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Function weighted $\mathcal{G}$-metric spaces and Hausdorff Δ-distances; an application to fixed point theory

Ehsan Lotfali Ghasab[a], Hamid Majani[a,*]

[a] Department of Mathematics, Shahid Chamran University of Ahvaz, Ahvaz, Iran

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Abstract

In this paper, we introduce a new space which is a generalization of function weighted metric space introduced by Jleli and Samet [On a new generalization of metric spaces, J. Fixed Point Theory Appl. 2018, 20:128] where namely function weighted $\mathcal{G}$-metric space. Also, a Hausdorff Δ-distance is introduced in these spaces. Then several fixed point results for both single-valued and multi-valued mappings in such spaces are proved. We also construct some examples for the validity of the given results and present an application to the existence of a solution of the Volterra-type integral equation.

Keywords: Function weighted $\mathcal{G}$-metric space, Hausdorff Δ-distance, fixed point, common coupled fixed point.
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1. Introduction and preliminaries

In the last century, nonlinear functional analysis has experienced many advances. One of these improvements is the introduction of various metric spaces and is the proof of fixed point results in these spaces along with its applications in engineering science. One of these spaces is G-metric space introduced by Mustafa [13]. This is a generalization of metric spaces in which every triple of elements is assigned to a non-negative real number. Another of these spaces is function weighted metric spaces defined by Jleli and Samet [9].

*Corresponding author
Email addresses: e-lotfali@stu.scu.ac.ir (Ehsan Lotfali Ghasab), h.majani@scu.ac.ir; majani.hamid@gmail.com (Hamid Majani)

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Definition 1.1. *[2, 9] Let $g : (0, +\infty) \to \mathbb{R}$ be a function such that for every sequence $\{b_n\} \subset (0, +\infty)$ we have $\lim_{n\to\infty} b_n = 0$ if and only if $\lim_{n\to\infty} g(b_n) = -\infty$. This function is called logarithmic-like and is called a non-decreasing function if for all $u, v \in (0, +\infty)$ where $u \leq v$ we have $g(u) \leq g(v)$.*

It the sequel, the set of all functions that are non-decreasing and logarithmic-like is denoted by $\mathcal{F}$. In 2019, some of researchers such as: Alqahtani et al. [2], Aydi et al. [3] and Bera et al. [4] discussed on the structure of this space and on the fixed points of mappings satisfying in various contractive conditions.

Definition 1.2. *[9] Let : $\eta : \mathcal{X} \times \mathcal{X} \to [0, +\infty)$ be a mapping and there exist a $g \in \mathcal{F}$ and a constant $B \in [0, +\infty)$ such that*

η1) $\eta(x, y) = 0 \Leftrightarrow x = y$ for all $x, y \in \mathcal{X}$;

η2) $\eta(x, y) = \eta(y, x)$ for all $x, y \in \mathcal{X}$;

η3) for all $(x, y) \in \mathcal{X} \times \mathcal{X}$ and for each $N \in \mathbb{N}$ with $N \geq 2$, we have

$$\eta(x, y) > 0 \Rightarrow g(\eta(x, y)) \leq g(\sum_{i=1}^{N-1} \eta(v_i, v_{i+1})) + B$$

for all $(v_i)_{i=1}^N \subset X$ with $(v_1, v_N) = (x, y)$.

Then, the function η is named as a function weighted metric or an $\mathcal{F}$-metric on $\mathcal{X}$, and the pair $(\mathcal{X}, \eta)$ is called a function weighted metric space or a $\mathcal{F}$-metric space.

On the other hands, Bhaskar and Lakshmikantham [5] defined the concept of coupled fixed point and presented some coupled fixed point results for a mixed monotone mapping in partially ordered matric spaces. Also, they studied the existence and uniqueness of a solution to a periodic boundary value problem. For more details on coupled, tripled and n-tupled fixed point theorems, we refer to [6, 7, 8, 10, 12, 14].

Definition 1.3. *[1, 11] An element $(x, y) \in \mathcal{X}^2$ is called a coupled coincidence point of two mappings $h : \mathcal{X}^2 \to \mathcal{X}$ and $r : \mathcal{X} \to \mathcal{X}$ if $h(x, y) = rx$ and $h(y, x) = ry$. Also, an element $(x, y) \in \mathcal{X}^2$ is called a common fixed point of a mapping $h : \mathcal{X}^2 \to \mathcal{X}$ and $r : \mathcal{X} \to \mathcal{X}$ if $h(x, y) = rx = x$ and $h(y, x) = ry = y$.*

Note that if r is the identity mapping, then (x, y) is called a coupled fixed point of h [5].

Definition 1.4. *[11] Let $\mathcal{X}$ be a nonempty set. For two given mappings $h : \mathcal{X}^2 \to \mathcal{X}$ and $r : \mathcal{X} \to \mathcal{X}$, h and r is said to be commutative if $h(rx, ry) = r(h(x, y))$ for all $x, y \in \mathcal{X}$.*

In this paper, we introduced the generalization of both G-metric spaces and function weighted metric spaces, namely, function weighted $\mathcal{G}$-metric space. Also, we define a Hausdorff Δ-distance in these spaces. Then, we introduce some common fixed point results in such spaces and prove them. For this goal, this paper is organized as follows. In the next section, we define a function weighted $\mathcal{G}$-metric space and introduce the definition of convergent in this space. Then we prove a common fixed point theorem and a common coupled fixed point result in this space. In Section 3, we introduce a Hausdorff Δ-distance and obtain a coincidence point results for multi-valued mappings concerning this distance. In the final section, an application of these results considered for the existence of solution of a Volterra-type integral equation.

2. Function weighted $\mathcal{G}$-metric space and fixed point theory

Let's start by introducing the following space.

Definition 2.1. *Let $\Delta : \mathcal{X} \times \mathcal{X} \times \mathcal{X} \to [0, +\infty)$ be a given mapping. Suppose that there exist a $f \in \mathcal{F}$ and a constant $C \in [0, +\infty)$ such that*

Δ1) $\Delta(x, y, z) = 0$ iff $x = y = z$ for all $x, y, z \in \mathcal{X}$;

Δ2) $\Delta(x, y, z) = \Delta(y, x, z) = \cdots$ for all $x, y, z \in \mathcal{X}$;

Δ3) $0 < \Delta(x, x, y)$ for all $x, y \in \mathcal{X}$ with $x \neq y$;

Δ4) $\Delta(x, x, y) \leq \Delta(x, y, z)$ for all $x, y, z \in \mathcal{X}$ with $z \neq y$;

Δ5) For all $(x, y, z) \in \mathcal{X} \times \mathcal{X} \times \mathcal{X}$ and for each $N \in \mathbb{N}$ with $N \geq 2$, we have

$$\Delta(x, y, z) > 0 \Longrightarrow f(\Delta(x, y, z)) \leq f(\sum_{i=3}^{N-1} \Delta(u_{i-2}, u_{i-1}, u_{i-1}) + \Delta(u_{i-1}, u_i, u_{i+1})) + C$$

for each $(u_i)_{i=1}^N \subset \mathcal{X}$ with $(u_1, u_{N-1}, u_N) = (x, y, z)$.

Then, the function Δ is said to be a function weighted generalized metric or an $\mathcal{FG}$-metric on $\mathcal{X}$, and the pair $(\mathcal{X}, \Delta)$ is said to be a function weighted generalized metric space or an $\mathcal{FG}$-metric space.

Example 2.2. *Let $\mathcal{X}$ be a set of all real number and $f(t) = \ln(t)$ be a non-decreasing function. Then, for all distinct $x, y, z \in \mathcal{X}$, for every $N \in \mathbb{N}$ with $N \geq 3$, for each $(u_i)_{i=1}^N \subset \mathcal{X}$ with $(u_1, u_{N-1}, u_N) = (x, y, z)$, and every G-metric on $\mathcal{X}$, we have*

$$G(x, y, z) > 0 \Longrightarrow \ln(G(x, y, z)) \leq \ln(\sum_{i=3}^{N-1} G(u_{i-2}, u_{i-1}, u_{i-1}) + G(u_{i-1}, u_i, u_{i+1})).$$

since $G(x, y, z) \leq \sum_{i=3}^{N-1} G(u_{i-2}, u_{i-1}, u_{i-1}) + G(u_{i-1}, u_i, u_{i+1})$. Here, we take $C = 0$. Thus, $(\mathcal{X}, G)$ is an $\mathcal{FG}$-metric space.

Definition 2.3. *Let $(\mathcal{X}, \Delta)$ be an $\mathcal{FG}$-metric space. A subset $\mathcal{O}$ of $\mathcal{X}$ is said to be open if for every $x \in \mathcal{O}$, there is some $r > 0$ such that $B(x, r) \subset \mathcal{O}$, where*

$$B(x, r) = \{y \in \mathcal{X} : \Delta(x, y, y) < r\}.$$

We say that a subset C of $\mathcal{X}$ is closed if $\mathcal{X} - C$ is open.

Now, we consider the definition of convergence, Cauchy and completeness of an $\mathcal{FG}$-metric space.

Definition 2.4. *Let $(\mathcal{X}, \Delta)$ be an $\mathcal{FG}$-metric space and $\{x_n\}$ be a sequence of points of $\mathcal{X}$.*

1. *$\{x_n\}$ is called convergent sequence if there exist $x \in \mathcal{X}$ such that*

$$\lim_{n\to\infty} \Delta(x, x_n, x_n) = 0.$$

2. *$\{x_n\}$ is called Δ-Cauchy if for every $\epsilon > 0$, there is a $N \in \mathbb{N}$ such that $\Delta(x_n, x_m, x_l) < \epsilon$ for all $n, m, l \geq N$; that is, if $\Delta(x_n, x_m, x_l) \to 0$, as $n, m, l \to \infty$.*

3. *$(\mathcal{X}, \Delta)$ is said to be Δ-complete (or complete Δ-metric space) if every Δ-Cauchy sequence in $(\mathcal{X}, \Delta)$ is convergent in $\mathcal{X}$.*

Theorem 2.5. *Let $(\mathcal{X}, \Delta)$ be a complete function weighted $\mathcal{G}$-metric space. Also, $r, H : \mathcal{X} \to \mathcal{X}$ be two arbitrary mappings such that H, r are commutative, $H(\mathcal{X}) \subset r(\mathcal{X})$, and $r(\mathcal{X})$ is closed. Suppose that there exists $k \in (0, 1)$ such that*

$$\Delta(Hx, Hy, Hz) \leq k\Delta(rx, ry, rz) \tag{2.1}$$

for all $x, y, z \in \mathcal{X}$. Then H and r have a unique common fixed point in $\mathcal{X}$.

Proof . Since $H(\mathcal{X}) \subset r(\mathcal{X})$, we can choose a point $x_1 \in \mathcal{X}$ such that $Hx_0 = rx_1$ for a given $x_0 \in \mathcal{X}$. We construct a sequence x_n in $\mathcal{X}$ such that $y_n = Hx_n = rx_{n+1}$ for $n = 0, 1, \cdots$. First, observe that H and r possess a unique coincidence point. Indeed, suppose on the contrary that $u, v \in \mathcal{X}$ is two distinct coincidence points of H and r. Thus, $\Delta(u, u, v) > 0$, $r(u) = H(u)$ and $r(v) = H(v)$. Then, from 2.1, we have

$$\Delta(u, u, v) = \Delta(Hu, Hu, Hv) \leq k\Delta(ru, ru, rv) = k\Delta(u, u, v) < \Delta(u, u, v),$$

which is a contradiction.

Suppose $(f, C) \in \mathcal{F} \times [0, +\infty)$ so that $(\Delta 5)$ is fulfilled. For a given $\epsilon > 0$ and on account of $(\Delta 5)$, there exists $\gamma > 0$ such that

$$0 < t < \gamma \Rightarrow f(t) < f(\epsilon) - C. \tag{2.2}$$

Consider the sequence $\{y_n\} \subset \mathcal{X}$. Now, without loss of generality, we assume that $\Delta(Hx_0, Hx_0, Hx_1) > 0$. Otherwise, x_1 will be a coincidence point of H and r. By 2.1, we obtain

$$\begin{aligned} \Delta(Hx_n, Hx_n, Hx_{n+1}) &\leq k\Delta(rx_n, rx_n, rx_{n+1}) \\ &= k\Delta(Hx_{n-1}, Hx_{n-1}, Hx_n) \\ &\leq k^2\Delta(rx_{n-1}, rx_{n-1}, rx_n), \end{aligned}$$

which implies by induction that

$$\Delta(Hx_n, Hx_n, Hx_{n+1}) \leq k^n\Delta(Hx_0, Hx_0, Hx_1)$$

for all $n \in \mathbb{N}$. Hence, for all $m, n \in \mathbb{N}$ with $m > n$, we have

$$\sum_{i=n}^{m-1} \Delta(Hx_i, Hx_i, Hx_{i+1}) \leq \frac{k^n}{1-k}\Delta(Hx_0, Hx_0, Hx_1).$$

Since

$$\lim_{n\to\infty} \frac{k^n}{1-k}\Delta(Hx_0, Hx_0, Hx_1) = 0,$$

there exists some $N \in \mathbb{N}$ such that $0 < \frac{k^n}{1-k}\Delta(Hx_0, Hx_0, Hx_1) < \gamma$ for all $n \geq N$. Hence, by 2.2 and $(\Delta 1)$, we have

$$f(\sum_{i=n}^{m-1} \Delta(Hx_i, Hx_i, Hx_{i+1})) \leq f(\frac{k^n}{1-k}\Delta(Hx_0, Hx_0, Hx_1)) < f(\epsilon) - C \tag{2.3}$$

for $m > n \geq N$. Employing (Δ5) together with 2.3, we find if $\Delta(Hx_n, Hx_n, Hx_m) > 0$, then

$$f(\Delta(Hx_n, Hx_n, Hx_m)) \leq f(\sum_{i=n}^{m-1} \Delta(Hx_i, Hx_i, Hx_{i+1})) + C < f(\epsilon)$$

which implies by (Δ1) that $\Delta(Hx_n, Hx_n, Hx_m) < \epsilon$. This proves that $\{y_n\} = \{Hx_n\}$ is Cauchy. Since $\{Hx_n\} = \{rx_{n+1}\} \subset r(X)$ and $r(\mathcal{X})$ is closed, there exists $z \in \mathcal{X}$ such that $\lim_{n,m\to\infty} \Delta(rx_n, rx_m, rz) = 0$. As a next step, we shall indicate that z is a coincidence point of H and r. On the contrary, assume that $\Delta(Hz, Hz, rz) > 0$. Then we have

$$\begin{aligned} f(\Delta(Hz, Hz, rz)) &\leq f(\Delta(Hz, Hz, Hx_n) + \Delta(Hx_n, Hx_n, rz)) + C \\ &\leq f(k\Delta(rz, rz, rx_n) + \Delta(rx_{n+1}, rx_{n+1}, rz)) + C. \end{aligned}$$

As $n \to \infty$ in the inequality above, we obtain

$$\lim_{n\to\infty} f(k\Delta(rz, rz, rx_n) + \Delta(rx_{n+1}, rx_{n+1}, rz)) + C = -\infty,$$

which is a contradiction. Hence, $\Delta(Hz, Hz, rz) = 0$; that is, z is a unique coincidence point of H and r. Therefore, r and H have a unique point of coincidence $w = rz = Hz$. By commutativity of H and r, we have $rw = r(rz) = rH(z) = Hr(z) = Hw$. This implies that rw is another point of coincidence of r and H. By uniqueness of point of coincidence of r and H, we have $w = rw = Hw$; that is r and H have a unique common fixed point. This completes the proof. □

In the sequel, denote for simplicity $\mathcal{X} \times \cdots \times \mathcal{X}$ by $\mathcal{X}^n$, where $\mathcal{X}$ is a non-empty set and $n \in \mathbb{N}$.

Lemma 2.6. *Let $(\mathcal{X}, \Delta)$ be an $\mathcal{FG}$-metric space. Then the following assertions hold:*

1. *$(\mathcal{X}^n, \mathcal{D})$ is an $\mathcal{FG}$-metric space with*

$$\begin{aligned} \mathcal{D}((x_1, \cdots, x_n), (y_1, \cdots, y_n), (z_1, \cdots, z_n)) &= \max[\Delta(x_1, y_1, z_1), \\ &\qquad \Delta(x_2, y_2, z_2), \cdots, \Delta(x_n, y_n, z_n)]. \end{aligned}$$

2. *The mapping $h : \mathcal{X}^n \to \mathcal{X}$ and $r : \mathcal{X} \to \mathcal{X}$ have a n-tuple common fixed point if and only if the mapping $H : \mathcal{X}^n \to \mathcal{X}^n$ and $R : \mathcal{X}^n \to \mathcal{X}^n$ defined by*

$$H(x_1, x_2, \cdots, x_n) = (h(x_1, x_2, \cdots, x_n), h(x_2, \cdots, x_n, x_1), \cdots, h(x_n, x_1, \cdots, x_{n-1}))$$

and

$$R(x_1, x_2, \cdots, x_n) = (rx_1, rx_2, \cdots, rx_n)$$

have a common fixed point in $\mathcal{X}^n$.

3. *$(\mathcal{X}, \Delta)$ is complete if and only if $(\mathcal{X}^n, \mathcal{D})$ is complete.*

Proof . 1. Clearly, $\mathcal{D}$ satisfies in (Δ1)-(Δ4). We show that $\mathcal{D}$ satisfies in (Δ5). For every $(u_{i,j}) \subset \mathcal{X}$ for $1 \leq i \leq N$ and $1 \leq j \leq n$, consider $(u_{i1}, u_{iN-1}, u_{iN}) = (x_1, y_{N-1}, z_{N-1})$. Suppose that

$$\Delta(x_j, y_j, z_j) = \max[\Delta(x_1, y_1, z_1), \Delta(x_2, y_2, z_2), \cdots, \Delta(x_n, y_n, z_n)].$$

Then, we have

$$f_j(\Delta(x_j, y_j, z_j)) \leq f_j(\sum_{i=3}^{N-1} \Delta(u_{i-2j}, u_{i-1j}, u_{i-1j}) + \Delta(u_{i-1j}, u_{ij}, u_{i+1j})) + C_j,$$

where $f_j \in \mathcal{F}$ and $C_j \in [0, +\infty)$. Therefore, we obtain

$$\begin{aligned}
& f_j(\mathcal{D}((x_1, x_2, \cdots, x_n), (y_1, y_2, \cdots, y_n), (z_1, z_2, \cdots, z_n))) \\
= \;& f_j(\max[\Delta(x_1, y_1, z_1), \Delta(x_2, y_2, z_2), \cdots, \Delta(x_n, y_n, z_n)]) \\
= \;& f_j(\Delta(x_j, y_j, z_j)) \\
\leq \;& f_j(\sum_{i=3}^{N-1} \Delta(u_{i-2,j}, u_{i-1,j}, u_{i-1,j}) + \Delta(u_{i-1,j}, u_{i,j}, u_{i+1,j})) + C_j) \\
\leq \;& f_j(\sum_{i=3}^{N-1} \mathcal{D}((u_{i-2,1}, u_{i-2,2}, \cdots, u_{i-2,n}), (u_{i-1,1}, u_{i-1,2}, \cdots, u_{i-1,n}), (u_{i-1,1}, u_{i-1,2}, \cdots, u_{i-1,n})) \\
& \quad + \mathcal{D}((u_{i-1,1}, u_{i-1,2}, \cdots, u_{i-1,n}), (u_{i,1}, u_{i,2}, \cdots, u_{i,n}), (u_{i+1,1}, u_{i+1,2}, \cdots, u_{i+1,n}))) + C_j.
\end{aligned}$$

The proof of 2. and 3. are straightforward and left to the reader. $\square$

Note that the Lemma 2.6 is a two-way relationship. Thus, we can obtain n-tuple fixed point results from fixed point theorems and conversely. Now, set $n = 2$ in Lemma 2.6. Then we have the following theorem.

Theorem 2.7. *Let $(\mathcal{X}, \Delta)$ be a complete function weighted $\mathcal{G}$-metric spaces. Also, let $r : \mathcal{X} \to \mathcal{X}$ and $H : \mathcal{X}^2 \to \mathcal{X}$ be two mappings such that H, r are commutative, $H(\mathcal{X}^2) \subset r(\mathcal{X})$ and $r(\mathcal{X})$ is closed. Suppose that there exists $k \in (0, 1)$ such that*

$$\Delta(H(x, y), H(u, v), H(w, z)) \leq \frac{k}{2}(\Delta(rx, ru, rw) + \Delta(ry, rv, rz)) \tag{2.4}$$

for all $(x, y), (u.v), (w, z) \in \mathcal{X}^2$. Then H and r have a unique common coupled fixed point in $\mathcal{X} \times \mathcal{X}$.

Proof . Let us define $\mathcal{D} : \mathcal{X}^2 \times \mathcal{X}^2 \times \mathcal{X}^2 \to \mathcal{X}$ by

$$\mathcal{D}((x_1, x_2), (y_1, y_2), (z_1, z_2)) = \max[\Delta(x_1, y_1, z_1), \Delta(x_2, y_2, z_2)],$$

$\mathcal{H} : \mathcal{X}^2 \to \mathcal{X}^2$ by $\mathcal{H}(x, y) = (H(x, y), H(y, x))$ and $\mathcal{R} : \mathcal{X}^2 \to \mathcal{X}^2$ by $\mathcal{R}(x, y) = (rx, ry)$. Using Lemma 2.6, $(\mathcal{X}^2, \mathcal{D})$ is a complete $\mathcal{FG}$-metric space. Also, $(x, y) \in \mathcal{X}^2$ is a common coupled fixed point of H and r if and only if it is a common fixed point of $\mathcal{H}$ and $\mathcal{R}$. On the other hands, from 2.4, we have either

$$\begin{aligned}
& \mathcal{D}(\mathcal{H}(x, y), \mathcal{H}(u, v), \mathcal{H}(w, z)) \\
= \;& \mathcal{D}((H(x, y), H(y, x)), (H(u, v), H(v, u)), (H(w, z), H(z, w))) \\
= \;& \max[\Delta(H(x, y), H(u, v), H(w, z)), \Delta(H(y, x), H(v, u), H(z, w))] \\
= \;& \Delta(H(x, y), H(u, v), H(w, z)) \\
\leq \;& \frac{k}{2}(\Delta(rx, ru, rw) + \Delta(ry, rv, rz)) \\
\leq \;& k \max[\Delta(rx, ru, rw), \Delta(ry, rv, rz)] \\
= \;& k\mathcal{D}(\mathcal{R}(x, y), \mathcal{R}(u, v), \mathcal{R}(w, z))
\end{aligned}$$

or

$$\begin{aligned}
& \mathcal{D}(\mathcal{H}(x,y),\mathcal{H}(u,v),\mathcal{H}(w,z)) \\
= \ & \mathcal{D}((H(x,y),H(y,x)),(H(u,v),H(v,u)),(H(w,z),H(z,w))) \\
= \ & \max[\Delta(H(x,y),H(u,v),H(w,z)),\Delta(H(y,x),H(v,u),H(z,w))] \\
= \ & \Delta(H(y,x),H(v,u),H(z,w)) \\
\leq \ & \frac{k}{2}(\Delta(ry,rv,rz)+\Delta(rx,ru,rw)) \\
\leq \ & k\max[\Delta(ry,rv,rz),\Delta(rx,ru,rw)] \\
= \ & k\mathcal{D}(\mathcal{R}(y,x),\mathcal{R}(v,u),\mathcal{R}(z,w)).
\end{aligned}$$

Now, by Theorem 2.5, $\mathcal{H}$ and $\mathcal{R}$ have a common fixed point and by Lemma 2.6, H and r have a common coupled fixed point. □

Example 2.8. *Let $\mathcal{X}=[0,1]$. Define $\Delta:\mathcal{X}\times\mathcal{X}\times\mathcal{X}\to[0,\infty)$ by*

$$\Delta(x,y,z)=|x-y|+|x-z|+|y-z|$$

for all $x,y,z\in\mathcal{X}$. Then Δ is a complete $\mathcal{FG}$-metric with $f(t)=Ln(t)$ and $C=0$. Consider $H:\mathcal{X}^2\to\mathcal{X}$ and $r:\mathcal{X}\to\mathcal{X}$ by

$$H(x,y)=\frac{x}{2}+\frac{y}{2} \quad \text{and} \quad r(x)=2x.$$

Clearly, H and r are commutative. Also, we have

$$\begin{aligned}
\Delta(H(x,y),H(u,v),H(w,z)) &= |\frac{x}{2}+\frac{y}{2}-(\frac{u}{2}+\frac{v}{2})|+|\frac{x}{2}+\frac{y}{2}-(\frac{w}{2}+\frac{z}{2})| \\
&\quad +|\frac{u}{2}+\frac{v}{2}-(\frac{w}{2}+\frac{z}{2})| \\
&= \frac{1}{2}(|x+y-(u+v)|+|x+y-(w+z)| \\
&\quad +|u+v-(w+z)|) \\
&\leq \frac{1}{2}\Delta(rx,ru,rw)+\frac{1}{2}\Delta(ry,rv,rz).
\end{aligned}$$

Therefore, by letting $k=\frac{1}{2}$, all the hypothesis of Theorem 2.7 are satisfied. Thus, H and r have a unique common coupled fixed point in $\mathcal{X}\times\mathcal{X}$.

3. Hausdorff $\boldsymbol{\Delta}$-distance and fixed point results

First, we introduce the following definition.

Definition 3.1. *Let $(\mathcal{X},\Delta)$ be an $\mathcal{FG}$-metric space and $CB(\mathcal{X})$ be the family of all nonempty closed bounded subsets of $\mathcal{X}$. We say $H(\cdot,\cdot,\cdot)$ is a Hausdorff Δ-distance on $CB(\mathcal{X})$, if*

$$H_\Delta(A,B,C)=\max\{\sup_{x\in A}\Delta(x,B,C),\sup_{x\in B}\Delta(x,C,A),\sup_{x\in C}\Delta(x,A,B)\},$$

where

$$\begin{aligned}
&\Delta(x,B,C)=d_\Delta(x,B)+d_\Delta(B,C)+d_\Delta(x,C),\\
&d_\Delta(x,B)=\inf\{d_\Delta(x,y),y\in B\},\\
&d_\Delta(A,B)=\inf\{d_\Delta(a,b),a\in A,b\in B\}.
\end{aligned}$$

Definition 3.2. *[15] Let $\mathcal{X}$ be a nonempty set. Assume that $g : \mathcal{X} \to \mathcal{X}$ and $T : \mathcal{X} \to CB(\mathcal{X})$. If $w = gx \in Tx$ for some $x \in \mathcal{X}$, then x is called a coincidence point of g and T and w is a point of coincidence of g and T*

Theorem 3.3. *Let $(\mathcal{X}, \Delta)$ be a complete $\mathcal{FG}$-metric space. Also, let $g : \mathcal{X} \to \mathcal{X}$ and $T : \mathcal{X} \to CB(\mathcal{X})$ be two function $T(\mathcal{X}) \subset g(\mathcal{X})$, $g(\mathcal{X})$ is closed and g is continuous. Assume that there exists $k \in (0, 1)$ such that*

$$H_\Delta(Tx, Ty, Tz) \leq k\Delta(gx, gy, gz) \tag{3.1}$$

for all $x, y, z \in \mathcal{X}$. Then T and g have coincidence point in $\mathcal{X}$.

Proof . Since $T(\mathcal{X}) \subset g(\mathcal{X})$, we can choose a point $x_1 \in X$ such that $gx_1 \in Tx_0$. We shall construct a sequence x_n in $\mathcal{X}$ such that $gx_{n+1} \in Tx_n$ for $n = 0, 1, \cdots$. Suppose $(f, C) \in \mathcal{F} \times [0, +\infty)$ so that $(\Delta 5)$ is fulfilled. For a given $\epsilon > 0$ and on account of $(\Delta 5)$, there exists $\gamma > 0$ such that

$$0 < t < \gamma \Rightarrow f(t) < f(\epsilon) - C. \tag{3.2}$$

Consider the sequence $\{gx_n\} \subset \mathcal{X}$. Now, without loss of generality, assume that $H_\Delta(Tx_0, Tx_0, Tx_1) > 0$. Otherwise, x_1 will be a coincidence point of T and g. Now, from 3.1, we have

$$\begin{aligned} \Delta(gx_{n+1}, gx_{n+1}, gx_{n+2}) &\leq H_\Delta(Tx_n, Tx_n, Tx_{n+1}) \\ &\leq k\Delta(gx_n, gx_n, gx_{n+1}) \\ &\leq kH_\Delta(Tx_{n-1}, Tx_{n-1}, Tx_n) \\ &\leq k^2\Delta(gx_{n-1}, gx_{n-1}, gx_n), \end{aligned}$$

which implies that $\Delta(gx_n, gx_n, gx_n) \leq k^n\Delta(Tx_0, Tx_0, Tx_1)$ for all $n \in \mathbb{N}$. Now, let $m, n \in \mathbb{N}$ with $m > n$. Then we have

$$\sum_{i=n}^{m-1} \Delta(gx_i, gx_i, gx_{i+1}) \leq \frac{k^n}{1-k}\Delta(gx_0, gx_0, gx_1).$$

On the other hand, since $\lim_{n\to\infty} \frac{k^n}{1-k}\Delta(gx_0, gx_0, gx_1) = 0$, there exists $N \in \mathbb{N}$ such that

$$0 < \frac{k^n}{1-k}\Delta(gx_0, gx_0, gx_1) < \gamma$$

for $n \geq N$. Hence, by 3.2 and $(\Delta 1)$, we have

$$f(\sum_{i=n}^{m-1} \Delta(gx_i, gx_i, gx_{i+1})) \leq f(\frac{k^n}{1-k}\Delta(gx_0, gx_0, gx_1)) < f(\epsilon) - C \tag{3.3}$$

for all $m > n \geq N$. Employing $(\Delta 5)$ together with 3.3, we obtain

$$\Delta(gx_n, gx_n, gx_m) > 0 \Rightarrow f(\Delta(gx_n, gx_n, gx_m)) \leq f(\sum_{i=n}^{m-1} \Delta(gx_i, gx_i, gx_{i+1})) + C < f(\epsilon),$$

which implies by $(\Delta 1)$ that $\Delta(gx_n, gx_n, gx_m) < \epsilon$. This proves that $\{gx_n\}$ is Cauchy. Since X is a complete $\mathcal{FG}$-metric space and $g(\mathcal{X})$ is closed, there exists $x \in \mathcal{X}$ such that $\lim\limits_{n\to\infty} gx_n = gx$. Now, we claim that $gx \in Tx$. For this, from 3.1, we have

$$\Delta(gx_{n+1}, Tx, Tx) \leq H_\Delta(T(x_n), Tx, Tx) \leq k\Delta(gx_n, gx, gx).$$

Thus,

$$\lim_{n\to\infty} \Delta(gx_{n+1}, Tx, Tx) = \Delta(gx, Tx, Tx) = 0.$$

Hence, $gx \in Tx$; that is, T and g have a point of coincidence. $\square$

Example 3.4. *Let $\mathcal{X} = [0,1]$, $T : \mathcal{X} \to CB(\mathcal{X})$ and $g : \mathcal{X} \to \mathcal{X}$ be defined by $Tx = [0, \frac{1}{16}x]$ and $gx = \sqrt{x}$. Define $\Delta : \mathcal{X} \times \mathcal{X} \times \mathcal{X} \to [0,\infty)$ by*

$$\Delta(x, y, z) = \max\{|x - y|, |x - z|, |y - z|\}.$$

Then Δ is an complete $\mathcal{FG}$-metric with $f(t) = Ln(t)$ and $C = 0$. Clearly, $T(\mathcal{X}) \subset g(\mathcal{X})$ and $g(\mathcal{X})$ is closed. If $x = y = z = 0$, then

$$H_\Delta(Tx, Ty, Tz) = 0 \leq k\Delta(gx, gy, gz).$$

Thus, we may assume that x, y and z are not all zero. Without loss of the generality, assume that $x \leq y \leq z$. Then

$$\begin{aligned} H_\Delta(Tx, Ty, Tz) &= H_\Delta([0, \frac{1}{16}x], [0, \frac{1}{16}y], [0, \frac{1}{16}z]) \\ &= \max[\sup_{0\leq a\leq \frac{1}{16}x} \Delta(a, [0, \frac{1}{16}y], [0, \frac{1}{16}z]), \sup_{0\leq b\leq \frac{1}{16}y} \Delta(b, [0, \frac{1}{16}z], [0, \frac{1}{16}x]), \\ &\quad \sup_{0\leq c\leq \frac{1}{16}z} \Delta(c, [0, \frac{1}{16}x], [0, \frac{1}{16}y])]. \end{aligned}$$

Since $x \leq y \leq z$, so $[0, \frac{1}{16}x] \subset [0, \frac{1}{16}y] \subset [0, \frac{1}{16}z]$. This implies that

$$d_\Delta([0, \frac{1}{16}x], [0, \frac{1}{16}y]) = d_\Delta([0, \frac{1}{16}y], [0, \frac{1}{16}z]) = d_\Delta([0, \frac{1}{16}x], [0, \frac{1}{16}z]) = 0.$$

For each $0 \leq a \leq \frac{1}{16}x$, we have

$$\Delta(a, [0, \frac{1}{16}y], [0, \frac{1}{16}z]) = d_\Delta(a, [0, \frac{1}{16}y]) + d_\Delta([0, \frac{1}{16}y], [0, \frac{1}{16}z]) + d_\Delta(a, [0, \frac{1}{16}z]) = 0.$$

Also, for each $0 \leq b \leq \frac{1}{16}y$, we have

$$\begin{aligned} \Delta(b, [0, \frac{1}{16}x], [0, \frac{1}{16}z]) &= d_\Delta(b, [0, \frac{1}{16}x]) + d_\Delta([0, \frac{1}{16}x], d_\Delta(a, [0, \frac{1}{16}z])) + d_\Delta(b, [0, \frac{1}{16}z]) \\ &= \begin{cases} 0, & b \leq \frac{x}{16} \\ 2b - \frac{x}{8}, & b \geq \frac{x}{16} \end{cases} \end{aligned}$$

This yields that

$$\sup_{0\leq b\leq \frac{1}{16}y} \Delta(b, [0, \frac{1}{16}z], [0, \frac{1}{16}x]) = \frac{y}{8} - \frac{x}{8}.$$

Moreover, for each $0 \leq c \leq \frac{1}{16}z$, we have

$$\begin{aligned} \Delta(c, [0, \frac{1}{16}x], [0, \frac{1}{16}y]) &= d_\Delta(c, [0, \frac{1}{16}x]) + d_\Delta([0, \frac{1}{16}x], d_\Delta(a, [0, \frac{1}{16}y])) + d_\Delta(c, [0, \frac{1}{16}y]) \\ &= \begin{cases} 0, & c \leq \frac{x}{16} \\ 2c - \frac{x}{8}, & \frac{x}{16} \leq c \leq \frac{y}{16} \\ 4c - \frac{x}{8} - \frac{y}{8}, & c \geq \frac{y}{16} \end{cases} \end{aligned}$$

This yields that

$$\sup_{0\le c\le \frac{1}{16}z} \Delta(c,[0,\frac{1}{16}x],[0,\frac{1}{16}y]) = \frac{z}{4}-\frac{y}{8}-\frac{x}{8}.$$

We deduce that

$$\begin{aligned} H_\Delta(Tx,Ty,Tz) &= \frac{z}{4}-\frac{x}{8}-\frac{y}{8} \le \frac{1}{4}(z-x) = \frac{1}{2}(\frac{1}{2}(z-x)) \\ &\le \frac{1}{2}(\frac{z-x}{\sqrt{x}+\sqrt{z}}) = \frac{1}{2}(\sqrt{z}-\sqrt{x}). \end{aligned}$$

On the other hand, it is obvious that all other hypotheses of Theorem 3.3 are satisfied and so g and T have a unique common fixed point.

4. An application to a Volterra integral equations

As an application of our results, we consider the following Volterra integral equation:

$$x(t) = \int_0^t K(t,s,x(s))ds + v(t), \tag{4.1}$$

where $t\in I=[0,1]$, $K\in C(I\times I\times\mathbb{R},\mathbb{R})$ and $v\in C(I,\mathbb{R})$.

Let $C(I,\mathbb{R})$ be the Banach space of all real continuous functions defined on I with norm $||x||_\infty = \max_{t\in I}|x(t)|$ for all $x\in C(I,\mathbb{R})$ and $C(I\times I\times C(I,\mathbb{R}),\mathbb{R})$ be the space of all continuous functions defined on $I\times I\times C(I,\mathbb{R})$. Alternatively, the Banach space $C(I,\mathbb{R})$ can be endowed with Bielecki norm $||x||_B = \sup_{t\in I}\{|x(t)|e^{-\tau t}\}$ for all $x\in C(I,\mathbb{R})$ and $\tau>0$, and the induced metric $\Delta_B(x,y,z) = ||x-y||_B+||x-z||_B+||y-z||_B$ for all $x,y,z\in C(I,\mathbb{R})$. Also, define $T: C(I,\mathbb{R})\to C(I,\mathbb{R})$ by

$$Tx(t) = \int_0^t K(t,s,x(s))ds + v(t), \quad v\in C(I,\mathbb{R}).$$

Theorem 4.1. *Let $(C(I,\mathbb{R}),\Delta_B)$ be a complete $\mathcal{FG}$-metric space by $f(t)=Ln(t)$, $T: C(I,\mathbb{R})\to C(I,\mathbb{R})$ be a operator with $Tx(t)=\int_0^t K(t,s,x(s))ds+v(t)$ and $rx=I(x)$. Assume that $K\in C(I\times I\times\mathbb{R},\mathbb{R})$ is an operator such that*

(i) K is continuous;

(ii) $\int_0^t K(t,s,\cdot)$ for all $t,s\in I$ is increasing;

(iii) there exists $\tau>0$ such that

$$|K(t,s,x(s))-K(t,s,y(s))| \le e^{-\tau}|x(s)-y(s)|$$

for all $x,y\in C(I,\mathbb{R})$ and $t,s\in I$.

Then, the Volterra-type integral equation 4.1 has a solution in $C(I,\mathbb{R})$.

Proof . By definition of T, we have

$$\begin{aligned}
\Delta_B(Tx,Ty,Tz) &= \| \int_0^t K(t,s,x(s))ds - \int_0^t K(t,s,y(s))ds\|_B \\
&\quad +\| \int_0^t K(t,s,x(s))ds - \int_0^t K(t,s,z(s))ds\|_B \\
&\quad +\| \int_0^t K(t,s,y(s))ds - \int_0^t K(t,s,z(s))ds\|_B \\
&= \sup_{t\in I}\{| \int_0^t K(t,s,x(s))ds - \int_0^t K(t,s,y(s))ds|\ e^{-\tau t}\} \\
&\quad + \sup_{t\in I}\{| \int_0^t K(t,s,x(s))ds - \int_0^t K(t,s,z(s))ds|\ e^{-\tau t}\} \\
&\quad + \sup_{t\in I}\{| \int_0^t K(t,s,y(s))ds - \int_0^t K(t,s,z(s))ds|\ e^{-\tau t}\} \\
&\leq \sup_{t\in I}\{\int_0^t |K(t,s,x(s)) - K(t,s,y(s))|e^{-\tau t}\ ds\} \\
&\quad + \sup_{t\in I}\{\int_0^t |K(t,s,x(s)) - K(t,s,z(s))|e^{-\tau t}\ ds\} \\
&\quad + \sup_{t\in I}\{\int_0^t |K(t,s,y(s)) - K(t,s,z(s))|e^{-\tau t}\ ds\} \\
&\leq \sup_{t\in I}\{\int_0^t e^{-\tau}|x(s)-y(s)|e^{-\tau t}\ ds\} + \sup_{t\in I}\{\int_0^t e^{-\tau}|x(s)-z(s)|e^{-\tau t}\ ds\} \\
&\quad + \sup_{t\in I}\{\int_0^t e^{-\tau}|y(s)-z(s)|e^{-\tau t}\ ds\} \\
&\leq (||x-y||_B + ||x-z||_B + ||y-z||_B) \sup_{t\in I}\{\int_0^t e^{-\tau} ds\} \\
&\leq e^{-\tau}\Delta_B(x,y,z).
\end{aligned}$$

Now, we consider that the function $f(t) = \ln(t)$ for each $t \in I$, $C = 0$ and $k = e^{-\tau}$. Therefore, all conditions of Theorem 2.5 are satisfied. Consequently, Theorem 2.5 ensures the existence of fixed point of T that this fixed point is the solution of the integral equation. □

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