k}, y_{(2n)_k-1}) \\ \leq & d(y_{(2m)_k+1}, y_{(2m)_k}) + d(y_{(2m)_k}, y_{(2n)_k}) + d(y_{(2n)_k}, y_{(2n)_k-1}). \end{aligned}$$

Letting limit as $k \to +\infty$ with (2.2) and (2.5), we obtain

$$\epsilon \leq \lim_{k\to+\infty} d(y_{(2m)_k}, y_{(2n)_k-1}) \leq \epsilon \Rightarrow \lim_{n\to+\infty} d(y_{(2m)_k}, y_{(2n)_k-1}) = \epsilon. \tag{2.7}$$

Now from condition (ii), we obtain

$$\begin{aligned} \mathcal{G}\Big(d(\mathcal{A}x_{(2m)_k+1}, \mathcal{B}x_{(2n)_k}), d(\mathcal{PF}x_{(2m)_k+1}, \mathcal{TI}x_{(2n)_k}), d(\mathcal{A}x_{(2m)_k+1}, \mathcal{PF}x_{(2m)_k+1}), d(\mathcal{B}x_{(2n)_k}, \mathcal{TI}x_{(2n)_k}), \\ d(\mathcal{A}x_{(2m)_k+1}, \mathcal{TI}x_{(2n)_k}), d(\mathcal{B}x_{(2n)_k}, \mathcal{PF}x_{(2m)_k+1})\Big) \geq 0 \\ \mathcal{G}\Big(d(y_{(2m)_k}, y_{(2n)_k-1}), d(y_{(2m)_k+1}, y_{(2n)_k}), d(y_{(2m)_k}, y_{(2m)_k+1}), d(y_{(2n)_k-1}, y_{(2n)_k}), d(y_{(2m)_k}, y_{(2n)_k}), \\ d(y_{(2n)_k-1}, y_{(2m)_k+1})\Big) \geq 0 \end{aligned}$$

Letting limit as $k \to +\infty$ with (2.2), (2.5), (2.6) and (2.7), we obtain

$$\mathcal{G}(\epsilon,\ \epsilon,\ 0,\ 0,\ \epsilon,\ \epsilon) \geq 0.$$

This is a contradiction to $(\mathcal{G}_3)$ for $\epsilon > 0$. Therefore, $\{y_n\}$ is a Cauchy sequence in X. Since X is complete metric space, the sequence $\{y_n\}$ converges to a point $w \in X$. Consequently, the subsequences $\{\mathcal{A}x_{2n+1}\}$, $\{\mathcal{B}x_{2n+2}\}$, $\{\mathcal{PF}x_{2n+1}\}$ and $\{\mathcal{TI}x_{2n}\}$ also converge to $w \in X$.
From (iii), $(\mathcal{B}, \mathcal{TI})$ is $\mathcal{S}_{\mathcal{A}}$-compatible, we obtain

$$\lim_{n\to\infty} \mathcal{BA}x_{2n+1} = \lim_{n\to\infty} \mathcal{TIA}x_{2n+1} = \mathcal{A}w.$$

We show $\mathcal{A}w = w$, otherwise by condition (ii), we obtain

$$\begin{aligned} \mathcal{G}\Big(d(\mathcal{A}x_{2n+1}, \mathcal{BA}x_{2n+1}), d(\mathcal{PF}x_{2n+1}, \mathcal{TIA}x_{2n+1}), d(\mathcal{A}x_{2n+1}, \mathcal{PF}x_{2n+1}), \\ d(\mathcal{BA}x_{2n+1}, \mathcal{TIA}x_{2n+1}), d(\mathcal{A}x_{2n+1}, \mathcal{TIA}x_{2n+1}), d(\mathcal{BA}x_{2n+1}, \mathcal{PF}x_{2n+1})\Big) \geq 0. \end{aligned}$$

Taking limit as $n \to +\infty$, we obtain

$$\begin{aligned} \mathcal{G}\Big(d(w, \mathcal{A}w), d(w, \mathcal{A}w), d(w, w), d(\mathcal{A}w, \mathcal{A}w), d(w, \mathcal{A}w), d(\mathcal{A}w, w)\Big) \geq & 0 \\ \mathcal{G}\Big(d(w, \mathcal{A}w), d(w, \mathcal{A}w),\ 0,\ 0\ , d(w, \mathcal{A}w), d(w, \mathcal{A}w)\Big) \geq & 0. \end{aligned}$$

This is a contradiction to $(\mathcal{G}_3)$ for $\mathcal{A}w \neq w$ and hence, $\mathcal{A}w = w$.
To show that $\mathcal{PF}w = \mathcal{A}w$, by condition (ii), we obtain

$$\mathcal{G}\Big(d(\mathcal{A}w, \mathcal{B}x_{2n}), d(\mathcal{PF}w, \mathcal{TI}x_{2n}), d(\mathcal{A}w, \mathcal{PF}w), d(\mathcal{B}x_{2n}, \mathcal{TI}x_{2n}), d(\mathcal{A}w, \mathcal{TI}x_{2n}), d(\mathcal{B}x_{2n}, \mathcal{PF}w)\Big) \geq 0.$$

Taking limit as $n \to +\infty$, we obtain,

$$\mathcal{G}\Big(d(\mathcal{A}w,w),d(\mathcal{PF}w,w),d(\mathcal{A}w,\mathcal{PF}w),d(w,w),d(\mathcal{A}w,w),d(w,\mathcal{PF}w)\Big)\geq 0$$
$$\mathcal{G}\Big(0,d(\mathcal{PF}w,\mathcal{A}w),d(\mathcal{A}w,\mathcal{PF}w),0,0,d(\mathcal{A}w,\mathcal{PF}w)\Big)\geq 0.$$

By virtue of $(\mathcal{G}_{2b})$, we obtain

$$0\geq F\Big(d(\mathcal{PF}w,\mathcal{A}w),\varphi(d(\mathcal{PF}w,\mathcal{A}w))\Big)\geq d(\mathcal{PF}w,\mathcal{A}w)$$

yields that $\mathcal{PF}w=\mathcal{A}w$.
Also from (iii), $(\mathcal{A},\mathcal{PF})$ is $\mathcal{S}_{\mathcal{B}}$- compatible, then

$$\lim_{n\to\infty}\mathcal{AB}x_{2n}=\lim_{n\to\infty}\mathcal{PFB}x_{2n}=\mathcal{B}w.$$

We claim that $\mathcal{B}w=w$, otherwise by condition (ii), we obtain

$$\mathcal{G}\Big(d(\mathcal{AB}x_{2n},\mathcal{B}x_{2n}),d(\mathcal{PFB}x_{2n},\mathcal{TI}x_{2n}),d(\mathcal{AB}x_{2n},\mathcal{PFB}x_{2n}),d(\mathcal{B}x_{2n},\mathcal{TI}x_{2n}),d(\mathcal{AB}x_{2n},\mathcal{TI}x_{2n}),$$
$$d(\mathcal{B}x_{2n},\mathcal{PFB}x_{2n})\Big)\geq 0.$$

Letting limit as $n \to +\infty$, we obtain

$$\mathcal{G}\Big(d(\mathcal{B}w,w),d(\mathcal{B}w,w),d(\mathcal{B}w,\mathcal{B}w),d(w,w),d(\mathcal{B}w,w),d(w,\mathcal{B}w)\Big)\geq 0$$
$$\mathcal{G}\Big(d(\mathcal{B}w,w),d(\mathcal{B}w,w),0,0,d(\mathcal{B}w,w),d(\mathcal{B}w,w)\Big)\geq 0.$$

This is a contradiction to $(\mathcal{G}_3)$ for $\mathcal{B}w\neq w$ and hence $\mathcal{B}w=w$.
Now, we show that $\mathcal{TI}w=\mathcal{B}w$. From condition (ii), we obtain

$$\mathcal{G}\Big(d(\mathcal{A}x_{2n+1},\mathcal{B}w),d(\mathcal{PF}x_{2n+1},\mathcal{TI}w),d(\mathcal{A}x_{2n+1},\mathcal{PF}x_{2n+1}),d(\mathcal{B}w,\mathcal{TI}w),d(\mathcal{A}x_{2n+1},\mathcal{TI}w),$$
$$d(\mathcal{B}w,\mathcal{PF}x_{2n+1})\Big)\geq 0.$$

Letting limit as $n \to +\infty$, we obtain

$$\mathcal{G}\Big(d(w,\mathcal{B}w),d(w,\mathcal{TI}w),d(w,w),d(\mathcal{B}w,\mathcal{TI}w),d(w,\mathcal{TI}w),d(\mathcal{B}w,w)\Big)\geq 0$$
$$\mathcal{G}\Big(0,d(\mathcal{B}w,\mathcal{TI}w),0,d(\mathcal{B}w,\mathcal{TI}w),d(\mathcal{B}w,\mathcal{TI}w),0\Big)\geq 0$$

By virtue of $(\mathcal{G}_{2a})$, we obtain

$$0\geq F\Big(d(\mathcal{B}w,\mathcal{TI}w),\ \varphi(d(\mathcal{B}w,TIw))\Big)\geq d(\mathcal{B}w,\mathcal{TI}w)$$

yields $\mathcal{TI}w=\mathcal{B}w$. Thus $\mathcal{A}w=\mathcal{B}w=\mathcal{PF}w=\mathcal{TI}w=w$.

Further, suppose that $(\mathcal{A},\mathcal{F})$, $(\mathcal{P},\mathcal{F})$, $(\mathcal{B},\mathcal{I})$, and $(\mathcal{T},\mathcal{I})$ are commuting at common fixed points of $\mathcal{A},\mathcal{B},\mathcal{PF}$ and $\mathcal{TI}$. From the pairs $(\mathcal{A},\mathcal{F})$ and $(\mathcal{P},\mathcal{F})$, we obtain

$$\mathcal{AF}w=\mathcal{FA}w=\mathcal{F}w\ \text{ and } \mathcal{PFF}w=\mathcal{FPF}w=\mathcal{F}w,\text{where } \mathcal{PF}w=w.$$

We show $\mathcal{F}w = w$. For this, from condition (ii), we obtain

$$\mathcal{G}\Big(d(\mathcal{A}\mathcal{F}w, \mathcal{B}w), d(\mathcal{P}\mathcal{F}\mathcal{F}w, \mathcal{T}\mathcal{I}w), d(\mathcal{A}\mathcal{F}w, \mathcal{P}\mathcal{F}\mathcal{F}w), d(\mathcal{B}w, \mathcal{T}\mathcal{I}w), d(\mathcal{A}\mathcal{F}w, \mathcal{T}\mathcal{I}w), d(\mathcal{B}w, \mathcal{P}\mathcal{F}\mathcal{F}w)\Big) \geq 0$$

$$\mathcal{G}\Big(d(\mathcal{F}w, w), d(\mathcal{F}w, w), d(\mathcal{F}w, \mathcal{F}w), d(w, w), d(\mathcal{F}w, w), d(w, \mathcal{F}w)\Big) \geq 0$$

$$\mathcal{G}\Big(d(\mathcal{F}w, w), d(\mathcal{F}w, w), 0, 0, d(\mathcal{F}w, w), d(\mathcal{F}w, w)\Big) \geq 0.$$

This is a contradiction to $(\mathcal{G}_3)$ for $\mathcal{F}w \neq w$ and hence $\mathcal{F}w = w$. Consequently $\mathcal{P}\mathcal{F}w = w$ yields $\mathcal{S}w = w$. Therefore $\mathcal{F}w = \mathcal{P}w = w$. Also from the pairs $(\mathcal{B}, \mathcal{I})$ and $(\mathcal{T}, \mathcal{I})$, we obtain

$$\mathcal{B}\mathcal{I}w = \mathcal{I}\mathcal{B}w = \mathcal{I}w \text{ and } \mathcal{T}\mathcal{I}\mathcal{I}w = \mathcal{I}\mathcal{T}\mathcal{I}w = \mathcal{I}w, \text{ where } \mathcal{T}\mathcal{I}w = \mathcal{B}w = w.$$

Now, we show $\mathcal{I}w = w$. Let on contrary, then by condition (ii), we obtain

$$\mathcal{G}\Big(d(\mathcal{A}w, \mathcal{B}\mathcal{I}w), d(\mathcal{P}\mathcal{F}w, \mathcal{T}\mathcal{I}\mathcal{I}w), d(\mathcal{A}w, \mathcal{P}\mathcal{F}w), d(\mathcal{B}\mathcal{I}w, \mathcal{T}\mathcal{I}\mathcal{I}w), d(\mathcal{A}w, \mathcal{T}\mathcal{I}\mathcal{I}w), d(\mathcal{B}\mathcal{I}w, \mathcal{P}\mathcal{F}w)\Big) \geq 0$$

$$\mathcal{G}\Big(d(w, \mathcal{I}w), d(w, \mathcal{I}w), d(w, w), d(\mathcal{I}w, \mathcal{I}w), d(w, \mathcal{I}w), d(\mathcal{I}t, t)\Big) \geq 0$$

$$\mathcal{G}\Big(d(w, \mathcal{I}w), d(w, \mathcal{I}w), 0, 0, d(w, \mathcal{I}w), d(\mathcal{I}w, w)\Big) \geq 0.$$

This is contradiction of $(\mathcal{G}_3)$ for $\mathcal{I}w \neq w$ and hence $\mathcal{I}w = w$. Consequently $\mathcal{T}\mathcal{I}w = w$ yields $\mathcal{T}w = w$. Therefore $\mathcal{T}w = \mathcal{I}w = w$. Thus, $\mathcal{A},\ \mathcal{B},\ \mathcal{S},\ \mathcal{T},\ \mathcal{F}$ and $\mathcal{I}$ have a common fixed point in X.
For the uniqueness, suppose $w^* \in X$ is an another fixed point of $\mathcal{A}, \mathcal{B}, \mathcal{P}, \mathcal{T}, \mathcal{F}, \mathcal{I}$ such that $w \neq w^*$. By condition (ii), we obtain

$$\mathcal{G}\Big(d(\mathcal{A}w, \mathcal{B}w^*), d(\mathcal{P}\mathcal{F}w, \mathcal{T}\mathcal{I}w^*), d(\mathcal{A}w, \mathcal{P}\mathcal{F}w), d(\mathcal{B}w^*, \mathcal{T}\mathcal{I}w^*), d(\mathcal{A}w, \mathcal{T}\mathcal{I}w^*), d(\mathcal{B}w^*, \mathcal{P}\mathcal{F}w)\Big) \geq 0$$

$$\mathcal{G}\Big(d(w, w^*), d(w, w^*),\ 0,\ 0,\ d(w, w^*), d(w, w^*)\Big) \geq 0.$$

This is a contradiction to $(\mathcal{G}_3)$ for $w \neq w^*$ and hence $w = w^*$. □

Corollary 2.2. *Let (X, d) be a complete metric space and $\mathcal{A}, \mathcal{B}, \mathcal{P}, \mathcal{T} : X \to X$ be self mappings that satisfy the conditions:*
(i) $\mathcal{A}$ and $\mathcal{B}$ are surjective;
(ii) there exists $\mathcal{G} \in \mathcal{G}_C$ such that

$$\mathcal{G}\Big(d(\mathcal{A}x, \mathcal{B}y), d(\mathcal{P}x, \mathcal{T}y), d(\mathcal{A}x, \mathcal{P}x), d(\mathcal{B}y, \mathcal{T}y), d(\mathcal{A}x, \mathcal{T}y), d(\mathcal{B}y, \mathcal{P}x)\Big) \geq 0$$

for all $x, y \in X$;
(iii) $(\mathcal{B}, \mathcal{T})$ is $\mathcal{S}_\mathcal{A}$-compatible and $(\mathcal{A}, \mathcal{P})$ is $\mathcal{S}_\mathcal{B}$-compatible.
Then $\mathcal{A}, \mathcal{B}, \mathcal{P}$ and $\mathcal{T}$ have a unique common fixed point in X.

Proof . Follows from Theorem 2.1 by taking $\mathcal{F} = \mathcal{I} = I_X$ (an identity mapping of X). □

Theorem 2.3. *Let (X,d) be a complete metric space. Suppose $\mathcal{A},\mathcal{B},\mathcal{P},\mathcal{T}:X\to X$ be self mappings that satisfy the conditions:*
(i) $\mathcal{A}$ and $\mathcal{B}$ are surjective;
(ii) there exists $\mathcal{G}\in\mathcal{G}_C$ such that

$$\mathcal{G}\Big(d(\mathcal{A}x,\mathcal{B}y),d(\mathcal{P}x,\mathcal{T}y),d(\mathcal{A}x,\mathcal{P}x),d(\mathcal{B}y,\mathcal{T}y),d(\mathcal{A}x,\mathcal{T}y),d(\mathcal{B}y,\mathcal{P}x)\Big)\geq 0$$

for all $x,y\in X$;
(iii) $(\mathcal{A},\mathcal{P})$ is $\mathcal{S}_{\mathcal{A}}$-compatible and $(\mathcal{B},\mathcal{T})$ is $\mathcal{S}_{\mathcal{B}}$-compatible. Then $\mathcal{A}$, $\mathcal{B}$, $\mathcal{P}$ and $\mathcal{T}$ have a unique common fixed point in X.

Proof . As in Theorem 2.1, for any $x_0\in X$ we define two sequences $\{x_n\}$ and $\{y_n\}$ in X such that $y_{2n}=\mathcal{A}x_{2n+1}=\mathcal{T}x_{2n}$ and $y_{2n+1}=\mathcal{B}x_{2n+2}=\mathcal{P}x_{2n+1}$, where $n\in\mathbb{N}$. Following the same step as in Theorem 2.1, one can show that $\{y_n\}$ is a Cauchy sequence in X. The sequence $\{y_n\}$ converges to a point $w\in X$ as (X,d) is complete. Consequently, all the sub sequences $\{\mathcal{A}x_{2n+1}\}$, $\{\mathcal{B}x_{2n+2}\}$, $\{\mathcal{P}x_{2n+1}\}$ and $\{\mathcal{T}x_{2n}\}$ converge to $w\in X$.
Form (iii), $(\mathcal{A},\mathcal{P})$ is $\mathcal{S}_{\mathcal{A}}$-compatible, then

$$\lim_{n\to\infty}\mathcal{A}\mathcal{A}x_{2n+1}=\lim_{n\to\infty}\mathcal{P}\mathcal{A}x_{2n+1}=\mathcal{A}w.$$

We show that $\mathcal{A}w=w$. From condition (ii), we obtain

$$\mathcal{G}\Big(d(\mathcal{A}\mathcal{A}x_{2n+1},\mathcal{B}x_{2n+2}),d(\mathcal{P}\mathcal{A}x_{2n+1},\mathcal{T}x_{2n+2}),d(\mathcal{A}\mathcal{A}x_{2n+1},\mathcal{P}\mathcal{A}x_{2n+1}),d(\mathcal{B}x_{2n+2},\mathcal{T}x_{2n+2}),$$
$$d(\mathcal{A}\mathcal{A}x_{2n+1},\mathcal{T}x_{2n+2}),d(\mathcal{B}x_{2n+2},\mathcal{P}\mathcal{A}x_{2n+1})\Big)\geq 0.$$

Letting limit as $n\to+\infty$, we obtain

$$\mathcal{G}\Big(d(\mathcal{A}w,w),d(\mathcal{A}w,w),d(\mathcal{A}w,\mathcal{A}w),d(w,w),d(\mathcal{A}w,w),d(w,\mathcal{A}w)\Big)\geq 0$$
$$\mathcal{G}\Big(d(\mathcal{A}w,w),d(\mathcal{A}w,w),0,0,d(\mathcal{A}w,w),d(w,\mathcal{A}w)\Big)\geq 0.$$

This is a contradiction to $(\mathcal{G}_3)$ for $\mathcal{A}w\neq w$ and hence $\mathcal{A}w=w$. Now we claim that $\mathcal{S}w=\mathcal{A}w$, otherwise from condition (ii), we obtain

$$\mathcal{G}\Big(d(\mathcal{A}w,\mathcal{B}x_{2n}),d(\mathcal{P}w,\mathcal{T}x_{2n}),d(\mathcal{A}w,\mathcal{P}w),d(\mathcal{B}x_{2n},\mathcal{T}x_{2n}),d(\mathcal{A}w,\mathcal{T}x_{2n}),d(\mathcal{B}x_{2n},\mathcal{P}w)\Big)\geq 0.$$

Letting limit as $n\to+\infty$, we obtain

$$\mathcal{G}\Big(d(\mathcal{A}w,w),d(\mathcal{P}w,w),d(\mathcal{A}w,\mathcal{P}w),d(w,w),d(\mathcal{A}w,w),d(w,\mathcal{S}w)\Big)\geq 0$$
$$\mathcal{G}\Big(0,d(\mathcal{P}w,\mathcal{A}w),d(\mathcal{A}w,\mathcal{P}w),0,0,d(\mathcal{A}w,\mathcal{P}w)\Big)\geq 0.$$

This is a contradiction to $(\mathcal{G}_{2b})$ for $\mathcal{P}w\neq\mathcal{A}w$ and hence $\mathcal{P}w=\mathcal{A}w=w$.Also, from (iii), $(\mathcal{B},\mathcal{T})$ is $\mathcal{S}_{\mathcal{B}}$-compatible, then

$$\lim_{n\to\infty}\mathcal{B}\mathcal{B}x_{2n+2}=\lim_{n\to\infty}\mathcal{T}\mathcal{B}x_{2n+2}=\mathcal{B}w.$$

We show that $\mathcal{B}w = w$, otherwise by condition (ii), we obtain

$$\mathcal{G}\Big(d(\mathcal{A}w, \mathcal{B}\mathcal{B}x_{2n+2}), d(\mathcal{P}w, \mathcal{T}\mathcal{B}x_{2n+2}), d(\mathcal{A}w, \mathcal{P}w), d(\mathcal{B}\mathcal{B}x_{2n+2}, \mathcal{T}\mathcal{B}x_{2n+2}), d(\mathcal{A}w, \mathcal{T}\mathcal{B}x_{2n+2}), d(\mathcal{B}\mathcal{B}x_{2n+2}, \mathcal{P}w)\Big) \geq 0.$$

Letting limit as $n \to +\infty$, we obtain

$$\mathcal{G}\Big(d(w, \mathcal{B}w), d(w, \mathcal{B}w), d(w, w), d(\mathcal{B}w, \mathcal{B}w), d(w, \mathcal{B}w), d(\mathcal{B}w, w)\Big) \geq 0$$
$$\mathcal{G}\Big(d(w, \mathcal{B}w), d(w, \mathcal{B}w), 0, 0, d(w, \mathcal{B}w), d(\mathcal{B}w, w)\Big) \geq 0.$$

This is a contradiction to $(\mathcal{G}_3)$ for $\mathcal{B}w \neq w$ and hence $\mathcal{B}w = w$. Again by condition (ii), we obtain

$$\mathcal{G}\Big(d(\mathcal{A}w, \mathcal{B}w), d(\mathcal{P}w, \mathcal{T}w), d(\mathcal{A}w, \mathcal{P}w), d(\mathcal{B}w, \mathcal{T}w), d(\mathcal{A}w, \mathcal{T}w), d(\mathcal{B}w, \mathcal{P}w)\Big) \geq 0$$
$$\mathcal{G}\Big(0, d(w, \mathcal{T}w), 0, d(w, \mathcal{T}w), d(w, \mathcal{T}w), 0\Big) \geq 0.$$

By (G_{2a}), we obtain that $0 \geq F\Big(d(w, \mathcal{T}w), \varphi(d(w, \mathcal{T}w))\Big) \geq d(\mathcal{P}w, w)$ yields $\mathcal{T}w = w$. Consequently $\mathcal{B}w = \mathcal{T}w = w$. Thus $\mathcal{A}w = \mathcal{B}w = \mathcal{P}w = \mathcal{T}w = w$. Uniqueness of common fixed point can be proved as in Theorem 2.1. □

Example 2.4. *Let $X = [0, 12]$ with usual metric $d(x, y) = |x - y|$. Define $\mathcal{A}, \mathcal{B}, \mathcal{P}, \mathcal{T} : X \to X$ as:*

$$\mathcal{A}x = \begin{cases} 2x, & 0 \leq x < 6 \\ x, & 6 \leq x \leq 12 \end{cases} \quad ; \quad \mathcal{B}x = \begin{cases} 2x, & 0 \leq x < 6 \\ \frac{x+6}{2}, & 6 \leq x < 12 \\ 12, & x = 12 \end{cases};$$

$$\mathcal{P}x = \begin{cases} \frac{x}{2}, & 0 \leq x < 6 \\ 6, & x = 6 \\ 0, & 6 < x \leq 12 \end{cases} \quad ; \quad \mathcal{T}x = \begin{cases} \frac{x}{2}, & 0 \leq x < 6 \\ 6, & x = 6 \\ 3, & 6 < x \leq 12. \end{cases}$$

Clearly, $\mathcal{A}$ and $\mathcal{B}$ are surjective. Let $\{x_n\}$ be a sequence in X, where $x_n = 6 + \frac{1}{n} > 6$, for all $n \in \mathbb{N}$ such that $x_n \to 6$ as $n \to +\infty$. Then, $\lim_{n\to+\infty} \mathcal{A}\mathcal{A}x_n = \lim_{n\to+\infty} \mathcal{P}\mathcal{A}x_n = \mathcal{A}w$, where $\mathcal{A}x_n \to 6 = w$ as $n \to +\infty$. Further, $\lim_{n\to+\infty} \mathcal{B}\mathcal{B}x_n = \lim_{n\to+\infty} \mathcal{T}\mathcal{B}x_n = \mathcal{B}w$, where $\mathcal{B}x_n \to 6 = w$ as $n \to +\infty$. Therefore, the pairs $(\mathcal{A}, \mathcal{P})$ and $(\mathcal{B}, \mathcal{T})$ are respectively $S_{\mathcal{A}}$- and $S_{\mathcal{B}}$-compatible. Setting $F(s, t) = hs$ and $\mathcal{G}(t_1, t_2, ..., t_6) = 1 - \frac{t_2(h+1)}{\max\{t_1, t_2, t_3, t_4, t_5+t_6\}}$, where $h \in (1, \infty)$. Then $F \in C_{inv}$ and $\mathcal{G} \in \mathcal{G}_C$. Taking with $h = \frac{1}{2}$, we have the following cases:

Case-1: For all $x, y \in [0, 6)$, we obtain

$$\begin{aligned} (h+1)d(\mathcal{P}x, \mathcal{T}y) =& (h+1)|\mathcal{P}x - \mathcal{T}y| = \frac{(h+1)}{2}|x - y| \\ =& \frac{3}{2}|x - y| \leq 2|x - y| = |\mathcal{A}x - \mathcal{B}y| \\ \leq& \max\Big(d(\mathcal{A}x, \mathcal{B}y), d(\mathcal{P}x, \mathcal{T}y), d(\mathcal{A}x, \mathcal{P}x), d(\mathcal{B}y, \mathcal{T}y), \quad d(\mathcal{A}x, \mathcal{T}y) + d(\mathcal{B}y, \mathcal{P}x)\Big). \end{aligned}$$

Case-2: For all $x \in [0,6)$ *and* $y = 6$*, we obtain*

$$\begin{aligned}(h+1)d(\mathcal{P}x,\mathcal{T}y) =&(h+1)|\mathcal{P}x-\mathcal{T}6| = \frac{3}{2}|x-6|\\ \leq& 2|x-6| = |\mathcal{A}x-\mathcal{B}y|\\ \leq& \max\Big(d(\mathcal{A}x,\mathcal{B}y), d(\mathcal{P}x,\mathcal{T}y), d(\mathcal{A}x,\mathcal{P}x), d(\mathcal{B}y,\mathcal{T}y),\quad d(\mathcal{A}x,\mathcal{T}y)+d(\mathcal{B}y,\mathcal{P}x)\Big).\end{aligned}$$

Case-3: For all $x \in [0,6)$ *and* $y \in (6,12)$*, we obtain*
$0 < 3|\frac{x}{2}-3| \leq 9$*,* $3 \leq |2x-3| < 9$ *and* $3 < |\frac{y+6}{2}-\frac{x}{2}| < 9$*. Consequently, we obtain*

$$3|\frac{x}{2}-3| \leq 9 = 3+3 \leq |2x-3| + |\frac{y+6}{2}-\frac{x}{2}|$$

Furthermore, we obtain

$$\begin{aligned}(h+1)d(\mathcal{P}x,\mathcal{T}y) =&(h+1)|\mathcal{P}x-\mathcal{T}y| = 3|\frac{x}{2}-3|\\ \leq&|2x-3| + |\frac{y+6}{2}-\frac{x}{2}|\\ =&d(\mathcal{A}x,\mathcal{T}y)+d(\mathcal{B}y,\mathcal{P}x)\\ \leq& \max\Big(d(\mathcal{A}x,\mathcal{B}y), d(\mathcal{P}x,\mathcal{T}y), d(\mathcal{A}x,\mathcal{P}x), d(\mathcal{B}y,\mathcal{T}y),\quad d(\mathcal{A}x,\mathcal{T}y)+d(\mathcal{B}y,\mathcal{P}x)\Big).\end{aligned}$$

Case-4: For all $x \in [0,6)$ *and* $y = 12$*, we obtain*

$$\begin{aligned}(h+1)d(\mathcal{P}x,\mathcal{T}y) =&3|\frac{x}{2}-3| = \frac{3}{2}|x-6|\\ \leq&2|x-6| = |2x-12| = |\mathcal{A}x-\mathcal{B}y|\\ \leq& \max\Big(d(\mathcal{A}x,\mathcal{B}y), d(\mathcal{P}x,\mathcal{T}y), d(\mathcal{A}x,\mathcal{P}x), d(\mathcal{B}y,\mathcal{T}y),\quad d(\mathcal{A}x,\mathcal{T}y)+d(\mathcal{B}y,\mathcal{P}x)\Big).\end{aligned}$$

Case-5: For all $x \in (6,12)$ *and* $y = 12$*, we obtain*

$$\begin{aligned}(h+1)d(\mathcal{P}x,\mathcal{T}y) =&3|0-3| = 9\\ \leq& \max(|x-12|, 3, |x|, 9, |x-3|+12)\\ =& \max\Big(d(\mathcal{A}x,\mathcal{B}y), d(\mathcal{P}x,\mathcal{T}y), d(\mathcal{A}x,\mathcal{P}x), d(\mathcal{B}y,\mathcal{T}y),\quad d(\mathcal{A}x,\mathcal{T}y)+d(\mathcal{B}y,\mathcal{P}x)\Big).\end{aligned}$$

Combining all cases for all $x, y \in X$*, we obtain*

$$\begin{aligned}(h+1)d(\mathcal{P}x,\mathcal{T}y) \leq& \max\Big(d(\mathcal{A}x,\mathcal{B}y), d(\mathcal{P}x,\mathcal{T}y), d(\mathcal{A}x,\mathcal{P}x), d(\mathcal{B}y,\mathcal{T}y),\\ & d(\mathcal{A}x,\mathcal{T}y)+d(\mathcal{B}y,\mathcal{P}x)\Big) = Q(x,y)(say)\\ \implies \mathcal{G}(d(\mathcal{A}x,\mathcal{B}y),\ & d(\mathcal{P}x,\mathcal{T}y),\ d(\mathcal{A}x,\mathcal{P}x), d(\mathcal{B}y,\mathcal{T}y), d(\mathcal{A}x,\mathcal{T}y)+d(\mathcal{B}y,\mathcal{P}x))\\ =&1-\frac{(h+1)d(\mathcal{P}x,\mathcal{T}y)}{Q(x,y)} \geq 0.\end{aligned}$$

where, $Q(x,y) \neq 0$*. Thus all the conditions of Theorem 2.3 are satisfied and hence,* $x = 6$ *is a unique common fixed point of* $\mathcal{A}, \mathcal{B}, \mathcal{P}$ *and* $\mathcal{T}$*.*

Theorem 2.5. *Let (X,d) be a complete metric space. Suppose $\mathcal{A},\mathcal{B},\mathcal{P},\mathcal{T}:X\to X$ be self mappings that satisfy the conditions:*
(i) $\mathcal{A}$ and $\mathcal{B}$ are surjective;
(ii) there exists $\mathcal{G}\in\mathcal{G}_C$ such that

$$\mathcal{G}\Big(d(\mathcal{A}x,\mathcal{B}y),d(\mathcal{P}x,\mathcal{T}y),d(\mathcal{A}x,\mathcal{P}x),d(\mathcal{B}y,\mathcal{T}y),d(\mathcal{A}x,\mathcal{T}y),d(\mathcal{B}y,\mathcal{P}x)\Big)\geq 0$$

for all $x,y\in X$. If either one of the following holds:
(a) $(\mathcal{A},\mathcal{P})$ is $\mathcal{S}_{\mathcal{B}}$-compatible and $(\mathcal{A},\mathcal{P})$ is weakly compatible;
(b) $(\mathcal{B},\mathcal{T})$ is $\mathcal{S}_{\mathcal{A}}$-compatible and $(\mathcal{B},\mathcal{T})$ is weakly compatible.
Then $\mathcal{A}$, $\mathcal{B}$, $\mathcal{P}$ and $\mathcal{T}$ have a unique common fixed point in X.

Proof . As in Theorem 2.1, for any $x_0\in X$ we define two sequences $\{x_n\}$ and $\{y_n\}$ in X such that $y_{2n}=\mathcal{A}x_{2n+1}=\mathcal{T}x_{2n}$ and $y_{2n+1}=\mathcal{B}x_{2n+2}=\mathcal{S}x_{2n+1}$, where $n\in\mathbb{N}$. Following the same step as in Theorem 2.1, one can show that $\{y_n\}$ is a Cauchy sequence in X. The sequence $\{y_n\}$ converges to a point $w\in X$ as (X,d) is complete. Consequently all the sub sequences $\{\mathcal{A}x_{2n+1}\}$, $\{\mathcal{B}x_{2n+2}\}$, $\{\mathcal{P}x_{2n+1}\}$ and $\{\mathcal{T}x_{2n}\}$ converge to $w\in X$.
To prove $\mathcal{A},\mathcal{B},\mathcal{P}$ and $\mathcal{T}$ have a common fixed point, it arises the following two cases:
Case (i): Form (a), $(\mathcal{A},\mathcal{P})$ is $\mathcal{S}_{\mathcal{B}}$-compatible, then

$$\lim_{n\to\infty}\mathcal{A}\mathcal{B}x_{2n+2}=\lim_{n\to\infty}\mathcal{P}\mathcal{B}x_{2n+2}=\mathcal{B}w.$$

From condition (ii), we obtain

$$\mathcal{G}\Big(d(\mathcal{A}\mathcal{B}x_{2n+2},\mathcal{B}x_{2n+2}),d(\mathcal{P}\mathcal{B}x_{2n+2},\mathcal{T}x_{2n+2}),d(\mathcal{A}\mathcal{B}x_{2n+2},\mathcal{P}\mathcal{B}x_{2n+2}),$$
$$d(\mathcal{B}x_{2n+2},\mathcal{T}x_{2n+2}),d(\mathcal{A}\mathcal{B}x_{2n+2},\mathcal{T}x_{2n+2}),d(\mathcal{B}x_{2n+2},\mathcal{P}\mathcal{B}x_{2n+2})\Big)\geq 0.$$

Letting limit as $n\to+\infty$, we obtain

$$\mathcal{G}\Big(d(\mathcal{B}w,w),d(\mathcal{B}w,w),d(\mathcal{B}w,\mathcal{B}w),d(w,w),d(\mathcal{B}w,w),d(w,\mathcal{B}w)\Big)\geq 0$$
$$\mathcal{G}\Big(d(\mathcal{B}w,w),d(\mathcal{B}w,w),0,0,d(\mathcal{B}w,w),d(w,\mathcal{B}w)\Big)\geq 0.$$

This is a contradiction to $(\mathcal{G}_3)$ for $\mathcal{B}w\neq w$ and hence $\mathcal{B}w=w$. Now we claim that $\mathcal{T}w=\mathcal{B}w$, otherwise from condition (ii), we obtain

$$\mathcal{G}\Big(d(\mathcal{A}x_{2n},\mathcal{B}w),d(\mathcal{P}x_{2n},\mathcal{T}w),d(\mathcal{A}x_{2n},\mathcal{P}x_{2n}),d(\mathcal{B}w,\mathcal{T}w),d(\mathcal{A}x_{2n},\mathcal{T}w),d(\mathcal{B}w,\mathcal{P}x_{2n})\Big)\geq 0$$

Letting limit as $n\to+\infty$, we obtain

$$\mathcal{G}\Big(d(w,\mathcal{B}w),d(w,\mathcal{T}w),d(w,w),d(\mathcal{B}w,\mathcal{T}w),d(w,\mathcal{T}w),d(\mathcal{B}w,w)\Big)\geq 0$$
$$\mathcal{G}\Big(0,d(\mathcal{B}w,\mathcal{T}w),0,d(\mathcal{B}w,\mathcal{T}w),d(\mathcal{B}w,\mathcal{T}w),0\Big)\geq 0.$$

This is a contradiction to $(\mathcal{G}_{2a})$ for $\mathcal{B}w\neq\mathcal{T}w$ and hence $\mathcal{B}w=\mathcal{T}w=w$. Since $\mathcal{A}$ is surjective and $\mathcal{B}w=w$, then there exists $u\in X$ such that $\mathcal{B}w=\mathcal{A}u=w$. We claim that $\mathcal{A}u=\mathcal{P}u$. For this, from condition (ii), we obtain

$$\mathcal{G}\Big(d(\mathcal{A}u,\mathcal{B}w),d(\mathcal{P}u,\mathcal{T}w),d(\mathcal{A}u,\mathcal{P}u),d(\mathcal{B}w,\mathcal{T}w),d(\mathcal{A}u,\mathcal{T}w),d(\mathcal{B}w,\mathcal{P}u)\Big)\geq 0$$
$$\mathcal{G}\Big(0,d(\mathcal{A}u,\mathcal{P}u),d(\mathcal{A}u,\mathcal{P}u),0,0,d(\mathcal{A}u,\mathcal{P}u)\Big)\geq 0.$$

By virtue of (G_{2b}), we obtain that $0 \geq F\Big(d(\mathcal{A}u, \mathcal{P}u), \varphi(d(\mathcal{A}u, \mathcal{P}u))\Big) \geq d(\mathcal{A}u, \mathcal{P}u)$ yields $\mathcal{A}u = \mathcal{P}u = w$. Suppose the pair $(\mathcal{A}, \mathcal{P})$ is weakly compatible, then $\mathcal{A}\mathcal{P}u = \mathcal{P}\mathcal{A}u$ i.e., $\mathcal{A}w = \mathcal{P}w$. We show that $\mathcal{A}w = w$, otherwise from condition (ii), we obtain

$$\mathcal{G}\Big(d(\mathcal{A}w, \mathcal{B}x_{2n}), d(\mathcal{P}w, \mathcal{T}x_{2n}), d(\mathcal{A}w, \mathcal{P}w), d(\mathcal{B}x_{2n}, \mathcal{T}x_{2n}), d(\mathcal{A}w, \mathcal{T}x_{2n}), d(\mathcal{B}x_{2n}, \mathcal{P}w)\Big) \geq 0.$$

Letting limit as $n \to +\infty$, we obtain

$$\mathcal{G}\Big(d(\mathcal{A}w, w), d(\mathcal{P}w, w), d(\mathcal{A}w, \mathcal{P}w), d(w, w), d(\mathcal{A}w, w), d(w, \mathcal{P}w)\Big) \geq 0$$
$$\mathcal{G}\Big(d(\mathcal{A}w, w), d(\mathcal{A}w, w), 0, 0, d(\mathcal{A}w, w), d(\mathcal{A}w, w)\Big) \geq 0.$$

This is a contradiction to (G_3) for $\mathcal{A}w \neq w$ and hence $\mathcal{A}w = \mathcal{P}w = w$. Therefore, $\mathcal{A}w = \mathcal{B}w = \mathcal{P}w = \mathcal{T}w = w$.
Case (ii): Form (b), $(\mathcal{B}, \mathcal{T})$ is $\mathcal{S}_{\mathcal{A}}$-compatible, then

$$\lim_{n\to\infty} \mathcal{B}\mathcal{A}x_{2n+1} = \lim_{n\to\infty} \mathcal{T}\mathcal{A}x_{2n+1} = \mathcal{A}w.$$

From condition (ii), , we obtain

$$\mathcal{G}\Big(d(\mathcal{A}x_{2n+1}, \mathcal{B}\mathcal{A}x_{2n+1}), d(\mathcal{P}x_{2n+1}, \mathcal{T}\mathcal{A}x_{2n+1}), d(\mathcal{A}x_{2n+1}, \mathcal{P}x_{2n+1}),$$
$$d(\mathcal{B}\mathcal{A}x_{2n+1}, \mathcal{T}\mathcal{A}x_{2n+1}), d(\mathcal{A}x, \mathcal{T}\mathcal{A}x_{2n+1}), d(\mathcal{B}\mathcal{A}x_{2n+1}, \mathcal{P}x_{2n+1})\Big) \geq 0.$$

Letting limit as $n \to +\infty$, we obtain

$$\mathcal{G}\Big(d(w, \mathcal{A}w), d(w, \mathcal{A}w), 0, 0, d(w, \mathcal{A}w), d(\mathcal{A}w, w)\Big) \geq 0.$$

By $(\mathcal{G}_3)$, we obtain $\mathcal{A}w = w$. Further from condition (ii), we obtain

$$\mathcal{G}\Big(d(\mathcal{A}w, \mathcal{B}x_{2n+2}), d(\mathcal{P}w, \mathcal{T}x_{2n+2}), d(\mathcal{A}w, \mathcal{P}w), d(\mathcal{B}x_{2n+2}, \mathcal{T}x_{2n+2}), d(\mathcal{A}w, \mathcal{T}x_{2n+2}),$$
$$d(\mathcal{B}x_{2n+2}, \mathcal{P}w)\Big) \geq 0.$$

Letting limit as $n \to +\infty$, we obtain

$$\mathcal{G}\Big(d(0, d(\mathcal{P}w, w), d(w, \mathcal{P}w), 0, 0, d(w, \mathcal{P}w)\Big) \geq 0.$$

By $G_{2b})$, we obtain that $0 \geq F\Big(d(\mathcal{P}w, w), \varphi(d(\mathcal{P}w, w))\Big) \geq d(\mathcal{P}w, w)$ yields $\mathcal{P}w = w$. Therefore $\mathcal{A}w = \mathcal{P}w = w$. Since B is surjective and $\mathcal{A}w = w$, there exists $u \in X$ such that $\mathcal{A}w = \mathcal{B}u = w$. To show $\mathcal{B}u = \mathcal{T}u$, from condition (ii), we obtain

$$\mathcal{G}\Big(d(\mathcal{A}w, \mathcal{B}u), d(\mathcal{P}w, \mathcal{T}u), d(\mathcal{A}w, \mathcal{P}w), d(\mathcal{B}u, \mathcal{T}u), d(\mathcal{A}w, \mathcal{T}u), d(\mathcal{B}u, \mathcal{P}w)\Big) \geq 0$$
$$\mathcal{G}\Big(0, d(\mathcal{B}u, \mathcal{T}u), 0, d(\mathcal{B}u, \mathcal{T}u), d(\mathcal{B}u, \mathcal{T}u), 0\Big) \geq 0.$$

By (G_{2a}), we obtain that $0 \geq F\Big(d(\mathcal{B}u,\mathcal{T}u),\varphi(d(\mathcal{B}u,\mathcal{T}u)\Big) \geq d(\mathcal{B}u,\mathcal{T}u)$ yields $\mathcal{B}u = \mathcal{T}u$. Since $(\mathcal{B},\mathcal{T})$ is weakly compatible, $\mathcal{B}\mathcal{T}u = \mathcal{T}\mathcal{B}u$ i.e., $\mathcal{B}w = \mathcal{T}w$. By condition (ii), we obtain

$$\mathcal{G}\Big(d(\mathcal{A}w,\mathcal{B}w), d(\mathcal{P}w,\mathcal{T}w), d(\mathcal{A}w,\mathcal{P}w), d(\mathcal{B}w,\mathcal{T}w), d(\mathcal{A}w,\mathcal{T}w), d(\mathcal{B}w,\mathcal{P}w)\Big) \geq 0$$
$$\mathcal{G}\Big(d(w,\mathcal{B}w), d(w,\mathcal{B}w), 0, 0, d(w,\mathcal{B}w), d(\mathcal{B}w,w)\Big) \geq 0.$$

By (G_3) it leads to obtain $\mathcal{B}w = w$ and hence $\mathcal{B}w = \mathcal{T}w = w$. Therefore, $\mathcal{A}w = \mathcal{B}w = \mathcal{P}w = \mathcal{T}w = w$. Thus we conclude from Case (i) and (ii) that $\mathcal{A},\mathcal{B},\mathcal{P}$ and $\mathcal{T}$ have a common fixed point in X. Uniqueness of common fixed point can be proved as in Theorem 2.1. □

References

[1] M. Akkouchi, *Common fixed point for weakly compatible maps satisfying implicit relations without continuity*, Demonstratio Math. 44(1) (2011) 151–158..

[2] M. A. Al-Thagafi and N. Shahzad, *Generalized I-nonexpansive selfmaps and invariant approximations*, Acta Math. Sinica 24(5) (2008) 287–296.

[3] A. H. Ansari, *Note on φ-ψ contractive type mappings and related fixed point*, The 2nd regional conference on Mathematics and Applications, Payame Noor University, 2014, pp. 377–380.

[4] A. H. Ansari, V. Popa, Y. Mahendra Singh and M. S. Khan, *Fixed point theorems of an implicit relation via C-class function in metric spaces*, J. Adv. Math. Stud. 13(1) (2020) 1–10.

[5] S. Banach, *Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales*, Fund. Math. 3 (1922) 133–181.

[6] J. Daneš, *Two fixed point theorems in topological and metric spaces*, Bull. Aust. Math. Sic. 14 (1976) 259–265.

[7] A. Djoudi, *A unique common fixed point for compatible mappings of type (B) satisfying an implicit relation*, Demonstratio Math. 36(3)(2003) 763–770.

[8] A. Djoudi, *General fixed point theorems for weakly compatible maps*, Demonstratio Math. 38(1) (2005) 197–205.

[9] G. Jungck, *Commuting mappings and fixed points*, Amer. Math. Month. 83 (1976) 261–263.

[10] G. Jungck, *compatible mappings and common fixed points*, Internat. J. Math. Math. Sci. 9(4) (1986) 771–779.

[11] G. Jungck, P. P. Murthy and Y. J. Cho, *Compatible mappings of type (A) and common fixed points*, Math. Japon. 36(2) (1993) 381–390.

[12] G. Jungck and B. E. Rhoades, *Fixed points for set valued functions without continuity*, Indian J. Pure Appl. Math. 29(3) (1998) 227–238.

[13] M. S. Khan, M. Swaleh and S. Sessa, *Fixed point theorems by altering distances between the points*, Bull. Aust. Math. Soc. 30(1) (1981) 1–9.

[14] M. K. Jain, E. Karapinar, Haseen Aydi and R. P. Agarwal, *$\mathcal{S}_\mathcal{T}$-compatibility and fixed point theorems via inverse C-class functions*, Dyn. Syst. Appl. Accepted.

[15] H. K. Pathak, S. S. Chang and Y. J. Cho, *Fixed point theorems for compatible mappings of type (P)*, Indian J. Math. 36(2) (1994) 151–166.

[16] H. K. Pathak, Y. J. Cho, S. M. Kang and B. Madharia, *Compatible mappings of type (C) and common fixed point theorem Greguš type*, Demonstratio Math. 31(3) (1998) 499–517.

[17] H. K. Pathak and M. S. Khan, *Compatible mappings of type (B) and common fixed point theorems of Greguš type*, Czechoslovak Math. J. 45(120) (1995) 685–698.

[18] H. K. Pathak and M. S. Khan, *A comparison of various types of compatible maps and common fixed points*, Indian J. Pure Appl. Math. 28(4) (1997) 477–485.

[19] N. Saleem, A. H. Ansari and M. K. Jain, *Some fixed point theorems of inverse C-class function under weak semi compatibility*, J. Fixed Point Theo. 9 (2018) 1–23.

[20] S. Sessa, *On a weak commutativity condition of mappings in fixed point considerations*, Publ. Inst. Math. 32 (1982) 149–153.

[21] M. R. Singh and Y. Mahendra Singh, *Compatible mappings of type (E) and common fixed point theorems of Meir-Keeler type*, Int. J. Math. Sci. Engg. Appl. 1(2)(2007) 299—315.

[22] M. R. Singh and Y. Mahendra Singh, *On various types of compatible maps and common fixed point theorems for non-continuous maps*, Hacet. J. Math. Stat. 40(4) (2011) 503–513.