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Walsh function for solving fractional partial differential equation

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Abstract

In this article, we extended an efficient computational method based on Walsh operational matrix to find an approximate solution of fractional diffusion equations, First, we present the fractional Walsh operational matrix of integration and differentiation. Then by applying this method, the Fractional diffusion equations are reduced into a system of an algebraic equation. The benefits of this method are the low cost of setting up the equations without applying any projection method such as collocation, Galerkin, etc. The results show that the method is very accurate and efficient.

Keywords: Fractional diffusion equations, Operational matrix, Walsh functions, Block-pulse functions, Fractional calculus
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1. Introduction

Fractional diffusion equations naturally appear in history-dependent problems such as fluid flow, damped vibrations, viscoelasticity, population dynamics, heat conduction and seismology [26, 23, 10, 31]. On the other hand, the multi-term time-fractional operator has been found useful in describing complex physical and physiological systems [18, 20, 32, 28], Several numerical methods have been proposed in the last few years for solving fractional order diffusion-wave equations. Fractional order diffusion equations were shown to provide an adequate and accurate description of these transport processes, which exhibit anomalous diffusion [21, 29]. However, fractional diffusion operator is a non-local operator, which generates computational and numerical difficulties that have not been encountered in the context of second-order diffusion equations. Jiang et al. [19] used some analytical

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techniques to solve three types of multi-term time-space Caputo Riesz fractional advection diffusion equations with nonhomogeneous Dirichlet boundary conditions. Li and Xu [15] improved their previous results and proposed a space–time spectral method for the equation. Reutskiy [25] have applied a combined method of separating variables and Fourier expansion with backward substitution to solve a wide class of fractional partial differential equations including diffusion-wave equations,

In recent years operational matrix have been extensively used in different context and emerged as a potential alternative in the field of numerical solution of partial differential equations. The operational matrix of integration has been determined for several types of orthogonal polynomials, such as Chebyshev polynomials [11], Boubaker functions [17], Bernoulli wavelet [16]. Laguerre polynomials [27], Jacobi operational matrix [4]. And Ebadian have applied fractional operational matrix for solving nonlinear Volterra integro-differential equations [12], Imran Aziz have extended the Haar wavelet for the numerical solution of two-dimensional nonlinear integral equations [1], Y. Yang in [30] also developed the Jacobi collocation method to solve the time-fractional diffusion wave equation and convergence analysis. Khajehnasiri in [14] have applied a triangular functions method for the solution of 2D nonlinear Volterra-Fredholm integro differential equations, M. H. Heydari [13] utilized the Hat function method for the time fractional diffusion-wave equation.

As we know, the Walsh Functions (WFs) are a powerful mathematical tool for solving various kinds of integral equations. V. Balakumar, and K. Murugesan also developed the single term Walsh series technique to solve the system of volterra integral equation [3], K. Murugesan have extended the single term Walsh series (STWS) method for the nonlinear volterra integral equations and system of linear volterra integro-differential equations [5, 6]. R. Chandra Guru Sekar in [7] and A. E. K. Pushpam in [**?**] have applied a STWS method for the solution of system of linear second order volterra integro differential equations and linear system of stiff delay differential equation, respectively. Chandra Guru Sekar et al. used STWS method for solving nonlinear delay volterra integro differential equation [5],

The time fractional diffusion and wave-diffusion equations can be written in the following form:

$$\frac{\partial^\alpha F(x,t)}{\partial t^\alpha} + \frac{\partial F(x,t)}{\partial t} = \frac{\partial^2 F(x,t)}{\partial x^2} + r(x,t) \quad (x,t) \in [0,1] \times [0,1], \tag{1.1}$$

subject to the initial conditions:

$$f(x,0) = u_0(x), \quad \frac{\partial f(x,0)}{\partial t} = u_1(x), \quad x \in [0,1], \tag{1.2}$$

and boundary conditions:

$$F(0,t) = h_0(t), \quad F(1,t) = h_1(t), \quad t \in [0,1], \tag{1.3}$$

where x and t are the space and time variables, k is an arbitrary positive constant, which will be described in the next section, R is a given function in $\Omega^2([0,1] \times [0,1])$ and f_0, f_1, h_0 and h_1 are given functions in $\Omega^2[0,1]$, $u(x,t)$ is a sufficiently smooth function, $1 < \alpha \leqslant 2$ and is a Caputo fractional derivative of order α defined as [22]. When $0 < \alpha < 1$, equation (3.7) is a fractional diffusion equation and when $1 < \alpha < 2$, equation (3.7) is the time fractional diffusion-wave equation. When $\alpha = 1$, it represents a traditional diffusion equation; while if $\alpha = 2$, it represents a traditional wave equation [9], also discussed fractional differential equations with multi-orders. However, they only considered the multi-orders lying in (0, 2) in these papers.

In this paper we intend to extend the application of the single-term Walsh series method to solve the fractional order diffusion-wave equation. Our main aim is to generalize the Walsh function operational matrix to fractional calculus. It is worthy to mention here that, the method based on using the operational matrix for solving fractional order diffusion-wave equation. The method is based on reducing the equation to the system of algebraic equation by expanding the solution as Walsh functions.

In the next section we will define some basic definitions and properties of the fractional integral and derivative. In the section 3 we will define and some properties of the Walsh function, After which in "Applying the method" section we will solve fractional order diffusion-wave equation by using Walsh functions. We apply this proposed method for some examples of fractional order diffusion-wave equation. Finally, some concluding remarks are given in "Conclusion" section.

2. Preliminaries and Basic Definitions

In this section, first we recall essential basic definitions and properties of the fractional integral and derivative.

Definition 2.1. *The Riemann-Liouville fractional integration, for a function $z(t)$, of order $\alpha > 0$ is given by*

$$I_{x_0}^{\alpha} z(x) = \frac{1}{\Gamma(\alpha)} \int_{x_0}^{x} (x-t)^{\alpha-1} z(t) dt, \quad \alpha > 0, \quad x > 0. \tag{2.1}$$

The following equations define Riemann-Liouville and Caputo fractional derivatives of order α, respectively:

$$D_{x_0}^{\alpha} z(x) = \frac{d^m}{dx^m} [I_{x_0}^{m-\alpha} z(x)], \tag{2.2}$$

$$D_{*x_0}^{\alpha} z(x) = I_{x_0}^{m-\alpha} [\frac{d^m}{dx^m} z(x)], \tag{2.3}$$

where $m-1 \leq \alpha < m$ and $n \in \mathbb{N}$. From (2.1) and (2.2), we have

$$D_{x_0}^{\alpha} z(x) = \frac{1}{\Gamma(m-\alpha)} \frac{d^m}{dx^m} \int_{x_0}^{x} (x-t)^{m-\alpha-1} z(t) dt, \quad x > x_0 \tag{2.4}$$

Lemma 2.2. *If $m-1 < \alpha \leq m$, $m \in \mathbb{N}$, then $D_*^{\alpha} I^{\alpha} z(x) = z(x)$, and:*

$$I^{\alpha} D_*^{\alpha} \mathrm{z}(x) = \mathrm{z}(x) - \sum_{k=0}^{m-1} \mathrm{z}^{(k)}(0^+) \frac{x^k}{k!}, \quad x > 0.$$

3. Definition and properties of Walsh function

A function $f(x)$ integrable in $[0, 1)$, The expansion of $f(x)$ with respect to Walsh series as

$$f(x) = \sum_{i=0}^{\infty} f_i \Psi_i(x), \tag{3.1}$$

where $\Psi_i(x)$ is the ith Walsh function and f_i is the corresponding coefficient [24]. In Walsh series approach, we consider only finite number of terms. Then,

$$f(x) \simeq F^T \Psi(x), \tag{3.2}$$

where $F = [f_0, \ldots, f_{m-1}]^T$ and

$$\Psi(t) = [\Psi_1(x), \Psi_2(x), \ldots, \Psi_{m-1}(x)]^T. \tag{3.3}$$

The coefficients f_i are chosen to minimize the mean integral square error

$$\varepsilon = \int_0^1 \Big[f(x) - F^T \Psi(x) \Big]^2 dx \tag{3.4}$$

and are given by

$$f_i = \int_0^1 f(x) \Psi_i(x) dx. \tag{3.5}$$

It has been proved that

$$\int_0^1 f(x) dx = F^T \Upsilon \Psi(x), \tag{3.6}$$

where Υ is called the operational matrix for integration Walsh series. In Single Term Walsh Series, $\Upsilon = \dfrac{1}{2}$ [3, 2].

Operational matrix of integration variable t is defined as

$$\int_0^t \Phi_m(x) dx \cong P_{m \times m} \Phi_m(t) \tag{3.7}$$

where $P_{m\times m}$ is called the operational matrix of WFs. This matrix can be expressed as follows

$$P_{(m\times m)} = \left[\begin{array}{c:c:c:c} O & & I & \\ \hdashline \frac{1}{2} & & -\frac{2}{m} I_{(\frac{m}{4})} & \\ \hdashline & \ddots & & -\frac{1}{2m} I_{(\frac{m}{4})} \\ \hdashline \frac{2}{m} I_{(\frac{m}{4})} & & O_{(\frac{m}{4})} & \\ \hdashline & \frac{1}{2m} I_{(\frac{m}{4})} & & O\left(\frac{m}{4}\right) \end{array} \right] \tag{3.8}$$

We define a set of Walsh functions (WFs) as following formula:

$$\Phi_m(t) = W_{m\times m} \Psi_m(t), \tag{3.9}$$

where the Ψ_i are called the BPFs with unity height and $1/m$ width. $W_{(m\times m)}$ is called the Walsh matrix. The block-pulse functions are a set of orthogonal functions with piecewise constant value which are defined in the time interval $[0, T_1]$ as

$$\Psi_i = \begin{cases} 1, & (i-1)\frac{T_1}{m} \leq t < \frac{iT_1}{m}, \\ 0, & \text{otherwise}, \end{cases}$$

where $i = 0, 1, \cdots, m-1$ with m as a positive integer. The following properties of the W matrix will be considered

$$W^2_{(m\times m)} = m I_m$$

or

$$W^{-1}_{(m\times m)} = \frac{1}{m} W_{m\times m}. \tag{3.10}$$

Substituting (3.9) into (3.7), yields

$$\int_0^t W_{(m\times m)} \Psi_m(t) dt = P_{(m\times m)} W_{(m\times m)} \Psi_{(m)} \tag{3.11}$$

Therefore,

$$\int_0^t \Psi_m(t)dt = W^{-1}_{(m\times m)}P_{(m\times m)}W_{(m\times m)}\Psi_{(m)} \tag{3.12}$$

We define the following:

$$W^{-1}_{(m\times m)}P_{(m\times m)}W_{(m\times m)} = \Upsilon_{m\times m} \tag{3.13}$$

Combining it with (3.10) gives

$$\Upsilon_{(m\times m)} = \frac{1}{m}WPW. \tag{3.14}$$

Evaluation of the similarity transformation yields

$$\Upsilon_{(m\times m)} = \frac{1}{m}\begin{pmatrix} \frac{1}{2} & 1 & 1 & \dots & 1 \\ 0 & \frac{1}{2} & 1 & \dots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \frac{1}{2} \end{pmatrix}, \tag{3.15}$$

where Υ is a operational matrix of integration for BPFs. Inspecting the Υ matrix, we make the following decomposition:

$$\begin{aligned} \Upsilon_{(m\times m)} &= \frac{1}{m}(\frac{1}{2}I_m + Q_{(m\times m)} + Q^2_{(m\times m)} + \cdots + Q^{m-1}_{(m\times m)}) \\ &= \frac{1}{m}\left(\frac{1}{2}I_m + \sum_{i=1}^{\infty} Q^i_{(m\times m)}\right) \\ &= \frac{1}{m}\left(-\frac{1}{2}I_m + (I_m - Q_{(m\times m)})^{-1}\right) \\ &= \frac{1}{2m}(I_m + Q_{(m\times m)})(I_m - Q_{(m\times m)})^{-1}, \end{aligned} \tag{3.16}$$

where

$$Q_{(m\times m)} = \frac{1}{m}\begin{pmatrix} 0 & 1 & 0 & 0 & \dots & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 0 \end{pmatrix}. \tag{3.17}$$

And it is easy to see the following important property:

$$Q^i_{(m\times m)} = \begin{pmatrix} O & I_{m-i} \\ O & O \end{pmatrix} \quad for \quad i < m, \tag{3.18}$$

and

$$Q^i_{(m\times m)} = O_m \quad for \quad i \geq m. \tag{3.19}$$

3.1. Operational Matrix for Differentiation

In this section, we want to derive an exploit formula for walsh function of mth deger operational matrix of differentiation. Let us denote the operational matrix for differentiation as $\Upsilon_{m\times m}$.

$$\begin{aligned} \Upsilon^{-1}_{(m\times m)} &= 2m(I_m + Q_{(m\times m)})(I_m - Q_{(m\times m)}) \\ &= 2m(I_m - 2Q_{(m\times m)} + 2Q^2_{(m\times m)} + \cdots (-1)^{m-1}Q^{m-1}_{(m\times m)}) \\ &= 4m(\frac{1}{2}I_m + \sum_{i=1}^{m-1}(-1)^i Q^i_{(m\times m)}). \end{aligned} \tag{3.20}$$

Similarly transformating back to the Walsh domain yields the operational matrix for differentiation, denoted by $D_{m\times m}$

$$D_{m\times m} = P^{-1}_{m\times m} = \frac{1}{m} W_{m\times m} H^{-1}_{m\times m} W_{m\times m}. \tag{3.21}$$

In general, the formula is

$$D_{(m\times m)} = 2m \left[\begin{array}{c:c:c:c:c} O_{(\frac{m}{2})} & I_{(\frac{m}{2})} & & & \\ \hdashline & m & & -4I_{\frac{m}{8}} & \\ \hdashline & & \ddots & & -\frac{1}{2m} I_{(\frac{m}{4})} \\ \hdashline -I(\frac{m}{2}) & 4I_{(\frac{m}{8})} & & O_{(\frac{m}{4})} & \\ \hdashline & 2I_{(\frac{m}{4})} & & & O\left(\frac{m}{4}\right) \end{array}\right]. \tag{3.22}$$

From (3.20) the eigenvalue, h^{-1}, of matrix $\Upsilon^{-1}_{(m\times m)}$ can be expressed as the eigenvalue, q, of $Q_{(m\times m)}$

$$b = 4m(\frac{1}{2} + \sum_{i=1}^{m-1} (-1)^i q^i) \tag{3.23}$$

$$b = 2m \frac{1-q}{1+q} \tag{3.24}$$

3.2. Operational Matrices for Fractional Differentiation

Now we try to find the operational matrix for fractional differentiation. The general form of (3.24) could be written as follows:

$$b^\alpha = \left(2m \frac{1-q}{1+q}\right)^\alpha \tag{3.25}$$

Equation (3.25) can be developed into polynomial of q and terminated at q^{m-1}. As the result, Eq. (3.25) becomes

$$b^\alpha = (2m)^\alpha \Lambda_{l,m}(q) \tag{3.26}$$

where $\Lambda_{l,m}$ is the polynomial of order $m-1$ for α differentiation. Thus the operational matrix for α differentiation from (3.20) is given by

$$B^\alpha_{(m\times m)} = (2m)^\alpha \Lambda_{l,m}(Q_{(m\times m)}) \tag{3.27}$$

In the Walsh domain, the corresponding α differentiation operational matrix is

$$D^\alpha_{(m\times m)} = (2m)^\alpha W^{-1}_{(m\times m)} \Lambda_{l,m}(Q_{(m\times m)}) W_{(m\times m)} \tag{3.28}$$

3.3. Operational Matrices for Fractional Integration

We rewrite (3.16) by expressing $\Upsilon_{(m\times m)}$ as a polynomial of $Q_{(m\times m)}$

$$\Upsilon_{(m\times m)} = h_m(Q_{m\times m}) \tag{3.29}$$

where

$$h_m(x) = \frac{1}{m}(\frac{1}{2} + x + x^2 + \cdots + x^{m-1}) \tag{3.30}$$

If q is an eigenvalue of $Q_{m\times m}$, it is known (3.19) that corresponding eigenvalue for $\Upsilon_{(m\times m)}$ is

$$h = h_m(q) = \frac{1}{2m} \frac{1+q}{1-q}. \tag{3.31}$$

Therefore, we state that the eigenvalues of $\Upsilon_{m\times m}$ is $1/2m$ with multiplicity m. For finding the operational matrix of fractional integration, we can use the reasoning similar to the fractional differentiation case. Generalizing (3.31), yields

$$h = \left[\frac{1-q}{2m(1+q)}\right]^{\alpha} = (\frac{1}{2m}\rho_{l,m}(q))^{\alpha} \tag{3.32}$$

where $\rho_{l,m}$ is the polynomial of order $m-1$ for α integration. The operational matrix for α-integration in terms of the BPF is given by

$$\Upsilon^{\alpha}_{(m\times m)} = \frac{1}{(2m)^{\alpha}}\rho_{l,m}(Q_{m\times m}) \tag{3.33}$$

and the corresponding α-integration operational matrix in the Walsh domain is easily found as

$$\begin{aligned} P^{\alpha}_{(m\times m)} &= \frac{1}{(2m)^{\alpha}}W^{-1}_{(m\times m)}\rho_{l,m}(Q_{(m\times m)})W_{(m\times m)} \\ &= \frac{1}{m(2m)^{\alpha}}W_{(m\times m)}\rho_{l,m}(Q_{(m\times m)})W_{(m\times m)}, \end{aligned} \tag{3.34}$$

Therefore, we have the following nonlinear system.

$$(I^{\alpha}\Phi)(t) = P^{\alpha}_{m\times m}\Phi(t) \tag{3.35}$$

4. Applying the method

In this section we solve time fractional diffusion-wave equation with damping by using the fractional Walsh operational matrix of intergration

$$\frac{\partial^{\alpha} f(x,t)}{\partial t^{\alpha}} + \frac{\partial f(x,t)}{\partial t} = \frac{\partial^2 f(x,t)}{\partial x^2} + r(x,t), \tag{4.1}$$

$$(x,t) \in [0,1]\times[0,1], \quad 1<\alpha\le 2,$$

with initial conditions

$$f(x,0) = u_0(x), \quad \frac{\partial f(x,0)}{\partial t} = u_1(x), \quad x\in[0,1], \tag{4.2}$$

and boundary conditions:

$$f(0,t) = h_0(t), \quad f(1,t) = h_1(t), \quad t\in[0,1].$$

In order to use the WFs with the fractional operational matrix for solving this equation, by applying the Riemann-Liouville fractional integration of order α with respect to t on both sides of (4.1), and using the initial conditions (4.2), we obtain:

$$f(x,t) - w(x,t) + (I_t^{\alpha-1}f)(x,t) = \left(I_t^{\alpha}\frac{\partial^2 f}{\partial x^2}\right)(x,t) + (I_t^{\alpha}r)(x,t), \tag{4.3}$$

where $w(x,t) = u_0(x) + tu_1(x) - \dfrac{t^{\alpha-1}}{\Gamma(\alpha)}u_0(x)$. Now we approximate $\frac{\partial^2 f(x,t)}{\partial x^2}$ by the WFs as follows:

$$\frac{\partial^2 f(x,t)}{\partial x^2} \simeq \Phi(x)^T F\Phi(t), \tag{4.4}$$

where $F = [u_{ij}]_{\hat{m}\times\hat{m}}$ is an unknown matrix which should be found and $\Phi(.)$ is the WFs vector that was defined in (3.3). Moreover, by integrating (4.4) two times with respect to x, we have

$$f(x,t) \simeq f(0,t) + x\left(\frac{\partial f(x,t)}{\partial x}\bigg|_{x=0}\right) + \Phi(x)^T (P^T)^2 F\Phi(t), \tag{4.5}$$

and by putting $x = 1$ in (4.4) and considering the boundary conditions (4.2). we obtain:

$$\frac{\partial f(x,t)}{\partial x}\bigg|_{x=0} \simeq h_1(t) - h_0(t) - \Phi(1)^T (P^T)^2 F\Phi(t). \tag{4.6}$$

We also expand $h_0(t)$ and $h_1(t)$ by the WFs as follows:

$$h_0(t) \simeq H_0^T\Phi(t), \quad h_1(t) \simeq H_1^T\Phi(t), \tag{4.7}$$

where H_0 and H_1 are the WFs coefficient vectors. by substituting (4.7) into (4.6), we obtain:

$$\begin{aligned} \frac{\partial f(x,t)}{\partial x}\bigg|_{x=0} &\simeq (H_1^T - H_0^T - \Phi(1)^T (P^T)^2 F)\Phi(t), \\ &\triangleq \tilde{F}^T\Phi(t). \end{aligned} \tag{4.8}$$

Now by substituting (4.8) into (4.5), we have:

$$\begin{aligned} f(x,t) &\simeq \Phi(x)^T[KH_0^T + X\tilde{F}^T + (P^T)^2 F]\Phi(t), \\ &\triangleq \Phi(x)^T\Theta\Phi(t), \end{aligned} \tag{4.9}$$

where X and K are the WFs coefficient vectors for x and the unit step function, respectively. Furthermore, we expand $w(x,t)$ and $r(x,t)$ by the WFs as follows:

$$w(x,t) \simeq \Phi(x)^T W\Phi(t), \quad r(x,t) \simeq \Phi(x)^T R\Phi(t), \tag{4.10}$$

where W and R are the known WFs coefficient matrices for $w(x,t)$ and $r(x,t)$, respectively. Then by substituting (4.4), (4.9) and (4.10) into (4.3), and using fractional operational matrix of integration of WFs, we have:

$$\Phi(x)^T[\Theta + \Theta P^{(\alpha-1)} - FP^\alpha]\Phi(t) \simeq \Phi(x)^T[W + RP^\alpha]\Phi(t),$$

so, by replacing $\simeq$ by $=$, we have

$$\Theta + \Theta P^{(\alpha-1)} - FP^\alpha = W + QP^\alpha.$$

Finally, by solving this system for the unknown matrix U, we obtain an approximate solution for the problem using (4.9). The algorithm of the proposed method is presented as follows:

5. Numerical examples

In order to reveal the effectiveness of proposed method, two fractional diffusion equations are solved. Walsh function is utilized to approximate the unknown functions computations. As uptill now there is a few alternative numerical method for these types of fractional diffusion equations, Also we report the absolute errors of the proposed computational method in some points $(x_j, t_j) \in [0,1]\times[0,1]$ as follows $\mid e(x_j,t_j) \mid = \mid \Phi(x_j)^T\Lambda\Phi(t_{t_j}) - f(x_j,t_j) \mid$.

Table 1: Results for Example 1

$m = 20$ $n = 20$				
$x = t$	$\alpha = 1.2$	$\alpha = 1.4$	$\alpha = 1.6$	$\alpha = 1.8$
0.0	3.502E-7	2.258E-5	4.357E-4	4.458E-6
0.1	5.564E-7	5.213E-5	4.157E-4	5.115E-6
0.2	3.159E-7	2.369E-5	1.241E-4	7.254E-6
0.3	8.478E-7	7.258E-5	2.259E-5	4.418E-6
0.4	4.321E-7	6.534E-5	2.367E-5	2.548E-7
0.5	4.212E-7	3.312E-6	2.457E-5	6.040E-7
0.6	3.954E-7	4.125E-6	2.107E-5	2.008E-7
0.7	1.874E-8	9.219E-6	2.441E-6	7.428E-7
0.8	5.257E-8	4.985E-6	2.417E-6	8.782E-8
0.9	6.210E-8	3.457E-6	7.858E-6	6.217E-8

Example 5.1. *Consider the following fractional diffusion-wave equation given in [8]*

$$\frac{\partial^\alpha f(x,t)}{\partial t^\alpha} + \frac{\partial f(x,t)}{\partial t} = \frac{\partial^2 f(x,t)}{\partial x^2} + \frac{6t^{3-\alpha}}{\Gamma(4-\alpha)} e^x + 3t^2 e^x - t^3 e^x$$

$$(x,t) \in [0,1] \times [0,1], \quad 1 < \alpha \leq 2,$$

subject to the initial conditions

$$f(x,0) = 0, \quad \frac{\partial f(x,0)}{\partial t} = 0,$$

and boundary conditions:

$$f(0,t) = t^3, \quad f(1,t) = et^3.$$

The exact solution is known and given by $f(x,t) = e^x t^3$. To solve this example, we implement the WFs method for $\alpha = 1.2$, $\alpha = 1.4$, $\alpha = 1.6$ and $\alpha = 1.8$. Numerical results is presented in Table (1).

Example 5.2. *The following fractional diffusion-wave equation with damping is considered [8]*

$$\frac{\partial^\alpha f(x,t)}{\partial t^\alpha} + \frac{\partial f(x,t)}{\partial t} = \frac{\partial^2 f(x,t)}{\partial x^2} + \frac{2x(1-x)}{\Gamma(3-\alpha)} t^\alpha + 2tx(1-x) + 2t^2.,$$

$$(x,t) \in [0,1] \times [0,1], \quad 1 < \alpha \leq 2,$$

subject to the homogenous initial and boundary conditions, The exact solution of this problem is $f(x,t) = t^2 x(1-x)$. To solve this example, we implement the WFs method for $\alpha = 1.2$, $\alpha = 1.4$, $\alpha = 1.6$ and $\alpha = 1.8$. Numerical results is presented in Table (2).

6. Conclusion

The purpose of this article is to extend a Walsh Functions for obtaining the approximate solution of Fractional diffusion equations. Frits, the Walsh Function fractional operational matrix of differentiation and integration has been presented. Then by using this matrix, the Fractional diffusion equations has been reduced to an algebraic system. The benefits of this method are the low cost of setting up the equations without applying any projection method such as collocation, Galerkin, etc. For more investigation, two example have been presented. As the numerical results showed the proposed are an effective and accuracy.

Table 2: Results for Example 2

$m = 15$ $n = 15$				
$x = t$	$\alpha = 1.2$	$\alpha = 1.4$	$\alpha = 1.6$	$\alpha = 1.8$
0.0	4.145E-6	4.251E-5	2.770E-6	5.029E-6
0.1	5.417E-6	5.741E-5	3.548E-6	4.589E-6
0.2	7.714E-6	7.021E-5	2.355E-6	2.650E-6
0.3	5.015E-6	6.741E-5	7.709E-6	9.148E-7
0.4	6.450E-6	3.213E-5	4.157E-6	2.254E-7
0.5	2.488E-6	9.025E-5	9.354E-6	4.956E-7
0.6	1.752E-6	1.852E-5	6.741E-6	9.451E-7
0.7	6.145E-7	4.254E-6	2.834E-7	6.659E-8
0.8	5.145E-7	6.025E-6	1.415E-7	3.041E-8
0.9	9.015E-7	3.268E-6	8.623E-7	3.240E-8

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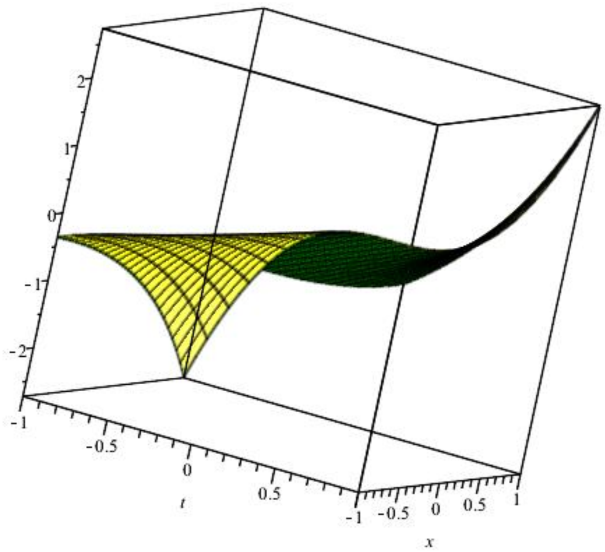

Figure 1: Exact and approximation solutions of Example 1.

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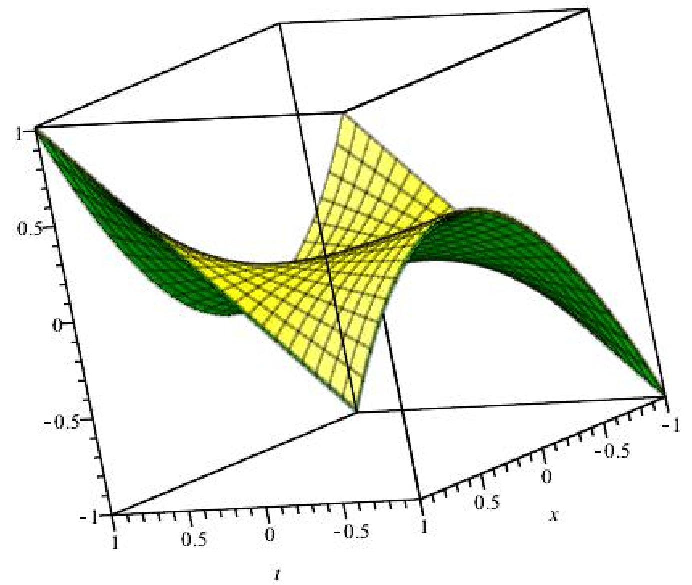

Figure 2: Comparison the exact solution and the presented method for Example 2.