Int. J. Nonlinear Anal. Appl. 12 (2021) No. 2, 1687-1697
ISSN: 2008-6822 (electronic)
http://dx.doi.org/10.22075/ijnaa.2021.5309

Mixed fractional partial differential equations by the base method

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(Communicated by Madjid Eshaghi Gordji)

Abstract

In this paper, the invariant subspace method is generalized and improved and is then used to have an exact solution for a wide class of the linear/ non-linear mixed fractional partial differential equations ($FPDEs$); with constant, non-constant coefficients. Some examples are given here to illustrate the efficiency of this method.

Keywords: Caputo fractional derivative, Mittag–Leffler function, Laplace transform, invariant subspace method, fractional partial differential equations.

1. Introduction

In the last decades shown that derivatives and integrals of arbitrary order are very convenient for describing properties of real materials. The new fractional- order models are more satisfying the former integer-order ones. In fact, a natural phenomenon may depend not only on the time instant but also on the previous time history, which can be modelled by fractional calculus. So motivated by this reasons, it is important to find efficient methods for solving fractional partial differential equations ($FPDEs$).

Recently, investigations have shown that a new method based on the invariant subspace provides an effective tool to find the exact solution of FDEs. This method was initially proposed by Galaktionov and Svirshchevskii [3, 13, 4]. The invariant subspace method was developed by Later Gazizov and Kasatkin [5], Harris and Garra [6, 7], Sahadevan and Bakkyaraj [11], and Ouhadan and El Kinani [9]. In [12], Sahadevan and Prakash showed how the invariant subspace method could be extended to time fractional partial differential equations ($FPDEs$), $\frac{\partial^\alpha u}{\partial t^\alpha} = F[u]$ and could construct their

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Received: January 2021 *Accepted:* April 2021

exact solutions. Where $F[.]$ is a nonlinear differential operator, $\frac{\partial^\alpha}{\partial\, t^\alpha}(.)$ is a fractional time derivative in the Caputo sense.

In [1], S. Choudhary and V. Daftardar-Gejji developed the invariant subspace method for deriving exact solutions of partial differential equations with fractional space and time derivatives.

$$\sum_{j=0}^{n} \lambda_j \frac{\partial^{\alpha+j}}{\partial\, t^{\alpha+j}} u(x,t) = N\Big(x, u, \frac{\partial^\beta}{\partial x^\beta}, \frac{\partial^{\beta+1}}{\partial x^{\beta+1}}, \dots, \frac{\partial^{\beta+m}}{\partial x^{\beta+m}}\Big).$$

All fractional partial derivatives are in Caputo sense, and $N[u]$ is a linear - nonlinear operator and $\alpha, \beta \ \in (0,1], \ \ m, n \ \in N.$

In [14], K.V. Zhukovsky used the inverse differential operational method to obtain solutions for differential equations with mixed derivatives of physical problems. In [8], Jun Jiang, Yuqiang Feng and Shougui Li develop the invariant subspace method for finding exact solutions to some nonlinear partial differential equations with fractional-order mixed partial derivatives (including both fractional space derivatives and time derivatives).

$$\sum_{j=0}^{n} \lambda_j \frac{\partial^{\alpha+j}}{\partial\, t^{\alpha+j}} u(x,t) = N\Big(x, u, \frac{\partial^\beta}{\partial x^\beta}, \frac{\partial^{\beta+1}}{\partial x^{\beta+1}}, \dots, \frac{\partial^{\beta+m}}{\partial x^{\beta+m}}\Big) + \mu \frac{\partial^\alpha}{\partial t^\alpha}\Big(\frac{\partial^\beta}{\partial x^\beta} u\Big) \tag{1.1}$$

$$a < \alpha \le a+1, \quad b < \beta \le b+1, \quad a, b \ \in N, \quad \lambda_j, \ \mu \ \in R.$$

In this paper, motivated by the above results, we improve this method by extension it to another forms through argue different cases of (1.1).
Also, we are going to argue the case λ_i is a function of t.

By invariant subspace method, the $FPDEs$ are reduced to the systems of $FDEs$ that can be solved by familiar analytical methods. This paper is as follows, in section 2 the preliminaries and notations are given. In Section 3, we develop the invariant subspace method for solving fractional space and time derivative nonlinear partial differential equations with fractional-order mixed derivatives. In Section 4, illustrative examples are given to explain the applicability of the method. Finally in Section 5, we give conclusions.

2. Preliminaries

In this section, we give some important definitions and notation which are needed in our work.

Definition 2.1. *[10] The Riemann–Liouville fractional integral of order α for a function f is defined as*

$$\mathbb{J}^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-x)^{\alpha-1} f(x)\, dx \qquad \alpha, \ t > 0$$

The $R-L$ fractional integral operator has the following properties:

- $\mathbb{J}^\alpha$ is a linear operator.
- $\mathbb{J}^0 = I.$
- $\lim_{\alpha\to 0} \mathbb{J}^\alpha = \mathbb{J}^0.$
- Has semigroup property; i.e $\mathbb{J}^\alpha\, \mathbb{J}^\beta = \mathbb{J}^\beta\, \mathbb{J}^\alpha = \mathbb{J}^{\alpha+\beta}.$

Definition 2.2. *[10] The Caputo fractional derivative of positive order α for a function f is defined as*

$$\frac{\partial^\alpha}{\partial t^\alpha} f(t) = \mathbb{J}^{n-\alpha} \mathrm{D}^n f(t) = \begin{cases} \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{f^{(n)}(x)}{(t-x)^{n-\alpha+1}} & n \neq \alpha \\ \\ f^{(n)}(x) & n = \alpha \end{cases}$$

Some properties of fractional Caputo derivative and fractional $R-L$ integral are:

- $\mathbb{J}^\alpha t^\beta = \frac{\Gamma(1+\beta)}{\Gamma(1+\alpha+\beta)}\, t^{\alpha+\beta} \qquad \alpha > 0,\ \beta > -1,\ \ t > 0$
- $\frac{d^\alpha}{dt^\alpha} t^\beta = \begin{cases} \frac{\Gamma(1+\beta)}{\Gamma(1+\beta-\alpha)} t^{\beta-\alpha} & n-1 < \alpha < n,\ \ \beta > n-1,\ \ \beta \in \mathbb{R} \\ \\ 0 & n-1 < \alpha < n,\ \ \beta \leq n-1,\ \ \beta \in \mathbb{N} \end{cases}$
- $\frac{\partial^\alpha}{\partial\, t^\alpha} \Big(\mathbb{J}^\alpha \Big) = I$
- $\mathbb{J}^\alpha \Big(\frac{\partial^\alpha}{\partial\, t^\alpha} f(t) \Big) = f(t) - \sum_{i=0}^{n-1} \frac{t^i}{i!} f^{(i)}(0)$
- $\frac{\partial^\alpha}{\partial\, t^\alpha} \Big(\frac{\partial^\beta}{\partial\, t^\beta} f(t) \Big) = \frac{\partial^\beta}{\partial\, t^\beta} \Big(\frac{\partial^\alpha}{\partial\, t^\alpha} f(t) \Big) = \frac{\partial^{\alpha+\beta}}{\partial\, t^{\alpha+\beta}} f(t)$ provided $f^{(i)} = 0,\ i = 0, 1, \ldots, n-1,\ \ \alpha + \beta \leq n,\ \ n \in N$

Definition 2.3. *[10] A two parameters Mittag-Leffler function is defined as:*

$$E_{\alpha,\beta}(x) = \sum_{k=0}^{\infty} \frac{x^k}{\Gamma(\alpha k + \beta)} \qquad \alpha,\ \beta\ \in \mathbb{C},\ \ R(\alpha),\ R(\beta) > 0$$

Remark 2.4. *A Mittag-Leffler function has an interesting properties [10]:*

- $E_{\alpha,\,1}(x) = E_\alpha(x)$
- $E_{1,\,1}(x) = e^x$
- $E_{2,1}(x^2) = \cosh x$
- $x\ E_{2,2}(x^2) = \sinh x$
- $E_{1,0}(x) = xe^x$
- $E_{2,1}(-x^2) = \cos x$
- $x\ E_{2,2}(-x^2) = \sin x$

Remark 2.5. *Fractional Caputo derivatives of Mittag-Leffler are given as [2]:*

- $E^{(n)}_{\alpha,\beta}(x) = \sum_{k=0}^{\infty} \frac{(k+n)!x^k}{k!\Gamma(\alpha k+\alpha n+\beta)} \qquad n\ \in \mathbb{N}$
- $\frac{d^\alpha}{dt^\alpha} \Big(E_\alpha(at^\alpha) \Big) = aE_\alpha(at^\alpha) \qquad \alpha > 0,\ \ a\ \in \mathbb{R}$
- $\frac{d^\gamma}{dt^\gamma} \Big(t^{\beta-1} E_{\alpha,\beta})(at^\alpha) \Big) = t^{\beta-\gamma-1} E_{\alpha,\beta-\gamma}(at^\alpha) \qquad \gamma > 0.$

If the Lapalce transform of the function $f(t)$ exist, then the Laplace transform of the $\alpha - th$ order Caputo derivative is given by [2] :

$$\mathcal{L}(D_t^\alpha f(t)) = s^\alpha \left[F(s) - \sum_{i=0}^{n-1} s^{-n-1} f^{(n)}(0) \right] \qquad n \in \mathbb{N},\ \ n-1 < \alpha < n,\ \ R(s) > 0$$

where $\mathcal{L}(f(t)) = F(s) = \int_0^\infty e^{-st}\ f(t)\ dt.$
Some important Laplace transformation of Mittag-Leffler function which are need are:

1. $\mathcal{L}\{t^{\alpha n+\beta-1}E^{(n)}_{\alpha,\beta}(\pm a t^{\alpha})\} = \frac{n! s^{\alpha-\beta}}{(s^{\alpha}\mp a)^{n+1}} \qquad n \in \mathbb{N}, \quad R(s) > |a|^{1/a}$
2. $\mathcal{L}\left\{t^{\alpha-\gamma-1}\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{(-b^n)\ (-a)^k\ \binom{n+k}{k}}{\Gamma[k(\alpha-\beta)+(n+1)\alpha-\gamma]}\ t^{k(\alpha-\beta)+n\alpha}\right\} = \frac{s^{\gamma}}{s^{\alpha}+a\ s^{\beta}+b}$

We call the finite dimension linear space I_n over $\mathbb{R}$ which spanned by n linearly independent functions $\phi_i(x),\ i = 0, 1, \ldots, n-1$, is invariant with respect to a differential operator $\mathbb{M}$ if $\mathbb{M}[u] \in I_n,\ \forall u \in I_n$

3. The fractional mixed partial differential equations

In this section we have generalized the invariant subspace method which states in [8] by given the cases of improvements to this method through adding new operators under some specific assumptions, such as the subspace has a base of Mittag-Leffler functions are finite members.

Case one

$$\sum_{j=0}^{m_1}\lambda_j\frac{\partial^{\alpha+j}}{\partial t^{\alpha+j}}u = N[u] + \mu_1\frac{\partial^{\alpha}}{\partial t^{\alpha}}\Big(\frac{\partial^{\beta}}{\partial x^{\beta}}u\Big) + \mu_2\frac{\partial^{\beta}}{\partial x^{\beta}}\Big(\frac{\partial^{\alpha}}{\partial t^{\alpha}}u\Big) \tag{3.1}$$

where

$$u = u(x,t), \quad N[u] = N\Big(x, u, \frac{\partial^{\beta}}{\partial x^{\beta}}u, \frac{\partial^{1+\beta}}{\partial x^{1+\beta}}u, \ldots, \frac{\partial^{m_2+\beta}}{\partial x^{m_2+\beta}}u\Big)$$
$$a-1 < \alpha < a, \quad b-1 < \beta < b, \quad a, b \in \mathbb{N}, \quad \lambda_j,\ \mu_1,\ \mu_2 \in \mathbb{R}$$

Theorem 3.1. *Suppose $I_{n+1} = L\{\phi_0(x), \phi_1(x), \ldots, \phi_n(x)\}$ is a finite- dimensional linear space, and it is invariant with respect to the operators $N[u]$, $\frac{\partial^{\beta}}{\partial x^{\beta}}u$ and $\frac{\partial^{\alpha}}{\partial t^{\alpha}}u$ then $FPDE$ (3.1) has an exact solution as follows:*

$$u(x,t) = \sum_{i=0}^{n}k_i(t)\phi_i(x) \tag{3.2}$$

where $\{k_i(t)\}$ satisfies the following $FDEs$:

$$\sum_{j=0}^{m_1}\lambda_j\frac{d^{\alpha+j}}{dt^{\alpha+j}}k_i(t) - \mu_1\frac{d^{\alpha}\psi_{n+1+i}}{dt^{\alpha}} - \mu_2\ \ \psi_{2n+2+i} = \psi_i, \qquad i = 0, \ldots, n \tag{3.3}$$

where $\{\psi_0, \psi_1, \ldots, \psi_n\}$, $\{\psi_{n+1}, \psi_{n+2}, \ldots, \psi_{2n+1}\}$, $\{\psi_{2n+2}, \psi_{2n+3}, \ldots, \psi_{3n+2}\}$ are the expansion coefficients of $N[u]$, $\frac{\partial^{\beta}}{\partial x^{\beta}}u$, $\frac{\partial^{\alpha}}{\partial t^{\alpha}}u$ respectively with respect to $\{\phi_0(x), \phi_1(x), \ldots, \phi_n(x)\}$.

Proof . *By the linearity of Caputo fractional derivative, equation (3.1) is as follows:*

$$\begin{aligned}\sum_{j=0}^{m_1}\lambda_j\frac{\partial^{\alpha+j}}{\partial t^{\alpha+j}}u &= \sum_{j=0}^{m_1}\lambda_j\frac{\partial^{\alpha+j}}{\partial t^{\alpha+j}}\sum_{i=0}^{n}k_i(t)\phi_i(x) = \sum_{j=0}^{m_1}\lambda_j\sum_{i=0}^{n}\frac{d^{\alpha+j}k_i(t)}{dt^{d+j}}\phi_i(x)\\ &= \sum_{i=0}^{n}\sum_{j=0}^{m_1}\lambda_j\frac{d^{\alpha+j}k_i(t)}{dt^{d+j}}\phi_i(x)\end{aligned} \tag{3.4}$$

As I_{n+1} is an invariant space under the operators $N[u]$, $\frac{\partial^\beta}{\partial x^\beta}u$, and $\frac{\partial^\alpha}{\partial t^\alpha}u$ there exist $3n+3$ functions $\psi_0,\psi_1,\dots,\psi_n$; $\psi_{n+1},\psi_{n+2},\dots,\psi_{2n+1}$; $\psi_{2n+2},\psi_{2n+3},\dots,\psi_{3n+2}$ all of $k_0(t),k_1(t),\dots,k_n(t)$ such that

$$N\Big(\sum_{i=0}^{n} k_i(t)\phi_i(x)\Big)=\sum_{i=0}^{n}\psi_i\ \ \phi_i(x) \tag{3.5}$$

$$\frac{\partial^\beta}{\partial x^\beta}u(x,t)=\sum_{i=0}^{n}\psi_{n+1+i}\ \ \phi_i(x) \tag{3.6}$$

$$\frac{\partial^\alpha}{\partial t^\alpha}u(x,t)=\sum_{i=0}^{n}\psi_{2n+2+i}\ \ \phi_i(x) \tag{3.7}$$

where $\{\psi_0,\psi_1,\dots,\psi_n\},\{\psi_{n+1},\psi_{n+2},\dots,\psi_{2n+1}\},\{\psi_{2n+2},\psi_{2n+3},\dots,\psi_{3n+2}\}$ are the expansion coefficients of the operators $N[u]$, $\frac{\partial^\beta}{\partial x^\beta}u$ and $\frac{\partial^\alpha}{\partial t^\alpha}u$ respectively with respect to $\{\phi_0(x),\phi_1(x),\dots,\phi_n(x)\}$. By equations (3.2), (3.5) , (3.6), and (3.7)

$$\begin{aligned}
&N[u]+\mu_1\frac{\partial^\alpha}{\partial t^\alpha}\Big(\frac{\partial^\beta}{\partial x^\beta}u\Big)+\mu_2\frac{\partial^\beta}{\partial x^\beta}\Big(\frac{\partial^\alpha}{\partial t^\alpha}u\Big)\\
&=\sum_{i=0}^{n}\psi_i\ \ \phi_i(x)+\mu_1\frac{\partial^\alpha}{\partial t^\alpha}\Big(\sum_{i=0}^{n}\psi_{n+1+i}\ \ \phi_i(x)\Big)+\mu_2\frac{\partial^\beta}{\partial x^\beta}\Big(\sum_{i=0}^{n}\psi_{2n+2+i}\ \ \phi_i(x)\Big)\\
&=\sum_{i=0}^{n}\psi_i\ \ \phi_i(x)+\mu_1\Big(\sum_{i=0}^{n}\frac{d^\alpha}{dt^\alpha}\psi_{n+1+i}\ \ \phi_i(x)\Big)+\mu_2\Big(\sum_{i=0}^{n}\psi_{2n+2+i}\ \ \frac{d^\beta}{dx^\beta}\phi_i(x)\Big)\\
&=\Big(\sum_{i=0}^{n}\psi_i\ \ +\mu_1\sum_{i=0}^{n}\frac{d^\alpha}{dt^\alpha}\psi_{n+1+i}\ \ +\mu_2\sum_{i=0}^{n}\psi_{2n+2+i}\ \Big)\phi_i(x)
\end{aligned}$$

By our assumption and properties of Mittag-Leffler function, we get $\frac{d^\beta}{dx^\beta}\phi_i(x)=\phi_i(x)$. So, equation (3.1) reads

$$\begin{aligned}
\sum_{i=0}^{n}\left[\sum_{j=0}^{m_1}\lambda_j\frac{d^{\alpha+j}k_i(t)}{dt^{d+j}}\right]\phi_i(x)\ \ &=\ \ \Big(\sum_{i=0}^{n}\psi_i\ \ +\mu_1\sum_{i=0}^{n}\frac{d^\alpha}{dt^\alpha}\psi_{n+1+i}\ \ +\mu_2\sum_{i=0}^{n}\psi_{2n+2+i}\ \ \Big)\phi_i(x)\\
\sum_{i=0}^{n}\Big(\sum_{j=0}^{m_1}\lambda_j\frac{d^{\alpha+j}k_i(t)}{dt^{d+j}}\ \ &-\ \ \psi_i\ \ -\mu_1\frac{d^\alpha}{dt^\alpha}\psi_{n+1+i}\ \ -\mu_2\psi_{2n+2+i}\Big)\phi_i(x)=0
\end{aligned}$$

Since $\phi_i(x)$ are linearly independent, we obtain the following $FDEs$:

$$\sum_{j=0}^{m_1}\lambda_j\frac{d^{\alpha+j}k_i(t)}{dt^{d+j}}=\mu_1\frac{d^\alpha}{dt^\alpha}\psi_{n+1+i}\ \ +\mu_2\psi_{2n+2+i}+\psi_i$$

□

Example 3.2. *Consider the following nonlinear fractional mixed partial differential equation:*

$$\frac{\partial^\alpha}{\partial t^\alpha}\ u=2\Big(u\ \ \frac{\partial^\beta}{\partial x^\beta}u\ -\ u^2\Big)+\frac{\partial^\alpha}{\partial t^\alpha}\Big(\frac{\partial^\beta}{\partial x^\beta}u\Big)+\frac{\partial^\beta}{\partial x^\beta}\Big(\frac{\partial^\alpha}{\partial t^\alpha}u\Big)\qquad 0<\alpha<1,\ \ 1<\beta\le 2. \tag{3.8}$$

Let $I_2 = \{1,\ E_\beta(x^\beta)\}$ is invariant subspace under the operators $N[u] = 2\Big(u\frac{\partial^\beta}{\partial x^\beta}u - u^2\Big), f[u] = \frac{\partial^\beta}{\partial x^\beta}u$ and $g[u] = \frac{\partial^\alpha}{\partial t^\alpha}u$ as, f $u \in I_2$, then

$$\begin{aligned} N[u] &= N[a + bE_\beta(x^\beta)] = 2\Big(a + bE_\beta(x^\beta)\Big)\Big(bE_\beta(x^\beta)\Big) - 2\Big(a + bE_\beta(x^\beta)\Big)^2 \\ &= -2a^2 - 2abE_\beta(x^\beta) \quad \in I_2 \\ f[u] &= b\ E_\beta(x^\beta) \quad \in I_2 \\ g[u] &= 0 \quad \in I_2. \end{aligned}$$

Then according to Theorem (3.1) and equation (3.3), we obtain the following solved $FDEs$:

$$\begin{aligned} &\frac{d^\alpha}{dt^\alpha}k_0(t) = -2k_0^2(t) + (0) + \frac{d^\alpha}{dt^\alpha}k_0(t) \Longrightarrow k_0(t) = 0 \\ &\frac{d^\alpha}{dt^\alpha}k_1(t) + 2k_0(t)k_1(t) = \frac{d^\alpha}{dt^\alpha}k_1(t) + \frac{d^\alpha}{dt^\alpha}k_1(t) \quad \Longrightarrow \quad \frac{d^\alpha}{dt^\alpha}k_1(t) = 0 \quad \Longrightarrow k_1(t) = a. \end{aligned}$$

Hence, (3.8) has the exact solution

$$u(x,t) = k_0(t) + k_1(t)E_\beta(x^\beta) = a\ E_\beta(x^\beta)$$

It is obovious that, when $\alpha = 1, \beta = 2$, the standard equation (3.8) $\dot{u} = 2u(u'' - u) + (\dot{u''}) + (\dot{u}\ )''$ has the exact solution $u(x,t) = a\cosh x$.

Case two

$$\sum_{j=0}^{m_1} \frac{\partial^{\alpha+j}}{\partial t^{\alpha+j}}u = N[u] + \Big(\mu_1\frac{\partial^\alpha}{\partial t^\alpha} + \mu_2\frac{\partial^\beta}{\partial t^\beta}\Big)\Big(\frac{\partial^\beta}{\partial x^\beta}u\Big) \tag{3.9}$$

$$\begin{gathered} where \ \ u = u(x,t), \ \ N[u] = N\Big(x, u, \frac{\partial^\beta}{\partial x^\beta}u, \frac{\partial^{1+\beta}}{\partial x^{1+\beta}}u, \ldots, \frac{\partial^{m_2+\beta}}{\partial x^{m_2+\beta}}u\Big) \\ a-1 < \alpha < a, \ \ b-1 < \beta < b, \ \ a,b \in \mathbb{N}, \ \ \lambda_j,\ \mu_1,\ \mu_2 \in \mathbb{R} \end{gathered}$$

Theorem 3.3. *Suppose $I_{n+1} = L\{\phi_0(x), \phi_1(x), \ldots, \phi_n(x)\}$ is a finite- dimensional linear space, and it is invariant with respect to the operators $N[u]$, $\frac{\partial^\beta}{\partial x^\beta}u$ then $FPDE$ (3.9) has an exact solution as follows:*

$$u(x,t) = \sum_{i=0}^{n} k_i(t)\phi_i(x) \tag{3.10}$$

where $\{k_i(t)\}$ satisfies the following $FDEs$:

$$\sum_{j=0}^{m_1} \lambda_j \frac{d^{\alpha+j}}{dt^{\alpha+j}}k_i(t) - \mu_1\frac{d^\alpha\psi_{n+1+i}}{dt^\alpha} - \mu_2\frac{d^\beta\psi_{n+1+i}}{dt^\beta} = \psi_i, \qquad i = 0, \ldots, n \tag{3.11}$$

where $\{\psi_0, \psi_1, \ldots, \psi_n\}$, and $\{\psi_{n+1}, \psi_{n+2}, \ldots, \psi_{2n+1}\}$, are the expansion coefficients of $N[u]$, $\frac{\partial^\beta}{\partial x^\beta}u$, respectively with respect to $\{\phi_0(x), \phi_1(x), \ldots, \phi_n(x)\}$.

Proof . *By the linearity of Caputo fractional derivative, the left hand side of equation (3.1) is as follows:*

$$\sum_{j=0}^{m_1}\lambda_j\frac{\partial^{\alpha+j}}{\partial t^{\alpha+j}}u=\sum_{j=0}^{m_1}\lambda_j\frac{\partial^{\alpha+j}}{\partial t^{\alpha+j}}\sum_{i=0}^{n}k_i(t)\phi_i(x)=\sum_{j=0}^{m_1}\lambda_j\sum_{i=0}^{n}\frac{d^{\alpha+j}k_i(t)}{dt^{d+j}}\phi_i(x)=\sum_{i=0}^{n}\sum_{j=0}^{m_1}\lambda_j\frac{d^{\alpha+j}k_i(t)}{dt^{d+j}}\phi_i(x) \tag{3.12}$$

As I_{n+1} is an invariant space under the operators $N[u]$, $\frac{\partial^\beta}{\partial x^\beta}u$, there exist $2n+2$ functions $\psi_0,\psi_1,\dots,\psi_n;\ \ \psi_{n+1},\psi_{n+2},\dots,\psi_{2n+1}$ all of $k_0(t),k_1(t),\dots,k_n(t)$ such that

$$N[u] = N\Big(\sum_{i=0}^{n}k_i(t)\phi_i(x)\Big)=\sum_{i=0}^{n}\psi_i\ \ \phi_i(x) \tag{3.13}$$

$$\frac{\partial^\beta}{\partial x^\beta}u(x,t) = \sum_{i=0}^{n}\psi_{n+1+i}\ \ \phi_i(x) \tag{3.14}$$

where $\{\psi_0,\psi_1,\dots,\psi_n\}$ and $\{\psi_{n+1},\psi_{n+2},\dots,\psi_{2n+1}\}$, are the expansion coefficients of $N[u]$, and $\frac{\partial^\beta}{\partial x^\beta}u$ respectively with respect to $\{\phi_0(x),\phi_1(x),\dots,\phi_n(x)\}$.
Substitute equations (3.12), (3.13) and (3.14) in (3.9)

$$\begin{aligned}
&N[u]+\Big(\mu_1\frac{\partial^\alpha}{\partial t^\alpha}+\mu_2\frac{\partial^\beta}{\partial t^\beta}\Big)\Big(\frac{\partial^\beta}{\partial x^\beta}u\Big)\\
&=\sum_{i=0}^{n}\psi_i\ \ \phi_i(x)+\mu_1\frac{\partial^\alpha}{\partial t^\alpha}\Big(\sum_{i=0}^{n}\psi_{n+1+i}\ \ \phi_i(x)\Big)+\mu_2\frac{\partial^\beta}{\partial t^\beta}\Big(\sum_{i=0}^{n}\psi_{n+1+i}\ \ \phi_i(x)\Big)\\
&=\sum_{i=0}^{n}\psi_i\ \ \phi_i(x)+\mu_1\Big(\sum_{i=0}^{n}\frac{d^\alpha}{dt^\alpha}\psi_{n+1+i}\ \ \phi_i(x)\Big)+\mu_2\Big(\sum_{i=0}^{n}\frac{d^\beta}{dt^\beta}\psi_{n+1+i}\ \ \phi_i(x)\Big)\\
&=\Big(\sum_{i=0}^{n}\psi_i\ \ +\mu_1\sum_{i=0}^{n}\frac{d^\alpha}{dt^\alpha}\psi_{n+1+i}\ \ +\mu_2\sum_{i=0}^{n}\frac{d^\beta}{dt^\beta}\psi_{n+1+i}\Big)\phi_i(x)
\end{aligned}$$

So, equation (3.1) reads

$$\begin{aligned}
&\sum_{i=0}^{n}\left[\sum_{j=0}^{m_1}\lambda_j\frac{d^{\alpha+j}k_i(t)}{dt^{d+j}}\right]\phi_i(x)=\Big(\sum_{i=0}^{n}\psi_i\ \ +\mu_1\sum_{i=0}^{n}\frac{d^\alpha}{dt^\alpha}\psi_{n+1+i}\ \ +\mu_2\sum_{i=0}^{n}\frac{d^\beta}{dt^\beta}\psi_{n+1+i}\Big)\phi_i(x)\\
&\sum_{i=0}^{n}\Big(\sum_{j=0}^{m_1}\lambda_j\frac{d^{\alpha+j}k_i(t)}{dt^{d+j}}-\psi_i\ \ -\mu_1\frac{d^\alpha}{dt^\alpha}\psi_{n+1+i}\ \ -\mu_2\frac{d^\beta}{dt^\beta}\psi_{n+1+i}\Big)\phi_i(x)=0
\end{aligned}$$

Since $\phi_i(x)$ are linearly independent, we obtain the following $FDEs$:

$$\sum_{j=0}^{m_1}\lambda_j\frac{d^{\alpha+j}k_i(t)}{dt^{\alpha+j}}\ \ -\mu_1\frac{d^\alpha}{dt^\alpha}\psi_{n+1+i}\ \ -\mu_2\frac{d^\beta}{dt^\beta}\psi_{n+1+i}=\psi_i$$

□

Example 3.4. *Consider the following fractional partial mixed derivatives:*

$$\frac{\partial^\alpha}{\partial t^\alpha}u=\frac{\partial^\beta}{\partial x^\beta}\Big(\frac{\partial^\beta}{\partial x^\beta}u\Big)-\Big(\frac{\partial^\alpha}{\partial t^\alpha}+\frac{\partial^\beta}{\partial t^\beta}\Big)\Big(\frac{\partial^\beta}{\partial x^\beta}u\Big)\qquad 0<\alpha\le 1,\ \ 1<\beta\le 2 \tag{3.15}$$

Consider $I_2 = \{1, E_\beta(x^\beta)\}$ is invariant subspace under the operators $N[u]$, $\frac{\partial^\beta}{\partial x^\beta}u$ as:

$$N[u] = \frac{\partial^\beta}{\partial x^\beta}\Big(\frac{\partial^\beta}{\partial x^\beta}u\Big) = \frac{\partial^\beta}{\partial x^\beta}\left[\frac{\partial^\beta}{\partial x^\beta}\Big(c_0 + c_1E_\beta(x^\beta)\Big)\right] = \frac{\partial^\beta}{\partial x^\beta}\Big(c_1E_\beta(x^\beta)\Big) = c_1E_\beta(x^\beta) \qquad \in I_2$$
$$\frac{\partial^\beta}{\partial x^\beta}u = c_1E_\beta(x^\beta) \qquad \in I_2$$

So, according Theorem (3.3), we have the following $FDEs$:

$$\frac{d^\alpha}{dt^\alpha}k_0(t) = 0 \tag{3.16a}$$
$$\frac{d^\alpha}{dt^\alpha}k_1(t) = -\frac{d^\alpha}{dt^\alpha}k_1(t) - \frac{d^\beta}{dt^\beta}k_1(t) + k_1(t) \tag{3.16b}$$

Equation (3.16a) yields $k_0(t) = a$, and substitute in equation (3.16b)

$$\begin{aligned}
\frac{d^\beta}{dt^\beta}k_1(t) + 2\frac{d^\alpha}{dt^\alpha}k_1(t) &= k_1(t) \qquad \text{By fractional Laplace transformation}\\
s^\beta\left[F(s) - \frac{b}{s} - \frac{c}{s^2}\right] + 2s^\alpha\left[F(s) - \frac{b}{s}\right] &= F(s) \qquad \text{where } k_1(0) = b,\ \acute{k}_1(0) = c\\
\Big(s^\beta + 2s^\alpha - 1\Big)F(s) &= bs^{\beta-1} + cs^{\beta-2} + 2bs^{\alpha-1}\\
F(s) &= \frac{bs^{\beta-1} + cs^{\beta-2} + 2bs^{\alpha-1}}{s^\beta + 2s^\alpha - 1}
\end{aligned}$$

Applying the inverse Laplace transformation and by properies of Laplace transform for the Mittag-Leffler function which states in section(2), we have

$$k_1(t) = A + B + C,$$

where

$$\begin{aligned}
A &= b\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{(-2)^k\binom{n+k}{k}}{\Gamma\left[k(\beta-\alpha) + n\beta + 1\right]}t^{n\beta+k(\beta-\alpha)}\\
B &= c\,t\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{(-2)^k\binom{n+k}{k}}{\Gamma\left[k(\beta-\alpha) + n\beta + 2\right]}t^{n\beta+k(\beta-\alpha)}\\
C &= 2b\,t^{\beta-\alpha}\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{(-2)^k\binom{n+k}{k}}{\Gamma\left[k(\beta-\alpha) + (n+1)\beta + 1 - \alpha\right]}t^{n\beta+k(\beta-\alpha)}
\end{aligned}$$

Thus, the exact solution of equation (3.15) is

$$u(x,t) = k_0(t) + k_1(t)E_\beta(x^\beta) = \left[a + \Big(A + B + C\Big)\right]E_\beta(x^\beta)$$

However, if we put $\alpha = 1$, and $\beta = 2$, then we have

$$u(x,t) = \left[a + b\,e^{-t}\cosh(\sqrt{2}\,t) + \frac{b+c}{\sqrt{2}}\,e^{-t}\sinh(\sqrt{2}\,t)\right]\cosh x$$

Case three

In the next theorem, we argue a new form of the invariant subspace method where the coefficients in the left hand side of Eq. (1.1) is a function.

$$\sum_{j=0}^{n} \lambda_j(t) \frac{\partial^{\alpha+j}}{\partial\, t^{\alpha+j}} u(x,t) = N\Big(x, u, \frac{\partial^{\beta}}{\partial x^{\beta}}, \frac{\partial^{\beta+1}}{\partial x^{\beta+1}}, \ldots, \frac{\partial^{\beta+m}}{\partial x^{\beta+m}}\Big) + \mu \frac{\partial^{\alpha}}{\partial t^{\alpha}}\Big(\frac{\partial^{\beta}}{\partial x^{\beta}} u\Big) \tag{3.17}$$

Theorem 3.5. *Suppose $I_{n+1} = L\{\phi_0(x), \phi_1(x), \ldots, \phi_n(x)\}$ is a finite- dimensional linear space, and it is invariant with respect to the operators $N[u]$ and $\frac{\partial^{\beta}}{\partial x^{\beta}} u$ then FPDE (3.17) has an exact solution as follows:*

$$u(x,t) = \sum_{i=0}^{n} k_i(t) \phi_i(x) \tag{3.18}$$

Where $\{k_i(t)\}$ satisfies the following system of FDEs with variable coefficients:

$$\sum_{j=0}^{m_1} \lambda_j(t) \frac{d^{\alpha+j}}{dt^{\alpha+j}} k_i(t) - \mu \frac{d^{\alpha}\psi_{n+1+i}}{dt^{\alpha}} = \psi_i, \qquad i = 0, \ldots, n \tag{3.19}$$

where $\{\psi_0, \psi_1, \ldots, \psi_n\}$, $\{\psi_{n+1}, \psi_{n+2}, \ldots, \psi_{2n+1}\}$ are the expansion coefficients of $N[u]$, $\frac{\partial^{\beta}}{\partial x^{\beta}} u$ respectively with respect to $\{\phi_0(x), \phi_1(x), \ldots, \phi_n(x)\}$

Proof . *By the linearity of Caputo fractional derivative, equation (3.17) is as follows:*

$$\begin{aligned}\sum_{j=0}^{m_1} \lambda_j(t) \frac{\partial^{\alpha+j}}{\partial t^{\alpha+j}} u &= \sum_{j=0}^{m_1} \lambda_j(t) \frac{\partial^{\alpha+j}}{\partial t^{\alpha+j}} \sum_{i=0}^{n} k_i(t)\phi_i(x) = \sum_{j=0}^{m_1} \lambda_j(t) \sum_{i=0}^{n} \frac{d^{\alpha+j} k_i(t)}{dt^{d+j}} \phi_i(x) \\ &= \sum_{i=0}^{n} \sum_{j=0}^{m_1} \lambda_j(t) \frac{d^{\alpha+j} k_i(t)}{dt^{d+j}} \phi_i(x)\end{aligned} \tag{3.20}$$

As I_{n+1} is an invariant space under the operators $N[u]$, $\frac{\partial^{\beta}}{\partial x^{\beta}} u$ there exist $2n+2$ functions $\psi_0, \psi_1, \ldots, \psi_n$; $\psi_{n+1}, \psi_{n+2}, \ldots, \psi_{2n+1}$ all of $k_0(t), k_1(t), \ldots, k_n(t)$ such that

$$N\Big(\sum_{i=0}^{n} k_i(t)\phi_i(x)\Big) = \sum_{i=0}^{n} \psi_i \;\; \phi_i(x) \tag{3.21}$$

$$\frac{\partial^{\beta}}{\partial x^{\beta}} u(x,t) = \sum_{i=0}^{n} \psi_{n+1+i} \;\; \phi_i(x) \tag{3.22}$$

where $\{\psi_0, \psi_1, \ldots, \psi_n\}$, $\{\psi_{n+1}, \psi_{n+2}, \ldots, \psi_{2n+1}\}$ are the expansion coefficients of $N[u]$, $\frac{\partial^{\beta}}{\partial x^{\beta}} u$ respectively with respect to $\{\phi_0(x), \phi_1(x), \ldots, \phi_n(x)\}$. Thus

$$\begin{aligned}N[u] + \mu \frac{\partial^{\alpha}}{\partial t^{\alpha}}\Big(\frac{\partial^{\beta}}{\partial x^{\beta}} u\Big) &= \sum_{i=0}^{n} \psi_i \;\; \phi_i(x) + \mu \frac{\partial^{\alpha}}{\partial t^{\alpha}}\Big(\sum_{i=0}^{n} \psi_{n+1+i} \;\; \phi_i(x)\Big) \\ &= \sum_{i=0}^{n} \psi_i \;\; \phi_i(x) + \mu \sum_{i=0}^{n} \frac{d^{\alpha}}{dt^{\alpha}} \psi_{n+1+i} \;\; \phi_i(x) \\ &= \Big(\sum_{i=0}^{n} \psi_i \;\; + \mu \sum_{i=0}^{n} \frac{d^{\alpha}}{dt^{\alpha}} \psi_{n+1+i}\Big) \phi_i(x)\end{aligned}$$

So, (3.17) reads

$$\sum_{i=0}^{n}\left[\sum_{j=0}^{m_1}\lambda_j(t)\frac{d^{\alpha+j}k_i(t)}{dt^{d+j}}\right]\phi_i(x)=\Big(\sum_{i=0}^{n}\psi_i\ +\mu\sum_{i=0}^{n}\frac{d^{\alpha}}{dt^{\alpha}}\psi_{n+1+i}\Big)\phi_i(x)$$
$$\sum_{i=0}^{n}\Big(\sum_{j=0}^{m_1}\lambda_j(t)\frac{d^{\alpha+j}k_i(t)}{dt^{d+j}}-\psi_i\ -\mu\frac{d^{\alpha}}{dt^{\alpha}}\psi_{n+1+i}\Big)\phi_i(x)=0$$

Since $\phi_i(x)$ are linearly independent, we obtain the following $FDEs$:

$$\sum_{j=0}^{m_1}\lambda_j(t)\frac{d^{\alpha+j}k_i(t)}{dt^{d+j}}\ -\mu\frac{d^{\alpha}}{dt^{\alpha}}\psi_{n+1+i}=\psi_i$$

□

Example 3.6. *Consider the following fractional partial differential equation*

$$t^{\alpha}D_t^{\alpha}u=\Gamma(1+\alpha)\Big(2u-D_x^{\beta}u\Big),\qquad 0<\alpha,\ \ \beta\le 1 \tag{3.23}$$

Let $I_2=\{1,E_\beta(x^\beta)$ be an invariant subspace under the operator $N[u]=\Gamma(1+\alpha)\Big(2u-D_x^{\beta}u\Big)$ as for $u\in I_2,\quad N[u]=\Gamma(1+\alpha)\left[2\Big(c_0+c_1E_\beta(x^\beta)\Big)-c_0E_\beta(x^\beta)\right]=\Gamma(1+\alpha)\Big(2c_0+c_1E_\beta(x^\beta)\Big)\ \in\ I_2$
So, according to Theorem (3.3), we have the following FDE system with variable coefficients:

$$t^{\alpha}\frac{d^{\alpha}}{dt^{\alpha}}c_0(t)=2\Gamma(1+\alpha)c_0(t) \tag{3.24a}$$
$$t^{\alpha}\frac{d^{\alpha}}{dt^{\alpha}}c_1(t)=\Gamma(1+\alpha)c_1(t) \tag{3.24b}$$

If we put $c_1(t)=bt^{\gamma}$ in (3.24b), then

$$t^{\alpha}\frac{d^{\alpha}}{dt^{\alpha}}c_1(t)=\frac{t^{\alpha}b\Gamma(\gamma+1)t^{\gamma-\alpha}}{\Gamma(1+\gamma-\alpha)}=\frac{b\Gamma(\gamma+1)t^{\gamma}}{\Gamma(1+\gamma-\alpha)}=b\Gamma(1+\alpha)t^{\gamma}\Longrightarrow\alpha=\gamma\Longrightarrow c_1(t)=bt^{\alpha}.$$

Also, we get $c_0(t)=0$ in (3.24a).
Consequently, the solution of (3.23) is $\qquad u(x,t)=c_0(t)+c_1(t)E_\beta(x^\beta)=bt^{\alpha}\ E_\beta(x^\beta).$
which easy to check that it is exact.

Conclusion

The invariant subspace method has been known as a powerful tool for solving many space, time, space-time and mixed non-linear fractional differential equations. In this paper, we have presented an extension of this method to solve some of non-homogeneous, variable–coefficients fractional partial differential equations with mixed Caputo derivatives.
All examples solved by using our new techniques gives an exact solutions for such problems.

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