Int. J. Nonlinear Anal. Appl. 12 (2021) No. 2, 1991-1996
ISSN: 2008-6822 (electronic)
http://dx.doi.org/10.22075/ijnaa.2020.20839.2205

C^*-metric spaces

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(Communicated by Madjid Eshaghi Gordji)

Abstract

The purpose of this article is to introducing the notion of an $\mathfrak{A}$-meter, as an operator valued distance mapping on a set X and investigating the theory of $\mathfrak{A}$-metric spaces, where $\mathfrak{A}$ is a noncommutative C^*-algebra. We demonstrate that each metric space may be seen as an $\mathfrak{A}$-metric space and that every $\mathfrak{A}$-metric space (X, δ) can be regarded as a topological space (X, τ_δ).

Keywords: C^*-algebra,C^*-metric space,allowance set,downward/upward direct, positive elements.
2010 MSC: 54E35,54E70,54A40,46A03,30L05.

1. Introduction

Mirzavaziri [7] obtained some generalizations of usual metrics as real- valued mappings, which have been given in the last century.His generalizations are in such away that the values of a metric can be mappings.

Let $\mathfrak{A} = C(\Omega)$ be a commutative unital C^*-algebra, where Ω is a compact Hausdorff topological space. A Hilbert $\mathfrak{A}$-module is a right $\mathfrak{A}$-module ξ equipped with an $\mathfrak{A}$-valued inner product. The reader is referred to [7] for more information on Hilbert C^*-modules.

Mirzavaziri [7], construct the ordered positive cone with family of positive elements of commutative unital C^*-algebra, $\mathcal{C}(\Omega)$,and named it $\mathfrak{A}^+$.Therefore, with the help of the usual definition of metric and Hilbert C^*-modules, he find an attractive idea to define the notion of an $\mathcal{A}$-valued C^*-metric $\mathfrak{A}$ on a set X as a mapping from $X \times X$ into the positive cone $\mathfrak{A}^+$ for a C^*-algebra $\mathfrak{A}$, and his notation is $\mathcal{F}$-metric.He showed that $\mathfrak{A}^+$ has the axiom of completeness and also the Archimedean

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Received: July 2020 *Accepted:* December 2020

property for non-empty bounded subsets of $\mathfrak{A}$,i.e. $\mathfrak{A}^+$ is an ordered complete set. This is very important for defining an induced topology by $\mathcal{F}$-meter.Partial order is necessary $\leq$ for triangle inequality of $\mathcal{F}$. He represented several examples to introduce an induced topology by an $\mathfrak{A}$-meter.This topology is defined by using elements of $\mathfrak{A}^+$ as radius of open neighborhoods and $\mathcal{R}$-extended topology, with allowance sets. He prove that any metric space is $\mathcal{F}$-metric space and the category of all metric spaces is a proper subset of the category of all extended $\mathcal{F}$-metric spaces. He proposed completion of $\mathcal{F}$-metric, and checked $\mathcal{F}$-metrizability of topological spaces. As an application of the concept of $\mathcal{F}$-metrics, he proved that each normal topological space is $\mathcal{F}$-metrizable.

In this article we want to introduce the concept of C^*-metric by using the positive elements of a noncommutative unital C^*-algebra, that is, $\mathfrak{A} = B(\mathcal{H})$, where $\mathcal{H}$ is a Hilbert space. We select projections, because the family of them is a partial ordered, lattice and it is downwards and upwards directed.The downwards directed property assists us to prove that the intersection of a finitely many open neighborhoods, is again an open neighborhood. Now we can construct the open neighborhood at a point $x \in X$ with radius $r \in \mathfrak{A}^+$. Not far from mind, each metric space is a C^*-metric space, for every commutative or noncommutative C^*-algebra $\mathfrak{A}$. We show that $\mathfrak{A}^+$ is ordered complete set, and then develop the theory of C^*-metric spaces. We will propose a few examples,to define topologies by using elements of $\mathfrak{A}^+$ as radius of open neighborhoods and $\mathcal{R}$-extended topology with allowance sets,as some applications.

2. Preliminaries

In all of the following section $\mathfrak{A} = B(\mathcal{H})$ is an unital noncommutative C^*-algebra.An element p in $\mathfrak{A}$ is called positive (denoted by $0 \leq p$ or $p \geq 0$) if $\{< px, x >: x \in \mathcal{H}\}$ is a subset of the nonnegative real numbers $\mathbb{R}^+$. It is strictly positive (denoted by $0 \prec p$ or $p \succ 0$) if $\{< px, x >: x \in \mathcal{H}\}$ is a subset of the positive real numbers $\mathbb{R}^{++}$. Note that $p \succ 0$ is not equivalent to $p > 0$ (i.e., $p \geq 0$ and $p \neq 0$). The set of all positive elements and the set of all strictly positive elements of $\mathfrak{A}$ are denoted by $\mathfrak{A}^+$ and $\mathfrak{A}^{++}$, respectively. Obviously, $p \in \mathfrak{A}^+$ is strictly positive if and only if it is invertible in $\mathfrak{A}$. (see Proposition 3.2.12 of[10]). It can viewed as the C^*-Archimedean property.

Let $P(\mathcal{H}) \subseteq \mathfrak{A}^+$ be the set of all projections $P_\alpha : \mathcal{H} \longrightarrow \mathcal{H}$ with norm $\|.\|_\infty$ defined by:

$$\begin{aligned} \|P_\alpha\|_\infty &= \sup\{\|P_\alpha x\| : x \in \mathcal{H}, \|x\| \leq 1\} \\ &= \sup\{(< P_\alpha x, x >)^{\frac{1}{2}} : x \in \mathcal{H}, \|x\| \leq 1\} \end{aligned}$$

For noncommutative C^*-algebra, it is important to show that $\mathfrak{A}^{++}$ is downwards directed,that is, if $S, T \in \mathfrak{A}^{++}$, then there is $Q \in \mathfrak{A}^{++}$ such that $Q \preceq T$ and $Q \preceq S$.

The set of closed subspaces can be partially ordered by inclusion,and it is complete lattice.Every family $\{Y_\alpha\}$ of closed subspaces in $\mathcal{H}$ possesses an infimum $\wedge\, Y_\alpha$ and a supremum $\vee Y_\alpha$, which are, respectively, the intersection of all Y_α and the closure of the subspace generated by all Y_α.Next theorem,help us to define a partial order $\leq$ in the set of projections. As the next theorem shows, these two orderings coincide, in summary:

$$P \leq Q \Longleftrightarrow Ran(P) \subseteq Ran(Q).$$

Theorem 2.1. *[9]*

Let S and T be in $P(\mathcal{H})$. The following conditions are equivalent:

a)Ran(S)$\subseteq$ Ran (T), that is, $S \leq T$.

b) $TS = S$.
c) $ST = S$.
d) $\|Sx\| \leq \|Tx\|$ *for all* $x \in \mathcal{H}$.
e)$S \leq T$.

There is, of course, a correspondent in terms of projections: every family $\{P_\alpha\}_{\alpha\in I}$ of projections has an infimum $\wedge\, P_\alpha$ and a supremum $\vee P_\alpha$, which are, respectively, the projection onto the intersection of all $Ran(P_\alpha)$ and the projection onto the closure of the subspace generated by all $\mathrm{Ran}(P_\alpha)$.

The above discussion clarifies that the set of projections in $B(\mathcal{H})$ has a lattice structure. In fact, the set of projections forms a complete lattice and it is downwards directed and upwards directed.

The sum of two strictly positive elements of $\mathfrak{A}^+$ is again strictly positive and if $T \in \mathfrak{A}^{++}$, then $\lambda T \in \mathfrak{A}^{++}$ for all $\lambda \in \mathfrak{A}^{++}$ [7]. This shows that $\mathfrak{A}^{++}$ is a cone. Let $T, S \in \mathfrak{A}^+$. The notation $T \lhd S$ used for $T(x) < S(x)$ for all $x \in \mathcal{H}$ with $S(x) \neq 0$. The relation $\lhd$ is transitive on $\mathfrak{A}^+$, and $T \lhd S$ implies $T + Q \lhd S + Q$ for all real valued mapping Q with $T + Q \in \mathfrak{A}^+$. Note that if $S \in \mathfrak{A}^{++}$, then $T \lhd S$ is equivalent to $T \prec S$.

Proposition 2.2. *[7] A positive element* p *of* $\mathfrak{A}$ *is invertible if and only if* $p \geq \lambda\iota$ *for some* $\lambda \in \mathbb{R}^{++}$.

Proposition 2.3. *[7] A pointwise infimum of any number of elements in* $P(\mathcal{H})$ *and a supremum of finitely many elements will again define an element in* $P(\mathcal{H})$*. Furthermore,* $P(\mathcal{H})$ *is stable under the addition and under the multiplication with positive real numbers. Finally,* $P(\mathcal{H})$ *is closed under uniform convergence.*

Let $\mathfrak{A}^+ = P(\mathcal{H})^+$, be the set of all positive projection operators on $\mathcal{H}$, and let $\mathfrak{A}^{++} = P(\mathcal{H})^{++}$ be the set of all strictly positive projection operators on $\mathcal{H}$. Clearly every operator is continuous with norm topology, will be continuous with weak operator topology.

Theorem 2.4. *[7] Let* $\mathcal{B}$ *be a nonempty subset of* $\mathfrak{A}^+$*. Then* $\inf \mathcal{B}$ *exists in* $\mathfrak{A}^+$*. In other words, there is* $T_0 \in \mathfrak{A}^+$ *such that* $T_0 \leq T$ *for each* $T \in \mathcal{B}$*, and if* S *is any lower bounded for* $\mathcal{B}$*, then* $S \leq T_0$.

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Proof . For each $T \in \mathcal{B}$ and $h \in \mathcal{H}$, we have $Th \geq 0$. Thus the set $\{Th : T \in \mathcal{B}\}$ is a nonempty bounded below subset of $\mathbb{R}^+$ and so its infimum exists. Let

$$T_0h = \inf\{Th : T \in \mathcal{B}\}.$$

Then $T_0 \in \mathcal{B}$. Now let S be a lower bound for $\mathcal{B}$. Hence $Sh \leq Th$ and so Sh is a lower bound for the set $\{Th : T \in \mathcal{B}\}$. Thus $Sh \leq T_0h$ or equivalently $S \leq T_0$. □

The set of positive elements of $\mathfrak{A}^+$ and $\mathfrak{A}^{++}$ are ordered, Archimedean, bounded below, downwards and upwards direct, and we use $\mathfrak{A}$ as the notation of noncommutative C^*-algebras.

3. C^*-Metric

Similarity between the cone of positive elements of $\mathfrak{A}$ and $\mathbb{R}^{++}$ and the notion of positive elements drives thought into introduce the following notion.

Definition 3.1. *Let X be a set. A mapping $\delta : X\times X \longrightarrow \mathfrak{A}^+$is called an $\mathfrak{A}$-metric, or an $\mathfrak{A}$-metric, if for all x,y,z $\in X$ the following conditions hold:*

(i) $\delta(x,y)= 0$ if and only if $x=y$;

(ii) $\delta(x,y)=\delta(y,x)$;

(iii) $\delta(x,y)\trianglelefteq \delta(x,z)+\delta(z,y)$ (triangle inequality).

In this case, (X,δ) is called a C^*-metric space, or an $\mathfrak{A}$-metric space.

Example 3.2. *Let T be a nonzero positive element of $\mathfrak{A}$. Then*

$$\delta(x,y) = \begin{cases} T & \text{if } x \neq y, \\ 0 & \text{otherwise}, \end{cases}$$

gives an $\mathfrak{A}$-metric δ on X, which is called the discrete $\mathfrak{A}$-metric on X constructed via T.

Example 3.3. *Let $X := B(\mathcal{H})$ and let $\delta(T,S) = | < (T-S)x, y > |$ define an $\mathfrak{A}$-metric on X. Then*

1- $\delta(T,T) = | < (T-T)x, y > | \trianglerighteq 0$,

2- $\delta(T,S) = | < (T-S)x, y > | = | < (S-T)x, y > | = \delta(S,T)$.

For the triangle inequality, we have

3-

$$\begin{aligned}\delta(T,S) &= | < (T-S)x, y > | = | < (T-S\pm Q)x, y > | \\ &\trianglelefteq | < (T-Q)x, y > | + | < (Q-S)x, y > | \\ &= \delta(T,Q) + \delta(Q,S)\end{aligned}$$

for all $x, y \in \mathcal{H}$ and $T, S, Q \in X$. Thus $\delta(T,S) \trianglelefteq \delta(T,Q) + \delta(Q,S)$.

4. TOPOLOGY

Let (X,δ) be an $\mathfrak{A}$-metric space. We can define the ball $N_r^\delta(x)$ centered at $x \in X$ with radius $r \in \mathfrak{A}^{++}$ by $N_r^\delta(x) = \{y \in X : \delta(x,y) \lhd r\}$. Open sets and interior points of a subset of X are defined in the usual manner[3]and [4]. Note that $N_r^\delta(x)$ is an open set. If $y \in N_r^\delta(x)$, then $\delta(x,y) \lhd r$ and for the strictly positive element $r_0 = r - \delta(x,y)$, we have $N_{r_0}^\delta(x) \subseteq N_r^\delta(x)$. This shows that y is an interior point of $N_r^\delta(x)$.

The topology mentioned in the above theorem is called the topology on X induced by the $\mathfrak{A}$-metric δ and is denoted by τ_δ.

We can consider a subset of $\mathfrak{A} \setminus \{0\}$ as a set of radiuses of our open balls as follows.

Definition 4.1. *Let (X,δ) be an $\mathfrak{A}$-metric space. Then $\mathcal{R}$, a nonempty subset of $\mathfrak{A}^+ \setminus \{0\}$, is called "allowance" with respect to δ if (a) it is downward directed, (b)$\lambda\mathcal{R} \subseteq \mathcal{R}$ for each $\lambda \in \mathbb{R}^{++}$,and (c)$\delta(x,y) \lhd r$ for some $r \in \mathcal{R}$ implies the existence of an element $r_0\mathcal{R}$ and $\lambda \in \mathcal{R}^{++}$ such that $r_0 + \delta(x,y) \lhd r \lhd \lambda r_0$. The "$\mathcal{R}$-extended topology" on X induced by δ, denoted by $\tau_\delta^\mathcal{R}$, is defined to be the topology on X generated by the topological base $\{N_r^\delta(x)\}_{r\in\mathcal{R},x\in X}$.*

Third condition is required to show that open balls are indeed open. We can view a metric space as an $\mathfrak{A}$-metric space,in the following theorem. We will show the balls in (X,d) by N^d and the balls in (X,δ) by N_r^δ.

The open subsets of an $\mathfrak{A}$-metric space form a topology and this will be shown in the following theorem.

Theorem 4.2. *Let (X, δ) be an $\mathfrak{A}$-metric space. Then the family of all open subsets of X with respect to δ forms a topology on X.*

Proof . We need to show that for an arbitrary family $\{N^{\delta}_{r_\gamma}(x_\gamma) : \gamma \in \Gamma\}$ of open balls, the set $U = \bigcup_{\gamma\in\Gamma} N^{\delta}_{r_\gamma}(x_\gamma)$ is open. If there is $\gamma \in \Gamma$ such that $x \in N^{\delta}_{r_\gamma}(x_\gamma)$. Then there is r_0 such that $N^{\delta}_{r_0}(x) \subseteq N^{\delta}_{r_\gamma}(x_\gamma) \subseteq U$. So x is an interior point of U.

We have to show that $V = \bigcap_{j=1}^{n} N^{\delta}_{r_j}(x_j)$. Let $x \in V$, too. Then $x \in N^{\delta}_{r_j}(x_j)$ and so there are $s_j \in \mathfrak{A}^{++}$ such that $N^{\delta}_{s_j}(x) \subseteq N^{\delta}_{r_j}(x_j)$. Pick an $r_0 \in \mathfrak{A}^{++}$ such that $r_0 \lhd s_j$ for all $1 \le j \le n$. (Note that $\mathfrak{A}^{++}$ is a lattice and downward directed, consequently, and so r_0 exists.) We then have $N^{\delta}_{r_0}(x) \subseteq V$. □

Theorem 4.3. *Suppose that (X, d) is a metric space and that $0 \neq T \in \mathfrak{A}^+$. If*

$$\delta_T : X \times X \longrightarrow \mathfrak{A}^+$$

is defined by $\delta_T(x, y) = d(x, y)T$, then (X, δ_T) is an $\mathfrak{A}$-metric space. Let $\mathcal{R}$ be an allowance set with respect to δ_T.
(i) If $\mathcal{R}$ has the Archimedean property, then $\tau^{\mathcal{R}}_{\delta_T} = \tau_d$.
(ii) If there is r in $\mathcal{R}$ such that $\lambda T \ntriangleleft r$ for any $\lambda \in \mathbb{R}^{++}$, then $\tau^{\mathcal{R}}_{\delta_T}$ is the discrete topology.

Proof . Put $\delta = \delta_T$, to prove the triangle inequality for δ. We recall that the set of a positive element of $\mathfrak{A}$ is a positive cone.
(i) We shall show that the family $\{N^{\delta}_{\lambda T}(x)\}_{\lambda\in\mathbb{R}^{++}, x\in X}$ forms a topological base for $\tau^{\mathcal{R}}_{\delta}$. To see this, let $N^{\delta}_{r}(x)$ be an arbitrary open ball in $\tau^{\mathcal{R}}_{\delta}$ and let $y \in N^{\delta}_{r}(x)$. Since $\mathcal{R}$ is allowance, there is $r_0 \in \mathcal{R}$ such that $r_0 \lhd r - \delta(x, y)$. By the Archimedean property of $r_0 \in \mathcal{R}$, there is $\lambda_0 \in \mathcal{R}^{++}$ such that $\lambda_0 T \lhd r_0$. Now $N^{\delta}_{\lambda_0 T}(y) \subseteq N^{\delta}_{r}(x)$, since $\delta(x, z) \lhd \delta(z, y) + \delta(y, x) \lhd \lambda_0 T + \delta(x, y) \lhd r_0 + \delta(x, y) \lhd r$ for $z \in N^{\delta}_{\lambda_0 T}(y)$.

Let $N^{\delta}_{\lambda T}(x)$ be an arbitrary open ball in $\tau^{\mathcal{R}}_{\delta}$ and let $y \in N^{\delta}_{\lambda T}(x)$. Let $\lambda_0 = \lambda - \delta(x, y)$. Then $\lambda_0 \in \mathbb{R}^{++}$, since $\delta(x, y) \lhd \lambda T$ implies that $d(x, y)T \lhd \lambda T$ and so $\lambda - d(x, y) > 0$. We assert that $N^{d}_{\lambda_0}(y) \subseteq N^{\delta}_{\lambda T}(x)$. To see this, let $z \in N^{d}_{\lambda_0}(y)$. We have $d(z, y) < \lambda_0$ and so

$$\begin{aligned}\delta(z, x) &\lhd \delta(z, y) + \delta(y, x) = d(z, y)T + d(y, x)T \\ &\lhd \lambda_0 T + d(y, x)T \\ &= (\lambda - d(x, y) + d(y, x))T \\ &= \lambda T.\end{aligned}$$

This shows that y is an interior point of $N^{\delta}_{\lambda T}(x)$ with respect to d. Thus $N^{\delta}_{\lambda T}(x) \in \tau_d$. On the other hand, if $N^{d}_{\lambda}(x)$ is an arbitrary basis element of the topology τ_d and $y \in N^{d}_{\lambda}(x)$, then for $\lambda_0 = \lambda - d(x, y)$, we have $N^{\delta}_{\lambda_0 T}(y) \subseteq N^{d}_{\lambda}(x)$ and so $N^{d}_{\lambda}(x)$ is open with respect to $\tau^{\mathcal{R}}_{\delta}$. Thus the topology coincide.
(ii) If $\delta(x, y) \lhd r$, then $d(x, y)T \lhd r$, and so $d(x, y) = 0$, which implies $x = y$. Hence $N^{\delta}_{r}(x) = \{x\}$ for each $x \in X$ and the topology $\tau^{\mathcal{R}}_{\delta}$ is then discrete.□

The following theorem states that if $\mathfrak{A}$ has the Archimedean property, then an $\mathfrak{A}$-extended $\mathfrak{A}$-metric space is nothing but a metric space.

Theorem 4.4. *Let (X, δ) be an $\mathcal{R}$-extended $\mathfrak{A}$-metric space. If we define $d : X \times X \longrightarrow \mathbb{R}^+$ by $d(x, y) = \|\delta(x, y)\|$, then (X, d) is a metric space. Furthermore, if $\mathcal{R}$ has the Archimedean property, then $\tau_d = \tau^{\mathcal{R}}_{\delta}$.*

Proof . For the triangle inequality, we have

$$\begin{aligned} d(x,y) = \|\delta(x,y)\| &\trianglelefteq \|\delta(x,z)+\delta(z,y)\| \\ &\trianglelefteq |\|\delta(x,z)\| + \|\delta(z,y)\| \\ &= d(x,z)+d(z,y), \end{aligned}$$

since $0 \leq T \leq S$ implies that $\|T\| \leq \|S\|$ for $T,S \in \mathfrak{A}$.

Now suppose that $\mathcal{R}$ has the Archimedean property. Let $N_r^\delta(x)$ be an arbitrary basis open set in $\tau_\delta^{\mathcal{R}}$ and let $y \in N_r^\delta(x)$. Since $\mathcal{R}$ is allowance, there is $T \triangleleft r - \delta(x,y)$ in $\mathcal{R}$ and so there is $\lambda_0 \in \mathbb{R}^{++}$ such that $\lambda_0 \iota \triangleleft T$. We have $N_{\lambda_0}^d(y) \subseteq N_r^\delta(x)$, since for $z \in N_{\lambda_0}^\delta(y)$,

$$\begin{aligned} \delta(z,x) &\leq \delta(z,y)+\delta(y,x) \\ &\leq \|\delta(z,y)\|\iota + \delta(x,y) \\ &\leq d(z,y)\iota + \delta(x,y) \\ &\triangleleft \lambda_0\iota + \delta(x,y) \\ &\triangleleft T + \delta(x,y) \\ &= r. \end{aligned}$$

Thus $N_r^\delta(x) \in \tau_d$.

Now let $N_\lambda^d(x)$ be an arbitrary basis open set in τ_d and let $y \in N_\lambda^d(x)$. Let r be a fixed element of $\mathcal{R}$ and let $r_0 = \frac{\lambda_0 r}{\|r\|}$, where $\lambda_0 = \frac{1}{2}(\lambda - d(x,y))$. Then $r_0 \in \mathcal{R} \bigcap \mathfrak{A}^{++}$ and $N_{r_0}^\delta(x) \subset N_\lambda^d(x)$. To see this, let $z \in N_{r_0}^\delta(x)$. We have $\delta(z,y) \triangleleft r_0$, and since $r_0(x) \neq 0$ $(x \in \mathcal{H})$, then $\delta(z,y)(x) < r_0(x)$ for all $x \in \mathcal{H}$. Thus $\|\delta(z,y)\| \leq \|r_0\|$. Hence

$$\begin{aligned} d(z,x) &\leq d(z,y)+d(y,x) \\ &\leq \|d(z,y)\| + d(x,y) \\ &\leq \|r_0\| + d(x,y) \\ &= \lambda_0 + d(x,y) \\ &< \lambda. \end{aligned}$$

This shows that y in an interior point of $N_\lambda^d(x)$ with respect to $\tau_\delta^{\mathcal{R}}$ and so $N_\lambda^d(x) \in \tau_\delta^{\mathcal{R}}$.□

According to the results obtained in this article and Mirzavaziri's article, it is possible to define the C^*-metric spaces with both commutative and noncommutative C^*-algebras.

References

[1] J. Dixmier, *C*-Algebras*, Translated from the French by Francis Jellett, North-Holland Mathematical Library, Vol. 15. North-Holland Publishing Co., Amsterdam-New York-Oxford, 1977.

[2] G. Dolinar and J. Marovt,*Star partial order on B($\mathcal{H}$)*, Linear Algebra Appl. 434(1) (2011) 319–326.

[3] K. Janich,*Topology*, Springer-Verlag, 1984.

[4] J.L. Kelley, *General Topology*, Van Nostrand Company, Inc., Toronto-New York-London, 1955.

[5] K. Menger,*Probabilistic theories of relations*, Proc. Nat. Acad. Sci. USA. 37 (1951) 178–180.

[6] K. Menger,*Probabilistic geometry*, Proc. Nat. Acad. Sci. USA. 37 (1951) 226–229.

[7] M. Mirzavaziri,*Function valued metric space*, Surv. Math. Appl. 5 (2010) 321–332.

[8] M.S. Moslehian, *On full Hilbert C*-modules*, Bull. Malays. Math. Sci. Soc. 24 (2001) 45–47.

[9] G.J. Murph,*C^*-Algebras and Operator Theory*, Academic Press, Inc., Boston, MA, 1990.

[10] G.K. Pedersen, *Analysis Now*, Springer Verlag, 1988.

[11] B. Schweizer and A. Sklar, *Probabilistic metric space*, North-Holland Series in Probability and Applied Mathematics. North-Holland Publishing Co., New York, (1983).