Superstability of the $p$-radical functional equations related to Wilson equation and Kim’s equation

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Abstract

In this paper, we solve and investigate the superstability of the $p$-radical functional equations related to the following Wilson and Kim functional equations

\[ f\left(\sqrt[p]{x^p + y^p}\right) + f\left(\sqrt[p]{x^p - y^p}\right) = \lambda f(x)g(y), \]
\[ f\left(\sqrt[p]{x^p + y^p}\right) + f\left(\sqrt[p]{x^p - y^p}\right) = \lambda g(x)f(y), \]

where $p$ is an odd positive integer and $f$ is a complex valued function. Furthermore, the results are extended to Banach algebras.

Keywords: stability, superstability, radical functional equation, cosine functional equation, Wilson functional equation, Kim functional equation.

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1. Introduction

In 1940, the stability problem of the functional equation was conjectured by Ulam [24]. In 1941, Hyers [14] obtained a partial answer for the case of additive mapping in this problem.


In 1979, Baker et al. [7] announced the superstability as the new concept as follows: If $f$ satisfies $|f(x+y) - f(x)f(y)| \leq \epsilon$ for some fixed $\epsilon > 0$, then either $f$ is bounded or $f$ satisfies the exponential functional equation $f(x+y) = f(x)f(y)$.

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D’Alembert [1] in 1769 (see Kannappan’s book [16]) introduced the cosine functional equation

\[ f(x + y) + f(x - y) = 2f(x)f(y), \quad (C) \]

and which superstability was proved by Baker [6] in 1980.

Baker’s result was generalized by Badora [4] in 1998 to a noncommutative group under the Kannappan condition [15]: \( f(x + y + z) = f(x + z + y) \), and it again was improved by Badora and Ger [5] in 2002 under the condition \(|f(x + y) + f(x - y) - 2f(x)f(y)| \leq \phi(x) \) or \( \phi(y) \).

The cosine (d’Alembert) functional equation \((C)\) was generalized to the following:

\[ f(x + y) + f(x - y) = 2f(x)g(y), \quad (W) \]
\[ f(x + y) + f(x - y) = 2g(x)f(y), \quad (K) \]

in which \((W)\) is called the Wilson equation, and \((K)\) arised by Kim was appeared in Kannappan and Kim’s paper (17).

The superstability of the cosine \((C)\), Wilson \((W)\) and Kim \((K)\) function equations were founded in Badora, Ger, Kannappan and Kim (8, 17, 18, 21).

In 2009, Eshaghi Gordji and Parviz [12] introduced the radical functional equation related to the quadratic functional equation

\[ f(\sqrt{x^2 + y^2}) = f(x) + f(y). \quad (R) \]

In [20], Kim introduced the trigonometric functional equation as the Pexider-type’s as following:

\[ f(x + y) + f(x - y) = \lambda f(x)f(y), \quad (1.1) \]
\[ f(x + y) + f(x - y) = \lambda f(x)g(y), \quad (1.2) \]
\[ f(x + y) + f(x - y) = \lambda g(x)f(y), \quad (1.3) \]
\[ f(x + y) \pm f(x - y) = \lambda g(x)h(y), \]
\[ f(x + y) \pm g(x - y) = \lambda h(x)k(y). \]

Recently, Almahalebiet et al. [3] obtained the superstability in Hyer’s sense for the \(p\)-radical functional equations related to Wilson equation and Kim’s equation.

The aim of this paper is to solve and investigate the superstability in Gavurta’s sense for the \(p\)-radical functional equations related to Wilson and Kim’s equations as following:

\[ f(\sqrt[p]{x^p + y^p}) + f(\sqrt[p]{x^p - y^p}) = 2f(x)f(y), \quad (C_r) \]
\[ f(\sqrt[p]{x^p + y^p}) + f(\sqrt[p]{x^p - y^p}) = \lambda f(x)f(y), \quad (C^{\lambda}_r) \]
\[ f(\sqrt[p]{x^p + y^p}) + f(\sqrt[p]{x^p - y^p}) = 2f(x)g(y), \quad (W_r) \]
\[ f(\sqrt[p]{x^p + y^p}) + f(\sqrt[p]{x^p - y^p}) = \lambda f(x)g(y), \quad (W^{\lambda}_r) \]
\[ f(\sqrt[p]{x^p + y^p}) + f(\sqrt[p]{x^p - y^p}) = 2g(x)f(y), \quad (K_r) \]
\[ f(\sqrt[p]{x^p + y^p}) + f(\sqrt[p]{x^p - y^p}) = \lambda g(x)f(y), \quad (K^{\lambda}_r) \]

In this paper, let \( \mathbb{R} \) be the field of real numbers, \( \mathbb{R}_+ = [0, \infty) \) and \( \mathbb{C} \) be the field of complex numbers. We may assume that \( f \) is a nonzero function, \( \varepsilon \) is a nonnegative real number, \( \varphi : \mathbb{R} \to \mathbb{R}_+ \) is a given nonnegative function and \( p \) is an odd nonnegative integer.
2. Superstability of the $p$-radical Wilson equation $(W_p^n)$ and Kim’s equation $(K_p^n)$.

In this section, we find a solution and investigate the superstability of $p$-radical functional equations related to the Wilson type equation $(W_p^n)$ and the Kim type equation $(K_p^n)$.

In the following lemmas, we obtain a solution of the functional equations $(C_1^n)$, $(W_p^n)$ and $(K_p^n)$, which can be easy.

**Lemma 2.1.** A function $f : \mathbb{R} \to \mathbb{C}$ satisfies $(C_1^n)$ if and only if $f(x) = F(x^n)$ for all $x \in \mathbb{R}$, where $F$ is a solution of $(1.1)$. In particular, for the case $\lambda = 2$, a function $f : \mathbb{R} \to \mathbb{C}$ satisfies $(C_1^n)$ if and only if $f(x) = \cos(x^n)$ for all $x \in \mathbb{R}$, namely, $F$ is a solution of $(1.1)$.

**Lemma 2.2.** A function $f, g : \mathbb{R} \to \mathbb{C}$ satisfies $(W_p^n)$ if and only if $f(x) = F(x^n)$ and $g(x) = G(x^n)$, where $F$ and $G$ are solutions of $(1.2)$. In particular, for the case $\lambda = 2$, a function $f, g : \mathbb{R} \to \mathbb{C}$ satisfies $(W_p^n)$ if and only if $f(x) = F(x^n) = \sin(x^n)$ and $g(x) = G(x^n) = \cos(x^n)$, where $F$ and $G$ are solutions of $(1.2)$.

**Lemma 2.3.** A function $f, g : \mathbb{R} \to \mathbb{C}$ satisfies the functional equation $(K_p^n)$ if and only if $f(x) = F(x^n)$ and $g(x) = G(x^n)$, where $F$ and $G$ are solutions of $(1.3)$. In particular, for the case $\lambda = 2$, a function $f, g : \mathbb{R} \to \mathbb{C}$ satisfies $(K_p^n)$ if and only if $f(x) = F(x^n) = G(x^n)$, where $F$ and $G$ are solutions of $(1.3)$.

Now we investigate the superstability of the Wilson equation $(W_p^n)$ and the Kim’s equation $(K_p^n)$.

**Theorem 2.4.** Assume that $f, g : \mathbb{R} \to \mathbb{C}$ satisfy the inequality

$$|f(\sqrt[n]{x^n + y^n}) + f(\sqrt[n]{x^n - y^n}) - \lambda g(y)f(y)| \leq \begin{cases} (i) & \varphi(x) \\ (ii) & \varphi(y) \text{ and } \varphi(x) \end{cases}.$$  \hspace{0.5cm} (2.1)

Then

(i) either $f$ is bounded or $g$ satisfies $(C_1^n)$,
(ii) either $g$ or $f$ is bounded or $g$ satisfies $(C_2^n)$, and $f$ and $g$ satisfy $(K_p^n)$ and $(W_p^n)$.

**Proof.** (i) Assume that $f$ is unbounded. Then we can choose $\{y_n\}$ such that $0 \neq |f(y_n)| \to \infty$ as $n \to \infty$.

Putting $y = y_n$ in $(2.1)$ and dividing both sides by $\lambda f(y_n)$, we have

$$\left| \frac{f(\sqrt[n]{x^n + y_n^n}) + f(\sqrt[n]{x^n - y_n^n})}{\lambda f(y_n)} - g(x) \right| \leq \frac{\varphi(x)}{\lambda f(y_n)}. \hspace{0.5cm} (2.2)$$

As $n \to \infty$ in $(2.2)$, we get

$$g(x) = \lim_{n \to \infty} \frac{f(\sqrt[n]{x^n + y_n^n}) + f(\sqrt[n]{x^n - y_n^n})}{\lambda f(y_n)} \hspace{0.5cm} (2.3)$$

for all $x \in \mathbb{R}$. 

Replacing \( y \) by \( \sqrt[p]{y^p + y_n^p} \) and \( \sqrt[p]{y^p - y_n^p} \) in (2.1), we obtain

\[
|f\left(\sqrt[p]{x^p + (y^p + y_n^p)}\right) + f\left(\sqrt[p]{x^p - (y^p + y_n^p)}\right) - \lambda g(x)f(\sqrt[p]{y^p + y_n^p})| \leq \varphi(x), \tag{2.4}
\]

\[
|f\left(\sqrt[p]{x^p + (y^p - y_n^p)}\right) + f\left(\sqrt[p]{x^p - (y^p - y_n^p)}\right) - \lambda g(x)f(\sqrt[p]{y^p - y_n^p})| \leq \varphi(x), \tag{2.5}
\]

for all \( x, y, y_n \in \mathbb{R} \). By (2.4) and (2.5), we obtain

\[
|f\left(\sqrt[p]{x^p + (y^p + y_n^p)}\right) + f\left(\sqrt[p]{x^p + (y^p - y_n^p)}\right) + f\left(\sqrt[p]{x^p - (y^p + y_n^p)}\right) + f\left(\sqrt[p]{x^p - (y^p - y_n^p)}\right) - \lambda g(x)[f(\sqrt[p]{y^p + y_n^p}) + f(\sqrt[p]{y^p - y_n^p})]| \leq 2\varphi(x)
\]

for all \( x, y, y_n \in \mathbb{R} \).

This implies that

\[
\frac{f\left(\sqrt[p]{x^p + y^p} + y_n^p\right) + f\left(\sqrt[p]{x^p + y^p} - y_n^p\right)}{\lambda f(y_n)} + \frac{f\left(\sqrt[p]{x^p - y^p} + y_n^p\right) + f\left(\sqrt[p]{x^p - y^p} - y_n^p\right)}{\lambda f(y_n)} - \frac{\lambda g(x)f(\sqrt[p]{y^p + y_n^p}) + f(\sqrt[p]{y^p - y_n^p})}{\lambda f(y_n)} \leq \frac{2\varphi(x)}{\lambda f(y_n)}
\]

for all \( x, y, y_n \in \mathbb{R} \).

Letting \( n \to \infty \) in (2.6), we obtain the desired result (2.2) by applying (2.3).

For the proof of the case (ii), first we show that \( f \) (or \( g \)) is unbounded if and only if \( g \) (or \( f \)) is also unbounded. Putting \( y = 0 \) in (2.1) (ii), we obtain

\[
|f(x) - \frac{\lambda}{2} g(x)f(0)| \leq \frac{\varphi(0)}{2},
\]

for all \( x \in \mathbb{R} \). If \( g \) is bounded, then by (2.7), we have

\[
|f(x)| = |f(x) - \frac{\lambda}{2} g(x)f(0) + \frac{\lambda}{2} g(x)f(0)| \leq \frac{\varphi(0)}{2} + \left| \frac{\lambda}{2} g(x)f(0) \right|
\]

which shows that \( f \) is also bounded.

On the other hand, if \( f \) is bounded, then we choose \( y_0 \in \mathbb{R} \) such that \( f(y_0) \neq 0 \), and then by (2.1) we can obtain

\[
|g(x)| - \left| \frac{f\left(\sqrt[p]{x^p + y_0^p}\right) + f\left(\sqrt[p]{x^p - y_0^p}\right)}{\lambda f(y_0)} \right| \leq \left| \frac{f\left(\sqrt[p]{x^p + y_0^p}\right) + f\left(\sqrt[p]{x^p - y_0^p}\right)}{\lambda f(y_0)} - g(x) \right| \leq \frac{\varphi(y_0)}{\lambda f(y_0)}
\]

and it follows that \( g \) is also bounded on \( \mathbb{R} \).

That is, if \( f \) (or \( g \)) is unbounded, then so is \( g \) (or \( f \)).
Let \( g \) be unbounded. Then \( f \) is also unbounded. So we can choose sequences \( \{x_n\} \) and \( \{y_n\} \) in \( \mathbb{R} \) such that \( g(x_n) \neq 0 \) and \( |g(x_n)| \to \infty \), \( f(y_n) \neq 0 \) and \( |f(y_n)| \to \infty \) as \( n \to \infty \).

For the case \( \varphi(y) \) in (2.1) (ii), taking \( x = x_n \), we deduce

\[
\lim_{n \to \infty} \frac{f \left( \sqrt[n]{x_n^p} + y^p \right) + f \left( \sqrt[n]{x_n^p} - y^p \right)}{\lambda g(x_n)} = f(y)
\]

for all \( y \in \mathbb{R} \). Using (2.1) we have

\[
|f \left( \sqrt[n]{x_n^p} + (x^p + y^p) \right) + f \left( \sqrt[n]{x_n^p} - (x^p + y^p) \right) - \lambda g(\sqrt[n]{x_n^p} + x^p) f(y) \\
+ f \left( \sqrt[n]{x_n^p} - (x^p - y^p) \right) + f \left( \sqrt[n]{x_n^p} - (x^p - y^p) \right) - \lambda g(\sqrt[n]{x_n^p} - x^p) f(y)| \leq 2\varphi(y)
\]

(2.10)

for all \( x, y \in \mathbb{R} \) and all \( n \in \mathbb{N} \).

Consequently,

\[
\frac{f \left( \sqrt[n]{x_n^p} + (x^p + y^p) \right) + f \left( \sqrt[n]{x_n^p} - (x^p + y^p) \right)}{\lambda g(x_n)} \\
+ \frac{f \left( \sqrt[n]{x_n^p} + (x^p - y^p) \right) + f \left( \sqrt[n]{x_n^p} - (x^p - y^p) \right)}{\lambda g(x_n)} \\
- \frac{\lambda g(\sqrt[n]{x_n^p} + x^p) + g(\sqrt[n]{x_n^p} - x^p) f(y)}{\lambda g(x_n)} | f(y) | \leq \frac{2\varphi(y)}{\lambda g(x_n)} .
\]

(2.11)

for all \( x, y \in \mathbb{R} \) and all \( n \in \mathbb{N} \).

Take the limit as \( n \to \infty \) with the use of \( |g(x_n)| \to \infty \) in (2.11). Since \( g \) satisfies \((K^*)\) by (i), we get that \( f \) and \( g \) are solutions of \((K^*)\).

Next, replace \((x, y)\) by \((\sqrt[n]{x_n^p} + y^p, x)\) and replace \((x, y)\) by \((\sqrt[n]{x_n^p} - y^p, x)\) for the case \( \varphi(y) \) in (2.1) (ii), respectively. Let us follows the same procedure as from (2.10) to (2.11). Then

\[
|f \left( \sqrt[n]{x_n^p} + y^p \right) + f \left( \sqrt[n]{x_n^p} - y^p \right) - \lambda g(\sqrt[n]{x_n^p} + y^p) f(x) \\
+ f \left( \sqrt[n]{x_n^p} + y^p \right) + f \left( \sqrt[n]{x_n^p} - y^p \right) - \lambda g(\sqrt[n]{x_n^p} - y^p) f(x)| \leq 2\varphi(y).
\]

Hence we have

\[
\frac{f \left( \sqrt[n]{x_n^p} + (x^p + y^p) \right) + f \left( \sqrt[n]{x_n^p} - (x^p + y^p) \right)}{\lambda g(x_n)} \\
+ \frac{f \left( \sqrt[n]{x_n^p} + (x^p - y^p) \right) + f \left( \sqrt[n]{x_n^p} - (x^p - y^p) \right)}{\lambda g(x_n)} \\
- \frac{\lambda g(\sqrt[n]{x_n^p} + y^p) + g(\sqrt[n]{x_n^p} - y^p) f(x)}{\lambda g(x_n)} | f(x) | \leq \frac{2\varphi(y)}{\lambda g(x_n)} ,
\]

(2.12)

for all \( x, y \in \mathbb{R} \) and all \( n \in \mathbb{N} \).

Then, by applying (2.9) and (i)’s result, it follows from (2.12) that \( f \) and \( g \) are solutions of \((W^*_2)\).

\( \Box \)

By a similar process of the proof of Theorem 2.1, we can prove the following theorem.
**Theorem 2.5.** Assume that \( f, g : \mathbb{R} \to \mathbb{C} \) satisfy the inequality
\[
|f\left(\sqrt[p]{x^n + y^p}\right) + f\left(\sqrt[p]{x^n - y^p}\right) - \lambda f(x)g(y)| \leq \begin{cases} 
(i) & \varphi(y) \\
(ii) & \varphi(x) \text{ and } \varphi(y). 
\end{cases}
\] (2.13)

Then

(i) either \( f \) is bounded or \( g \) satisfies \( (C^n_2) \),

(ii) either \( g(\varphi(f)) \) is bounded or \( g \) satisfies \( (C^n_2) \), and \( f \) and \( g \) satisfy \( (K^n_2) \) and \( (W^n_2) \).

**Proof.** The proof follows from that of Theorem 2.4. Let us choose \( \{x_n\} \) in \( \mathbb{R} \) such that \( 0 \neq |f(x_n)| \to \infty \) as \( n \to \infty \).

Taking \( x = x_n \) (with \( n \in \mathbb{N} \)) in (2.13), dividing both sides by \( |x| \), and passing to the limit as \( n \to \infty \), we obtain that
\[
g(y) = \lim_{n \to \infty} \frac{f\left(\sqrt[p]{x^n + y^p}\right) + f\left(\sqrt[p]{x^n - y^p}\right)}{\lambda f(x_n)}
\] (2.14)
for all \( y \in \mathbb{R} \).

(i) Replace \( (x, y) \) by \( \left(\sqrt[p]{x^n + y^p}, x\right) \) and replace \( (x, y) \) by \( \left(\sqrt[p]{x^n - y^p}, x\right) \) in (2.13). Thereafter we go through the same procedure as in (2.4) \( \sim \) (2.6) of Theorem 2.4. Then we obtain
\[
\left| f\left(\sqrt[p]{x^n + y^p} + x^p\right) + f\left(\sqrt[p]{x^n + y^p} - x^p\right) \right|
\]
\[
\quad + \frac{f\left(\sqrt[p]{x^n - y^p} + x^p\right) + f\left(\sqrt[p]{x^n - y^p} - x^p\right)}{\lambda f(x_n)}
\]
\[
\quad - \lambda f\left(\sqrt[p]{x^n + y^p} + f\left(\sqrt[p]{x^n - y^p}\right)\right|g(x)| \leq \frac{2\varphi(x)}{\lambda f(x_n)}.
\] (2.15)

Since the right-hand side of the inequality converges to zero as \( n \to \infty \) in (2.15), \( g \) satisfies \( (C^n_2) \).

(ii) As (2.8), we can see by some calculation that if \( f \) is bounded, then \( g \) is also bounded.

Assume that \( g \) is unbounded. Then \( f \) is unbounded and hence \( g \) satisfies \( (C^n_2) \).

Let us choose \( \{y_n\} \) in \( \mathbb{R} \) such that \( 0 \neq |g(y_n)| \to \infty \) as \( n \to \infty \).

As before, for the chosen sequence \( \{y_n\} \), we obtain that
\[
f(x) = \lim_{n \to \infty} \frac{f\left(\sqrt[p]{x^n + y_n^p}\right) + f\left(\sqrt[p]{x^n - y_n^p}\right)}{\lambda g(y_n)}
\] (2.16)
for all \( x \in \mathbb{R} \).

First, replace \( (x, y) \) by \( (x, \sqrt[p]{y^p + y_n^p}) \) and replace \( (x, y) \) by \( (x, \sqrt[p]{y^p - y_n^p}) \) for the case \( \varphi(x) \) in (2.13). Thereafter we go through the same procedure as in (2.4) \( \sim \) (2.6) of Theorem 2.4. Then we obtain
\[
\left| f\left(\sqrt[p]{x^n + y^p + y_n^p}\right) + f\left(\sqrt[p]{x^n + y^p - y_n^p}\right) \right|
\]
\[
\quad + \frac{f\left(\sqrt[p]{x^n - y^p + y_n^p}\right) + f\left(\sqrt[p]{x^n - y^p - y_n^p}\right)}{\lambda g(y_n)}
\]
\[
\quad - \lambda f(x)g\left(\sqrt[p]{y^p + y_n^p} + g\left(\sqrt[p]{y^p - y_n^p}\right)\right| \leq \frac{2\varphi(x)}{\lambda g(y_n)}.
\] (2.17)
Since the right-hand side of the inequality converges to zero as $n \to \infty$ in (2.17), by applying (i)'s result and (2.16), (2.17) implies that $f$ and $g$ satisfy (W₂)

Finally, for (K₂), we also apply the same procedures as above.

Replace $(x,y)$ by $(\sqrt[p]{x^p+y_n^p}, y_n)$ and replace $(x,y)$ by $(y, \sqrt[p]{y^p - y_n^p})$ for the case $\varphi(y)$ in (2.13). As above, let us go through the same procedure as in (2.16) ~ (2.17), then we obtain

$$\left| f \left( \sqrt[p]{x^p + y^p + y_n^p} \right) + f \left( \sqrt[p]{x^p + y^p - y_n^p} \right) \right| \lambda g(y_n)$$
$$+ \left| f \left( \sqrt[p]{x^p - y^p + y_n^p} \right) + f \left( \sqrt[p]{x^p - y^p - y_n^p} \right) \right| \lambda g(y_n)$$
$$- \lambda g(\sqrt[p]{x^p + y_n^p}) + g(\sqrt[p]{x^p - y_n^p}) f(y) \leq \frac{2 \varphi(y)}{\lambda g(y_n)}. \quad (2.18)$$

Taking the limit as $n \to \infty$ in (2.18), applying (i)'s result and (2.16) and (2.18), we obtain the required result that $f$ and $g$ satisfy (K₂). \Box

Notice that, in Theorems 2.4 and 2.5, the second term $\varphi(x)$ and $\varphi(y)$ of (ii) in (2.1) and (2.13) can be replaced by the fact that $g$ satisfies (C₂), respectively.

The following corollaries follow immediate from Theorems 2.4 and 2.5.

**Corollary 2.6.** Assume that $f, g : \mathbb{R} \to \mathbb{C}$ satisfy the inequality

$$|f \left( \sqrt[p]{x^p + y^p} \right) + f \left( \sqrt[p]{x^p - y^p} \right) - \lambda g(x) f(y)| \leq \varepsilon.$$  

Then

(i) either $f$ is bounded or $g$ satisfies (C₂),

(ii) either $g$ or $f$ is bounded or $g$ satisfies (C₂), also $f$ and $g$ satisfy (K₂) and (W₂).

**Corollary 2.7.** Assume that $f, g : \mathbb{R} \to \mathbb{C}$ satisfy the inequality

$$|f \left( \sqrt[p]{x^p + y^p} \right) + f \left( \sqrt[p]{x^p - y^p} \right) - \lambda f(x) g(y)| \leq \varepsilon.$$  

Then

(i) either $f$ is bounded or $g$ satisfies (C₂),

(ii) either $g$ or $f$ is bounded or $g$ satisfies (C₂), also $f$ and $g$ satisfy (K₂) and (W₂).

**Corollary 2.8.** Assume that $f, g : \mathbb{R} \to \mathbb{C}$ satisfy the inequality

$$|f \left( \sqrt[p]{x^p + y^p} \right) + f \left( \sqrt[p]{x^p - y^p} \right) - \lambda f(x) f(y)| \leq \begin{cases} (i) \varphi(x), \\ (ii) \varphi(y), \\ (iii) \varepsilon. \end{cases}$$  

Then either $f$ is bounded or $f$ satisfies (C₂).

**Remark 2.9.** In results, letting $p = 1$ or $\lambda = 2$, one can obtain (C₁), (M₁), (M₀), (1.1), (1.2), (1.3) (C₂), (W₂), (K₂). Hence they can be applied to stability results of cosine, Wilson, Kim, trigonometric functional equations, etc. See Badora [12], Badara and Ger [13], Baker [14], Fassi, et al. [15], Kannappan and Kim [16], Kim [17], [18], [19], [20], [21], [22], and Almahalebi, et al. [23]. Letting $p = 2, 3, 4$ and $\lambda = 1, 2$, we can obtain the other functional equations. If the obtained results can be extend to them, then it will be applied similarly to stability results.
3. Applications of the case $\tilde{f}(x) := f(x)f(0)^{-1}$ in $(W^p_2)$ and $(K^p_2)$

Let $\tilde{f}(x) := f(x)f(0)^{-1}$. The following lemmas show that similar arguments hold without assuming the continuity. To make it easy to write, we continue using this notation $\tilde{f}$ and note that it is legal only when $f(0) \neq 0$.

The following lemmas can be easy to check.

**Lemma 3.1.** Let $f : \mathbb{R} \to \mathbb{C}$ be a function satisfying

$$f\left(\sqrt[p]{x^p + y^p}\right) + f\left(\sqrt[p]{x^p - y^p}\right) = \lambda f(x)f(y)$$

for all $x, y \in \mathbb{R}$. If $f$ is an even function such that $f(0) \neq 0$, then $\tilde{f}$ satisfies $(C_4)$.

**Lemma 3.2.** Let $f, g : \mathbb{R} \to \mathbb{C}$ be functions satisfying

$$f\left(\sqrt[p]{x^p + y^p}\right) + f\left(\sqrt[p]{x^p - y^p}\right) = \lambda f(x)g(y)$$

for all $x, y \in \mathbb{R}$. If $f$ is an even function such that $f(0) \neq 0$, then $\tilde{f}$ satisfies $(C_4)$.

**Lemma 3.3.** Let $f, g : \mathbb{R} \to \mathbb{C}$ be functions satisfying

$$f\left(\sqrt[p]{x^p + y^p}\right) + f\left(\sqrt[p]{x^p - y^p}\right) = \lambda g(x)f(y)$$

for all $x, y \in \mathbb{R}$. Then, for $f(0) \neq 0$, $\tilde{f}$ satisfies $(C_4)$.

**Theorem 3.4.** Assume that $f, g : \mathbb{R} \to \mathbb{C}$ satisfy the inequality

$$|f\left(\sqrt[p]{x^p + y^p}\right) + f\left(\sqrt[p]{x^p - y^p}\right) - \lambda g(x)f(y)| \leq \begin{cases} (i) \varphi(x) \\ (ii) \varphi(y) \end{cases} \quad \varphi \text{ and } \varphi \text{ (3.1)}$$

(i) If $f$ is unbounded, then $\tilde{g}$ satisfies $(C_4)$.

(ii) If $g(\text{ or } f)$ is unbounded, then $\tilde{f}$ and $\tilde{g}$ satisfy $(C_4)$.

**Proof.** (i) It follows trivially from Theorem 2.4 (i) and Lemma 3.3.

(ii) Assume that $g(\text{ or } f)$ is unbounded. Then $f$ is unbounded. By (i), $\tilde{g}$ satisfies $(C_4)$. From Theorem 2.4 (ii), $f$ and $g$ satisfy $(K^p_4)$ and $(W^p_2)$. By Lemma 3.3, $\tilde{f}$ satisfies $(C_4)$. □

**Theorem 3.5.** Assume that $f, g : \mathbb{R} \to \mathbb{C}$ satisfy the inequality

$$|f\left(\sqrt[p]{x^p + y^p}\right) + f\left(\sqrt[p]{x^p - y^p}\right) - \lambda f(x)g(y)| \leq \begin{cases} (i) \varphi(y) \\ (ii) \varphi(x) \end{cases} \quad \varphi \text{ and } \varphi \text{ (3.1)}$$

(i) If $f$ is unbounded, then $\tilde{g}$ satisfies $(C_4)$.

(ii) If $g(\text{ or } f)$ is unbounded, then $\tilde{f}$ and $\tilde{g}$ satisfy $(C_4)$.

**Proof.** (i) It follows trivially from Theorem 2.5 (ii) and Lemma 3.2.

(ii) Assume that $g(\text{ or } f)$ is unbounded. Then $f$ is unbounded. By (i), $\tilde{g}$ satisfies $(C_4)$. From Theorem 2.5 (ii), $f$ and $g$ satisfy $(K^p_4)$ and $(W^p_2)$. By Lemma 3.3, $\tilde{f}$ satisfies $(C_4)$. □
Corollary 3.6. Assume that $f, g : \mathbb{R} \to \mathbb{C}$ satisfy the inequality

$$|f(\sqrt{x^p} + y^p) + f(\sqrt{x^p} - y^p) - \lambda g(x)f(y)| \leq \varepsilon,$$

$$|f(\sqrt{x^p} + y^p) + f(\sqrt{x^p} - y^p) - \lambda f(x)g(y)| \leq \varepsilon.$$

(i) If $f$ is unbounded, then $\tilde{g}$ satisfies $(C)$.

(ii) If $g(\text{or } f)$ is unbounded, then $\tilde{f}$ and $\tilde{g}$ satisfy $(C')$.

Corollary 3.7. Assume that $f : \mathbb{R} \to \mathbb{C}$ satisfies the inequality

$$|f(\sqrt{x^p} + y^p) + f(\sqrt{x^p} - y^p) - \lambda f(x)f(y)| \leq \begin{cases} (i) \varphi(x) \\ (ii) \varphi(y) \\ (iii) \varepsilon \end{cases}$$

If $f$ is unbounded, then $\tilde{f}$ satisfies $(C').$

Remark 3.8. As Remark 2.9, letting $p = 1$ or $\lambda = 2$, we obtain some results, which are the results given in Fassie[11].

4. Extension to Banach algebras

In this section, we will extend our main results to Banach algebras.

Theorem 4.1. Let $(E, ||\cdot||)$ be a semisimple commutative Banach algebra. Assume that $f, g : \mathbb{R} \to E$ satisfy the inequality

$$||f(\sqrt{x^p} + y^p) + f(\sqrt{x^p} - y^p) - \lambda g(x)f(y)\|| \leq \begin{cases} (i) \varphi(x) \\ (ii) \varphi(y) \text{ and } \varphi(x) \end{cases}$$

Let $z^* \in E^*$ be an arbitrary linear multiplicative functional.

(i) If $z^* \circ f$ is unbounded, then $g$ satisfies $(C')$.

(ii) If $z^* \circ g$ or $z^* \circ f$ is unbounded, then $g$ satisfies $(C')$, and $f$ and $g$ satisfy $(K')$ and $(W')$.

Proof. Assume that (4.1) holds and let $z^* \in E^*$ be a linear multiplicative functional. Since $||z^*|| = 1$, for all $x, y \in \mathbb{R}$, we have

$$\varphi(x) \geq ||f(\sqrt{x^p} + y^p) + f(\sqrt{x^p} - y^p) - \lambda g(x)f(y)|| = \sup_{||w|| = 1} |w^*(f(\sqrt{x^p} + y^p) + f(\sqrt{x^p} - y^p) - \lambda g(x)f(y))| \geq |z^*(f(\sqrt{x^p} + y^p)) + z^*(f(\sqrt{x^p} - y^p)) - \lambda \cdot z^*(g(x)) \cdot z^*(f(y))|,$$

which states that the superpositions $z^* \circ f$ and $z^* \circ g$ yield solutions of the inequality (2.1) in Theorem 2.4.

Hence we can apply to Theorem 2.4 (i).

(i) Since, by assumption, the superposition $z^* \circ f$ is unbounded, an appeal to Theorem 2.4 shows that the superposition $z^* \circ g$ is a solution of $(C')$, that is,

$$(z^* \circ g)(\sqrt{x^p} + y^p) + (z^* \circ g)(\sqrt{x^p} - y^p) = \lambda(z^* \circ g)(x)(z^* \circ g)(y).$$
Since \( z^* \) is a linear multiplicative functional, we get
\[
z^* \left( g\left( \sqrt{x^p + y^p} \right) + g\left( \sqrt{x^p - y^p} \right) - \lambda g(x)g(y) \right) = 0.
\]

Hence an unrestricted choice of \( z^* \) implies that
\[
g\left( \sqrt{x^p + y^p} \right) + g\left( \sqrt{x^p - y^p} \right) - \lambda g(x)g(y) \in \bigcap \{ \ker z^* : z^* \in E^* \}.
\]

Since \( E \) is a semisimple Banach algebra, \( \bigcap \{ \ker z^* : z^* \in E^* \} = 0 \), which means that \( g \) satisfies the claimed equation \((\text{C}_4)\).

(ii) By assumption, the superposition \( z^* \circ g \) is unbounded, an appeal to Theorem 2.4 shows that the results hold.

From a similar process as in \((\text{2.8})\) of Theorem 2.4, we can show that the unboundedness of the superposition \( z^* \circ g \) implies the unboundedness of the superposition \( z^* \circ f \).

First, it follows from the above result (i) that \( g \) satisfies the claimed equation \((\text{C}_4)\).

Next, an appeal to Theorem 2.4 shows that \( z^* \circ f \) and \( z^* \circ g \) are solutions of the equations \((\text{K}_4)\) and \((\text{W}_4)\), that is,
\[
(z^* \circ f)\left( \sqrt{x^p + y^p} \right) + (z^* \circ f)\left( \sqrt{x^p - y^p} \right) = \lambda (z^* \circ g)(x)(z^* \circ f)(y),
\]
\[
(z^* \circ f)\left( \sqrt{x^p + y^p} \right) + (z^* \circ f)\left( \sqrt{x^p - y^p} \right) = \lambda (z^* \circ f)(x)(z^* \circ g)(y).
\]

This means by a linear multiplicativity of \( z^* \) that the differences
\[
\mathcal{D}K^\lambda (x, y) := f\left( \sqrt{x^p + y^p} \right) + f\left( \sqrt{x^p - y^p} \right) - \lambda g(x)f(y),
\]
\[
\mathcal{D}W^\lambda (x, y) := f\left( \sqrt{x^p + y^p} \right) + f\left( \sqrt{x^p - y^p} \right) - \lambda f(x)g(y)
\]
fall into the kernel of \( z^* \). That is, \( z^* (\mathcal{D}K^\lambda (z, w)) = 0 \) and \( z^* (\mathcal{D}W^\lambda (z, w)) = 0 \).

Hence an unrestricted choice of \( z^* \) implies that
\[
\mathcal{D}K^\lambda (x, y), \mathcal{D}W^\lambda (x, y) \in \bigcap \{ \ker z^* : z^* \in E^* \}.
\]

Since the algebra \( E \) is semisimple, \( \bigcap \{ \ker z^* : z^* \in E^* \} = 0 \), which means that \( f \) and \( g \) satisfy the claimed equations \((\text{K}_4)\) and \((\text{W}_4)\). \(\square\)

By a similar procedure, we can prove the next theorem as an extension of Theorem 2.5. So we will skip the proof.

**Theorem 4.2.** Let \((E, \| \cdot \|)\) be a semisimple commutative Banach algebra. Assume that \( f, g : \mathbb{R} \to E \) satisfy the inequality
\[
\| f\left( \sqrt{x^p + y^p} \right) + f\left( \sqrt{x^p - y^p} \right) - \lambda f(x)g(y) \| \leq \left\{ \begin{array}{ll}
(i) \varphi(y) \\
(ii) \varphi(x) \text{ and } \varphi(y).
\end{array} \right.
\]

Let \( z^* \in E^* \) be an arbitrary linear multiplicative functional.

(i) If \( z^* \circ f \) is unbounded, then \( g \) satisfies \((\text{C}_4)\).

(ii) If \( z^* \circ g \) (or \( z^* \circ f \)) is unbounded, then \( g \) satisfies \((\text{C}_4)\), and \( f \) and \( g \) satisfy \((\text{K}_4)\) and \((\text{W}_4)\).

**Corollary 4.3.** Let \((E, \| \cdot \|)\) be a semisimple commutative Banach algebra. Assume that \( f, g : \mathbb{R} \to E \) satisfy the inequality
\[
\| f\left( \sqrt{x^p + y^p} \right) + f\left( \sqrt{x^p - y^p} \right) - \lambda g(x)f(y) \| \leq \varepsilon.
\]

Let \( z^* \in E^* \) be an arbitrary linear multiplicative functional.

(i) If \( z^* \circ f \) is unbounded, then \( g \) satisfies \((\text{C}_4)\).

(ii) If \( z^* \circ g \) (or \( z^* \circ f \)) is unbounded, then \( g \) satisfies \((\text{C}_4)\), and \( f \) and \( g \) satisfy \((\text{K}_4)\) and \((\text{W}_4)\).
Corollary 4.4. Let \( (E, \|\cdot\|) \) be a semisimple commutative Banach algebra. Assume that \( f, g : \mathbb{R} \to E \) satisfy the inequality
\[
\| f \left( \sqrt[p]{x^p + y^p} \right) + f \left( \sqrt[p]{x^p - y^p} \right) - \lambda f(x)g(y) \| \leq \varepsilon.
\]
Let \( z^* \in E^* \) be an arbitrary linear multiplicative functional.

(i) If \( z^* \circ f \) is unbounded, then \( g \) satisfies \((E^2)\).

(ii) If \( z^* \circ g \) (or \( z^* \circ f \)) is unbounded, then \( g \) satisfies \((E_1^2)\), and \( f \) and \( g \) satisfy \((E_2^2)\) and \((W_2^2)\).

Corollary 4.5. Let \( (E, \|\cdot\|) \) be a semisimple commutative Banach algebra. Assume that \( f, g : \mathbb{R} \to E \) satisfy the inequality
\[
\| f \left( \sqrt[p]{x^p + y^p} \right) + f \left( \sqrt[p]{x^p - y^p} \right) - \lambda f(x)g(y) \| \leq \begin{cases} 
(i) & \varphi(x) \\
(ii) & \varphi(y) \\
(iii) & \varepsilon.
\end{cases}
\]

Then either the superposition \( z^* \circ f \) is bounded for each linear multiplicative functional \( z^* \in E^* \) or \( f \) satisfies \((E_1^2)\).

Remark 4.6. (1) Letting \( p = 1 \) or \( \lambda = 2 \), we can get \((C), (II), (K), (C_2), (1.1), (1.2), (1.3)\). Hence they can be applied to stability results of cosine, Wilson, Kim, trigonometric functional equations. See \[ 2, 3, 4, 7, 11, 14, 15, 20, 21 \].

(2) The results of Section 3 also can be extended to Banach algebras. By applying \( p = 1 \) or \( \lambda = 2 \), some results can be derived.

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References