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Analysis of a harvested discrete-time biological models

Sadiq Al-Nassir[a,*], Alaa Hussein Lafta[a]

[a]Department of Mathematics, College of Science, University of Baghdad, Baghdad, Iraq

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Abstract

This work aims to analyze a three-dimensional discrete-time biological system, a prey-predator model with a constant harvesting amount. The stage structure lies in the predator species. This analysis is done by finding all possible equilibria and investigating their stability. In order to get an optimal harvesting strategy, we suppose that harvesting is to be a non-constant rate. Finally, numerical simulations are given to confirm the outcome of mathematical analysis.

Keywords: Prey-predator model; Optimal control; Stage structure model; optimal Harvesting.
2010 MSC: 92D25; 39D20; 39A30;49J15;

1. Introduction

In the real world, the individuals of many species have a life cycle which goes through two or more than two stages, namely the non-reproducing stage (Juvenile stage) and the reproducing stage(adult stage). In recent years, many authors have investigated prey-predator systems that contain stage-structure in prey or in predator species. Other researchers have studied age-structure models [1, 8, 20, 21, 23]. Mathematical models are used to understand the interactions or fluctuations among different species in the real life. These models can be formulated by a set of differential equations or a set of difference equations or partial differential equations as well as by a fractional-order derivative. In discrete-time, models are more suitably used to describe the life of the population than continuous-time models due to their efficiency for computation and numerical simulations [6, 14]. Harvesting is an important issue in managing the renew resources, so that many researchers have studied different harvesting strategies in their systems [9, 15, 18, 19]. In [3], authors investigated a stage-structured system of one species growth consisting of immature and mature members. In

*Corresponding author
Email addresses: sadiq.n@sc.uobaghdad.edu.iq (Sadiq Al-Nassir), alaa.h@sc.uobaghdad.edu.iq (Alaa Hussein Lafta)

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[5] Cui et al. analyzed and studied a stage-structured single-species population model without time delay. In this work, we are concerned with discrete prey-predator model with stage-structured in the predator species. we also study the influence of constant harvesting on the dynamics of the model, then the system is extended to the optimal control problem to find the optimal harvesting policy. We use Pontraygin's maximum principle to solve the optimal control problem for more details see [2, 4, 12, 16, 17]. This work is divided into five sections: In section 2 the mathematical model is formulated and its behavior dynamics are discussed. In section 3 the optimal control problem is considered and the optimal solution is got. The numerical results have presented in section 4. Finally, the conclusion is presented in the last section.

2. The model and the stability of its fixed points

We investigated the dynamics behavior of a three dimensional discrete time prey-predator model with stage-structured in predator species. The model is as follows:

$$\begin{aligned} U_{t+1} &= U_t(r - AU_t) - \frac{a_1 U_t W_t}{1 + b_1 U_t} - hU_t \\ V_{t+1} &= eW_t - (r_1 + D)V_t \\ W_{t+1} &= DV_t - r_2 W_t + \frac{a_2 U_t W_t}{1 + b_1 U_t} \end{aligned} \tag{2.1}$$

Where $U_t, V_t,$ and W_t represent the size of the prey species, the size of juvenile (immature) predator, and the size of adult (mature) predator species at time t respectively. The model parameters $r, A, a_1, b_1, h, e, r_1, D, r_2,$ anda_2 are assumed only positive values in which r the increase rate of prey species, the parameter A measures the intensity of competition among individuals of prey species a_1 denotes the capturing rate of the predator, a_2 represents the rate of conversing prey species that becomes adult predator species, b_1 measures the protection to prey species. The parameter h is the rate of harvesting such that $0 \leq h < 1$. The parameter e denotes to the birth rate of predator, r_1 is the death rate of the juvenile predator , D is the rate of juvenile becoming adult predator, r_2 refers to the mortality rate of adult predator. To reduce the number of parameters of the model (2.1), we use following dimensionless variables $x_t = AU_t, y_t = DV_t$ and $z_t = W_t$. Therefore the model(2.1) becomes :

$$\begin{aligned} x_{t+1} &= x_t(r - x_t) - \frac{a x_t z_t}{1 + b x_t} - h x_t \\ y_{t+1} &= c z_t - (r_1 + D) y_t \\ z_{t+1} &= y_t - r_2 z_t + \frac{f x_t z_t}{1 + b x_t} \end{aligned} \tag{2.2}$$

Where $a = a_1, b = \frac{b_1}{A}, c = De,$ and$f = \frac{a_2}{A}$ To find the fixed points of the model (2.2), we have to solve the following algebraic system:

$$\begin{aligned} x &= x(r - x) - \frac{axz}{1 + bx} - hx \\ y &= cz - (r_1 + D)y \\ z &= y - r_2 z + \frac{fxz}{1 + bx} \end{aligned}$$

Therefore we have the next Theorems:

Theorem 2.1. *The model (2.2) fixed points are*

1 - The trivial fixed point $e_0 = (0, 0, 0)$*, which always exists.*

2 -The boundary fixed point $e_1 = (x_1, 0, 0)$*, where* $x_1 = r - (1 + h)$*, which exists only when* $r > (1 + h)$*.*

3 -The fixed point $e_2 = (0, \frac{c}{1+r_1+D}s, s)$ *where* $s > 0$*, which exists only when* $c = (1+r_2)(1+r_1+D)$*.*

4 -The unique interior fixed point $e_3 = (X, Y, Z)$ *exists if the following conditions hold:*
i)- $c > r_2m+m$*, and* $bm(1+r_2) > (fm+cbm)$*. or* $c < r_2m+m$*, and* $bm(1+r_2) < (fm+cbm)$*.*
ii) $-\ 0 < X < r - h - 1$*. where* $X = \frac{(c-r_2m-m)}{(bm(1+r_2)-(fm+cb))}$, $Y = \frac{c}{m}Z$, $Z = \frac{(k(r-X-h-1))}{a}$, $m = 1 + r_1 + D$*, and* $k = 1 + bX$*.*

We will use the linearization procedure to study the behavior of all possible fixed points of model(2.2), around each one. This will be done by computation the Jacobian matrix of model(2.2) at point (x, y, z). This is given as follows:

$$J\left(x,y,z\right)=\begin{bmatrix} r-2x-\frac{az}{k^2}-h & 0 & -\frac{ax}{k} \\ 0 & -(r_1+D) & c \\ \frac{fz}{k^2} & 1 & -r_2+\frac{fx}{k} \end{bmatrix}$$

We have to look at $F(\lambda) = \lambda^3 + a_2\lambda^2 + a_1\lambda + a_0$, which represents the characteristic polynomial of $J(x, y, z)$ that evaluated at fixed point. The sufficient and necessary conditions to a fixed point to have all the eigenvalues inside the unit circle are[13]:

$$\begin{aligned} &1)\ |a_2+a_0| < a_1+a_0 \\ &2)\ |a_2-3a_0| < 3-a_1 \\ &3)\ \left|a_0^2+a_1-a_0a_2\right| < 1 \end{aligned} \tag{2.3}$$

For the two dimension system: $\overline{x}_{t+1} = A_{2x2}\overline{x}_t \quad , \ t \in Z^+$. The next lemma is needed.

Lemma 2.2. *[7]: Let* $f(\lambda) = \lambda^2 + p\lambda + q$*. Assume that* $f(1) > 0$*, and* $\lambda_1,\ \lambda_2$ *are the roots of f then:*
1-$|\lambda_1| < 1$ *and* $|\lambda_2| < 1$ *if and only if* $f(-1) > 0$ *and* $q < 1$
2-$|\lambda_1| > 1$ *and* $|\lambda_2| < 1$*(or* $|\lambda_1| < 1$ *and* $|\lambda_2| > 1$ *) if and only if* $f(-1) < 0$
3-$|\lambda_1| > 1$ *and* $|\lambda_2| > 1$*if and only if* $f(-1) > 0$ *and* $q > 1$
4-$\lambda_1 = -1$ *and* $|\lambda_2| \neq 1$ *if and only if* $f(-1) = 0$ *and* $p \neq 0, 2$

For the proof see[7].

The local stability of the fixed points e_0, and e_1, are given by next theorems.

Theorem 2.3. *The fixed point* $e_0 = (0, 0, 0)$ *of the model (2.2) is :*
1)Sink (stable) point if $c \in (k_3, Min\{k_1, k_2\})$ *when* $k_3 < Min\{k_1, k_2\}$ *and* $r < 1 + h$*.*

2) Source point if $c < Min\{k_1, k_2,\ k_3\}$ *and* $r > 1 + h$.

3) Saddle point if one of the conditions holds:

i) $c \in (k_3, Min\{k_1, k_2\})$, *when* $k_3 < Min\{k_1, k_2\}$ *and* $r > 1 + h$.

ii) $k_2 < c < k_1$ *and* $r < 1 + h$.

4) Non-hyperbolic point if one of the conditions holds:

i) $r = 1 + h$.

ii) $c = k_2$ *and* $r_1 + D + r_2 \neq 2$

where $k_1 = (1 + r_2)m$, $k_2 = (1 - r_2)(1 - r_1 - D)$, $k_3 = r_2(r_1 + D) - 1$ *and* $m = 1 + r_1 + D$.

Proof . The Jacobain matrix of the system (2.2) at e_0 is

$$J_{e_0} = \begin{bmatrix} r-h & 0 & 0 \\ 0 & -(r_1+D) & c \\ 0 & 1 & -r_2 \end{bmatrix}$$

Therefore the characteristic polynomial of J_{e_0} is $F(\lambda) = (r - h - \lambda)(\lambda^2 + p\lambda + q) = (r - h - \lambda)f(\lambda)$ where $p = r_1 + D + r_2$ and $q = r_1r_2 + Dr_2 - c$. Now $f(1) > 0$ if and only if $k_1 > c$ and $f(1) > 0$ if and only if $k_2 > c$. It is clear that $q < 1$ if and only if $k_3 < c$. Therefore if $c \in (k_3, Min\{k_1, k_2\})$, then by Lemma (2.2)(1) we have $|\lambda_i| < 1,\ i = 1, 2,$ and $|\lambda_3| < 1$ if and only if $r < 1 + h$. Therefore the point e_0 is locally stable if the condition(1)holds. The proof of 2 , 3, and 4 can be easily obtained from lemma (2.2). □

Theorem 2.4. *The fixed point* $e_1 = (r - h - 1, 0, 0)$ *of the system (2.2) is:*

1) Stable (sink) point if $c \in (m_3, Min\{m_1, m_2\})$, *when* $m_3 < Min\{m_1, m_2\}$ *and* $1+h < r < 3+h$.

2) Source point if $c < Min\{m_1, m_2,\ m_3\}$ *and* $r < 1 + h$ *or* $r > 3 + h$.

3) Saddle point if one of the conditions holds:

i) $c \in (m_3, Min\{m_1, m_2\})$, *when* $m_3 < Min\{m_1, m_2\}$ *and* $r < 1 + h$ *or* $r > 3 + h$.

ii) $m_2 < c < m_1$ *and* $1 + h < r < 3 + h$.

4) Non-hyperbolic point if one of the conditions holds:

i) $r = 1 + h$, *or* $r = 3 + h$.

ii) $c = m_2$ *and* $r_1 + D + r_2 \neq 2$.

Where $m_1 = (1 + r_2 - \frac{fx_1}{k})m$, $m_2 = (1 - r_2)(1 - r_1 - D + \frac{fx_1}{k})$ *and* $m_3 = (r_2 - \frac{fx_1}{k})(r_1 + D) - 1$.

Proof . The Jacobain matrix of the system (2.2) at e_1 is

$$J_{e_1} = \begin{bmatrix} 2-r+h & 0 & -\frac{ax_1}{k} \\ 0 & -(r_1+D) & c \\ 0 & 1 & -r_2+\frac{fx_1}{k} \end{bmatrix}$$

Therefore the characteristic polynomial of J_{e_1} is $F(\lambda) = (2-r+h-\lambda)(\lambda^2+p\lambda+q) = (2-r+h-\lambda)f(\lambda)$ where $p = r_1+D+r_2-\frac{fx_1}{k}$,and $q = r_1r_2+Dr_2-\frac{r_1fx_1}{k}-\frac{Dfx_1}{k}-c$. Hence

$f(1) > 0$ if and only if $m_1 > c$ and $f(-1) > 0$ if and only if $m_2 > c$. It is quite clear that $q < 1$ if and only if $m_3 < c$. According to lemma(2.2) the proof of 1 , 2, 3 and 4 can be easily obtained. □

Theorem 2.5. *For the fixed point e_2 we have the following:*

1 The fixed point e_2 is never to be sink point

2 It is saddle point if $as+h-1 < r < 1+as+h$.

3 The fixed point e_2 is non-hyperbolic point if $r = as+h-1$ or $r = 1+as+h$.

Where $s > 0$.

Proof . The Jacobain matrix of the fixed point e_2 is given by

$$J_{e_2} = \begin{bmatrix} r-as-h & 0 & 0 \\ 0 & -(r_1+D) & c \\ \frac{fs}{k} & 1 & -r_2 \end{bmatrix} = \begin{bmatrix} r-as-h & 0 & 0 \\ 0 & -(r_1+D) & (1+r_2)(1+r_1+D) \\ fs & 1 & -r_2 \end{bmatrix}.$$

So that the characteristic polynomial of J_{e_2} is $F(\lambda) = (r-as-h-\lambda)(\lambda^2+p\lambda+q) = (r-as-h-\lambda)f(\lambda)$ where $p = r_1+D+r_2$ and $q = r_1r_2+Dr_2-(1+r_2)(1+r_1+D), s > 0$. We can see that $f(1)$ is always equal to 0 . Hence by Lemma2.2, we always have at least one root lies outside unit circle then the fixed point e_2 is never to be sink point, while it is saddle point if $as+h-1 < r < 1+as+h$, and the point is non-hyperbolic point if $r = as+h-1$ or $r = 1+as+h$. □ The next Theorem gives the local stability of the unique positive fixed point $e_3 = (X, Y, Z) = (\frac{c-r_2m-m}{bm(1+r_2)-(fm+cb)}, \frac{c}{m}Z , \frac{k(r-X-h-1)}{a}$) where $m = 1+r_1+D$ and $k = 1+bX$.

Theorem 2.6. *The unique interior fixed point e_3 is locally stable if the following inequalities hold ;*

1) $M_1-(w_1+w_2-1-3c+3w_2w_1)w_3 < 0$, and $(-w_1-w_2-1-3c+3w_2w_1)w_3-M_2 < 0$.

2) $S_1-(-w_1-w_2-1+c-w_2w_1)w_3 < 0$, and $(-(w_1+w_2)-1+c-w_2w_1)w_3-S_2 < 0$.
3) $N_1-N_2 <1$.

Where $M_1 = -3+w_2w_1+\frac{aZ(w_1+w_2)}{k^2}-c+\frac{aXfZ}{k^3}-(w_1+w_2)-\frac{aZ}{k^2}+\frac{3aZw_1w_2}{k^2}-\frac{3acZ}{k^2}+\frac{3afXZw_2}{k^3}$, $M_2 = 3-w_2w_1-\frac{aZ(w_1+w_2)}{k^2}+c-\frac{aXfZ}{k^3}-(w_1+w_2)-\frac{aZ}{k^2}+\frac{3aZw_1w_2}{k^2}-\frac{3acZ}{k^2}+\frac{3afXZw_2}{k^3}$, $S_1 = -1-w_2w_1-\frac{aZ(w_1+w_2)}{k^2}+c-\frac{aXfZ}{k^3}-(w_1+w_2)-\frac{aZ}{k^2}-\frac{aZw_1w_2}{k^2}+\frac{acZ}{k^2}-\frac{afXZw_2}{k^3}$, $S_2 = (\frac{aZ}{k^2}-1)(w_1+w_2-w_1w_2+1-c)-\frac{aXfZ}{k^3}(1-w_2)$, $w_1 = r_2-\frac{fX}{k}$, $w_2 = r_1+D$, $w_3 = r-2X-h$, $k = 1+bX$, $N_1 = (c-w_1w_2^2z_1^2+(c-w_1w_2)(2z_1-1)-(w_1+w_2)z_1+(\frac{aXfZw_2}{k^3})^2+\frac{aXfZ}{k^3}w_2$, $N_2 = (w_1+w_2)[(c-w_1w_2)z_1+\frac{aXfZw_2}{k^3}]-\frac{aXfZz_1w_2}{k^3}-(c-w_1w_2)z_1^2$, *and* $z_1 = (w_3-\frac{aZ}{k^2})$.

Proof . The Jacobain matrix of the model(2.2) at e_3 is given by

$$J_{e_3} = \begin{bmatrix} r-2X-\frac{aZ}{k^2}-h & 0 & -\frac{aX}{k} \\ 0 & -(r_1+D) & c \\ \frac{fZ}{k^2} & 1 & -r_2+\frac{fX}{k} \end{bmatrix}.$$

Then the characteristic polynomial of J_{e_3} is
$f(\lambda) = \lambda^3 + a_2\lambda^2 + a_1\lambda + a_0$, where $a_2 = w_1+w_2 - w_3 + \frac{aZ}{k^2}$, $a_1 = w_2w_1 - (w_1+w_2)w_3 + \frac{aZ(w_1+w_2)}{k^2} - c + \frac{aXfZ}{k^3}$ and $a_0 = cw_3 - w_1w_2w_3 + \frac{aZw_1w_2}{k^2} - \frac{caZ}{k^2} + \frac{aXfZw_2}{k^3}$.
Assume that (1) holds then $M_1-(w_1+w_2-1-3c+3w_2w_1)w_3 < 0$ this gives $-3+w_2w_1+\frac{aZ(w_1+w_2)}{k^2} - c + \frac{aXfZ}{k^3} - (w_1+w_2) - \frac{aZ}{k^2} + \frac{3aZw_1w_2}{k^2} - \frac{3acZ}{k^2} + \frac{3afXZw_2}{k^3} < (w_1+w_2-1-3c+3w_2w_1)w_3$. By a simple arrangement we get $-3+a_1 < a_2 - 3a_0$ ....(*).
From (1) we also have $(-w_1-w_2-1-3c+3w_2w_1)w_3 - M_2 < 0$, hence $-(w_1+w_2)w_3 - w_3 - 3cw_3 + 3w_2w_1w_3 < 3 - w_2w_1 - \frac{aZ(w_1+w_2)}{k^2} + c - \frac{aXfZ}{k^3} - (w_1+w_2) - \frac{aZ}{k^2} + \frac{3aZw_1w_2}{k^2} - \frac{3acZ}{k^2} + \frac{3afXZw_2}{k^3}$.
Therefore we get $a_2 - 3a_0 < 3 - a_1$ ....(**).
So that by (*) and (**) the condition (2) in equation (2.3) holds.
Assume (2) holds then we have $S_1-(-(w_1+w_2)-1+c-w_2w_1)w_3\ ) < 0$ and $-1-w_2w_1-\frac{aZ(w_1+w_2)}{k^2} + c - \frac{aXfZ}{k^3} - (w_1+w_2) - \frac{aZ}{k^2} - \frac{aZw_1w_2}{k^2} + \frac{acZ}{k^3} - \frac{afXZw_2}{k^3} < -(w_1+w_2)w_3 - w_3 + cw_3 - w_2w_1w_3$.
Therefore $-1-a_1 < a_2 + a_0$ ....(***).
We also have $(-(w_1+w_2)-1+c-w_2w_1)w_3 - S_2 < 0$ this gives $-(w_1+w_2)w_3 - w_3 + cw_3 - w_2w_1w_3 < (\frac{aZ}{k^2}-1)(w_1+w_2-w_1w_2+1-c) - \frac{aXfZ}{k^3}$, hence $a_2 + a_0 < 1 + a_1$ ....(****).
Then from (***), and (****) the condition (1) in equation (2.3) satisfies.
Assume (3) holds then $N_1 = (c-w_1w_2)^2z_1^2 + (c-w_1w_2)(2z_1-1) - (w_1+w_2)z_1 + (\frac{aXfZw_2}{k^3})^2 + \frac{aXfZw_2}{k^3}$, after simple steps we get $N_1 = a_0^2 + a_1$. If $N_2 = (w_1+w_2)((c-w_1w_2)z_1 + \frac{aXfZw_2}{k^3}) - \frac{aXfZz_1w_2}{k^3} - (c - w_1w_2)z_1^2$ then $N_2 = a_0a_2$. So that $N_1 - N_2 < 1$ gives the condition (3) in equation (2.3) satisfies. Therefore the unique fixed point is locally stable. □

3. Optimal Harvesting

The aim of this section is to get optimal harvesting gain so we consider the rate of harvesting is not constant. Thus the system(2.2) is extended to the following system

$$\begin{aligned} x_{t+1} &= x_t(r-x_t) - \frac{ax_tz_t}{1+bx_t} - h_tx_t \\ y_{t+1} &= cz_t - (r_1+D)y_t \\ z_{t+1} &= y_t - r_2z_t + \frac{fx_tz_t}{1+bx_t} \end{aligned} \tag{3.1}$$

The definitions of parameters in system(3.1) are the same as in section 2. We want to maximize the following cost function

$$J(h_t) = \sum_{t=0}^{T-1} c_1h_tx_t - c_2h_t^2 \tag{3.2}$$

Where $c_1h_tx_t$ is the money that we get from harvesting and $c_2h_t^2$ is the cost function. To solve the optimal control problem we use the discrete maximum principle of Pontrygin [4, 11, 22]. Therefore

the Hamiltonian functional for $t = 0, 1, 2, ..., T - 1$ is given by :
$H(t, x_t, y_t, z_t, h_t) = c_1 h_t x_t - c_2 h_t^2 + \lambda_{t+1}[x_t(r - x_t) - \frac{a x_t z_t}{1+b x_t} - h_t x_t] + \mu_{t+1}[c z_t - (r_1 + D) y_t] + V_{t+1}(y_t - r_2 z_t + \frac{f x_t z_t}{1+b x_t})$ where , λ_t μ_t and V_t are the adjoint functions that are known in the literature by shadow price [4]. Therefore the necessary conditions are:

$$\begin{aligned}
\lambda_t &= \frac{\partial H_t}{\partial x_t} c_1 h_t + \lambda_t + 1(r - 2x_t) - \frac{a z_t}{(1 + b x_t)^2} - h_t) + V_{t+1}(\frac{f z_t}{(1 + b x_t)^2}) \\
\mu_t &= \frac{\partial H_t}{\partial y_t} - (r_1 + D)\mu_{t+1} + V_{t+1} \\
V_t &= \frac{\partial H_t}{\partial z_t} \lambda_{t+1}(-\frac{a x_t}{(1 + b x_t)}) + c\mu_{t+1} + V_{t+1}(-r_2 + \frac{f x_t}{1 + b x_t}).
\end{aligned} \tag{3.3}$$

with $\lambda_T = \mu_T = V_T = 0$, and the optimality condition will be:
$\frac{\partial H_t}{\partial h_t} = c_1 x_t - 2c_2 h_t - \lambda_{t+1} x_t = 0$, and $h_t^* = \frac{(c_1 - \lambda_{t+1})x_t}{2c_2}$. Then the characteristic optimal control solution is :

$$h_t^* = Max\{Min\{\frac{(c_1 - \lambda_{t+1})x_t}{2c_2}, M\}, 0\} \quad t = 0, 1, 2, ..., T - 1 \tag{3.4}$$

For solving the optimal problem (3.1)-(3.2), we use an iterative method to compute the optimal harvesting gain for more details see[4, 10, 11, 21]. We choose an initial solution of the control variable with an initial of state variable ,then we solve the state system(3.1) and the adjoint system(3.3) forward and backward respectively , then combine the new result variable with the previous one to update the control variable. This procedure continuous until finding the optimal solutions with corresponding optimal state solutions.

4. Numerical results

To confirm the dynamics of our theoretical findings of the system(2.2), the parametric values have been used as follows: $r = 0.9,\ a = 0.4,\ \ r_1 = 0.01,\ h = 0.02,\ e = 0.1,\ b = 0.5\ D = 0.005, f = 0.3\ \ c = 0.8,\ and\ r_2 = 0.01$. The initial point is $(0.1, 0.1, 0.1)$. According to the Theorem(2.3) the trivial fixed point is locally stable. u For the boundary fixed point e_1, the parametric values are used as $r = 1.5,\ a = 0.45,\ \ r_1 = 0.02,\ h = 0.02,\ e = 0.1,\ b = 0.5\ D = 0.1,\ c = 0.5,\ f = 0.3$ and $r_2 = 0.02$ with$(x_0, y_0, z_0) = (0.4, 0.2, 0.2)$. According to the theorem(2.4) the boundary fixed point e_1 is locally stable. For the unique positive fixed point we use the following parametric values $r = 2.5,\ a = 0.5,\ \ r_1 = 0.1,\ h = 0.1,\ e = 0.1,\ b = 0.45\ D = 0.1,\ \ c = 0.8,\ f = 0.45\ and\ r_2 = 0.02$ with $(x_0, y_0, z_0) = (0.8743, 2.4569, 3.96853)$. By theorem(2.6) the positive fixed point is locally stable. Figure 1, Figure 2, and Figure 3 illustrate the local stability of the trivial fixed point, the boundary fixed point, and the unique positive fixed point respectively. Figure 4 indicates the time series of the prey and the predator species.

For the optimality problem an iterative procedures is applied to solve the control problem for more details see [13]. An initial guess is used for the control then the state equations and the adjoint equations are solved forward and backward respectively, after that the optimal control solution is updated by using the characterization in (3.4) and the newly found adjoint and the state values. The procedures continues until the successive iterates are sufficiently close. We use the following set of parameters values $r = 2.5,\ a = 0.5,\ \ r_1 = 0.1,\ e = 0.1,\ b = 0.4\ D = 0.1,\ \ c = 0.8,\ f = 0.4, c_1 = 0.1, c_2 = 0.1$ and $r_2 = 0.02$ with $(x_0, y_0, z_0) = (1.2, 0.38, 0.57)$. While $h_t = h_t^*$. The optimal harvesting gain J is computed from equation (3.2), and its value is 2.0468. The other constant harvesting strategies are computed as follows $h_t = 0.35,\ 0.36, 0.37,$ and $0.38,$ then the values of $J =$

2.0360, 2.0417, 2.0443 and 2.0438, respectively. Figures 5-6 show the prey and the predator species with control , without control and the constant control harvesting . Figure 7 indicates the optimal control harvesting as function of time.

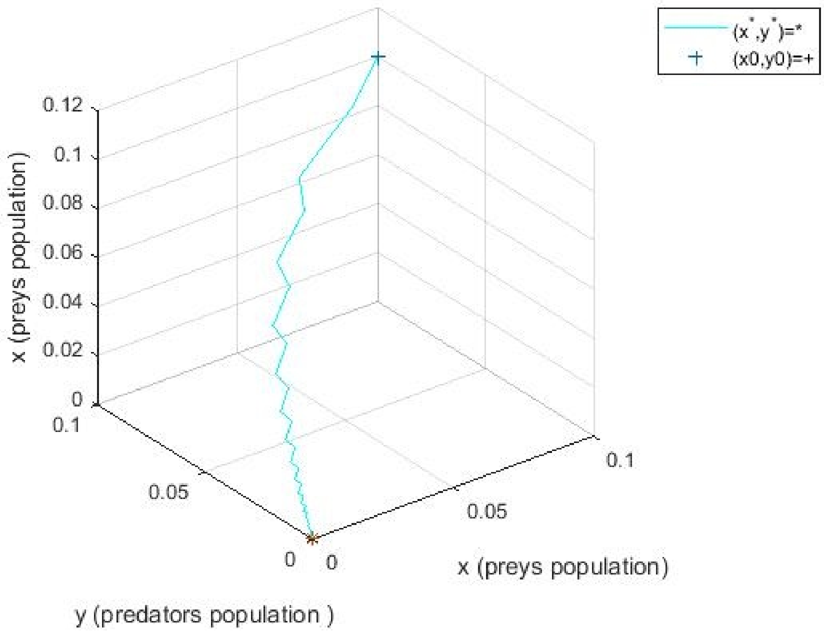

Figure 1: *This figure indicates the local stability of the trivial fixed point* e_1 .

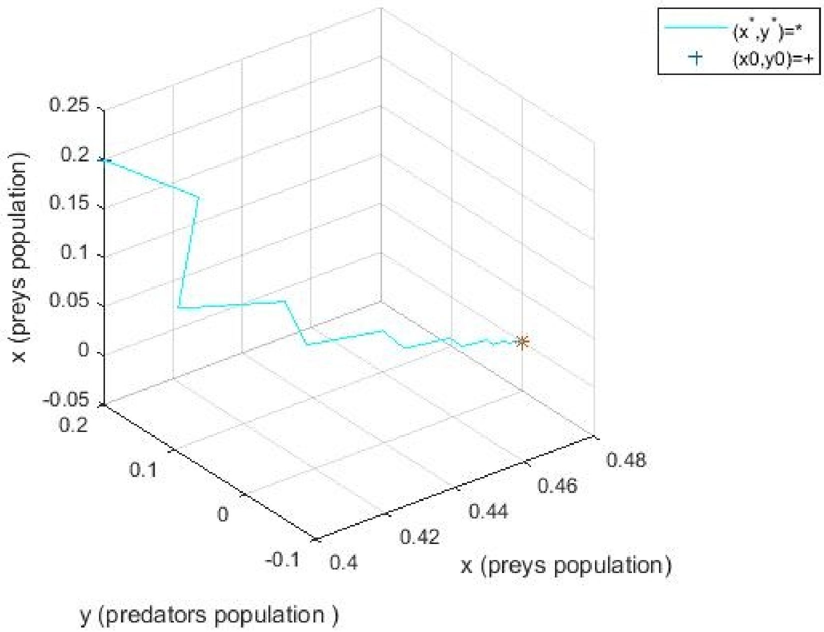

Figure 2: *This figure shows the local stability of the boundary fixed point e_2 ..*

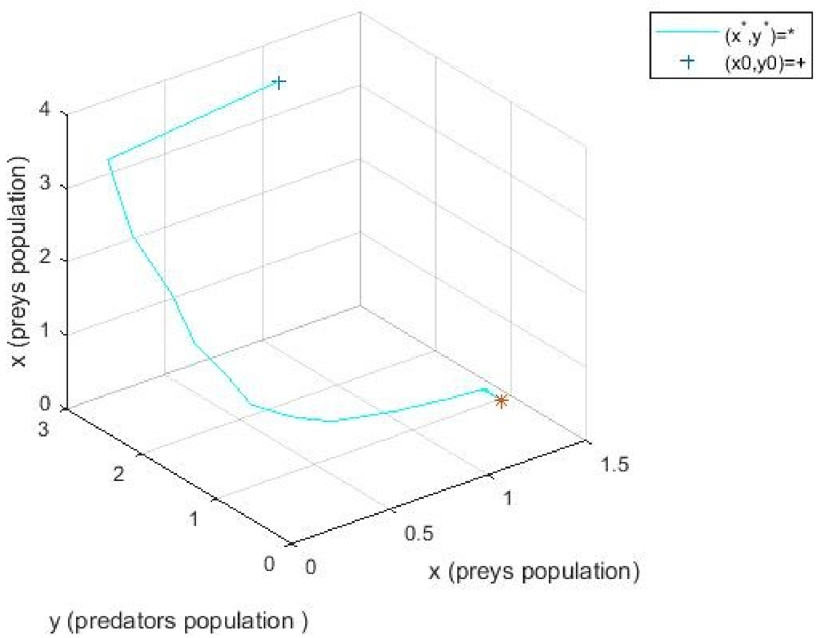

Figure 3: *The local stability of the unique interior fixed point e_3 is shown.*

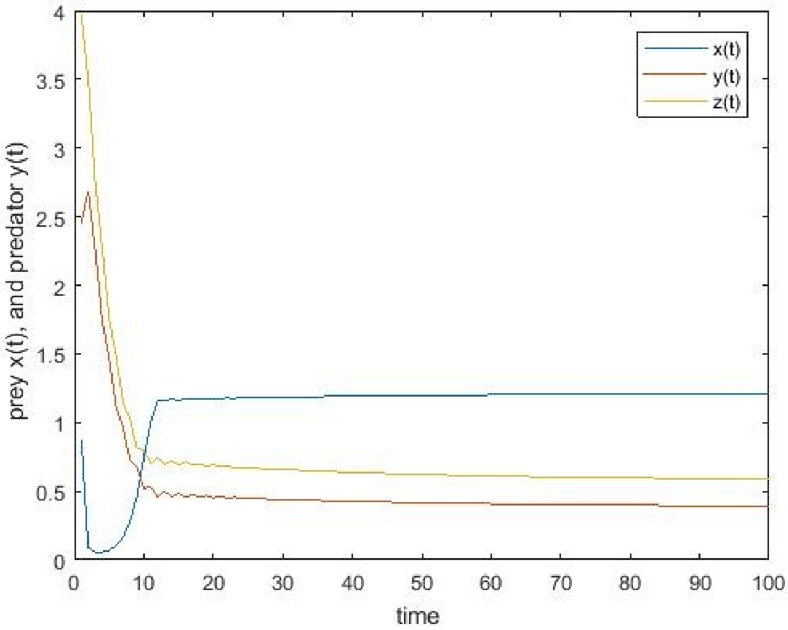

Figure 4: *This figure indicates the time series of the prey and the predator species.*

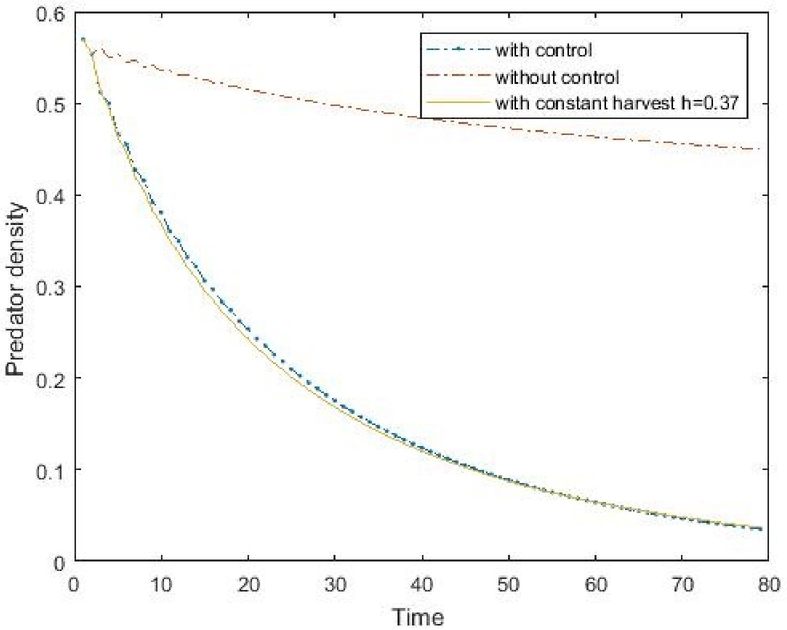

Figure 5: *This figure shows the effect of the optimal harvesting ,constant harvesting on the predator size. Note that we used the same values of parameters.*

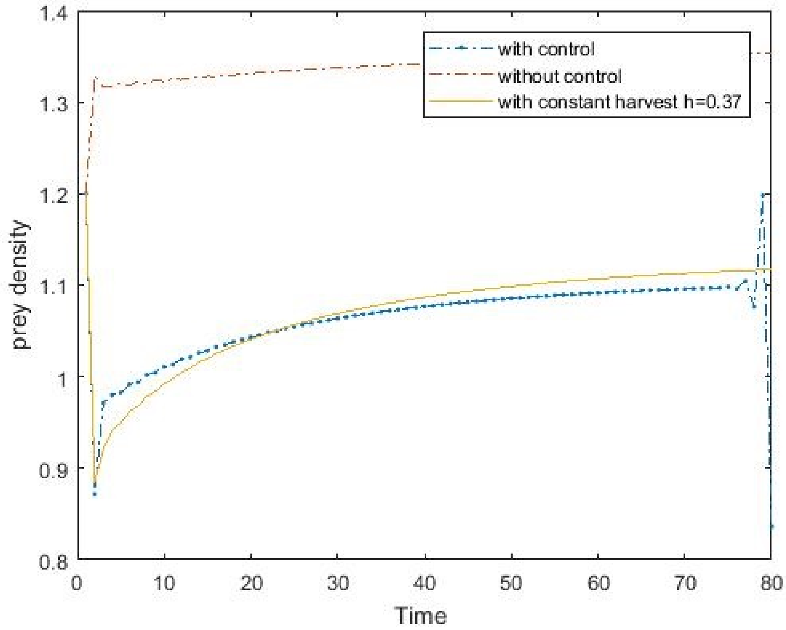

Figure 6: *In this figure the effect of the optimal harvesting ,constant harvesting on the prey size are shown. Note that we used the same values of parameters.*

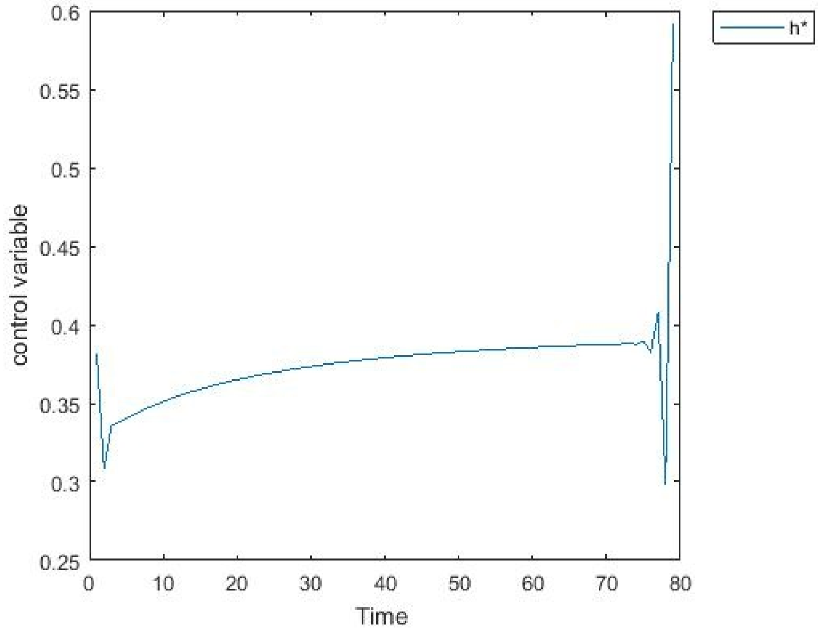

Figure 7: *This figure indicates the optimal control as a function of time.*

5. Conclusions

A three dimensional stage-structured model, immature and mature in predator species, with harvesting for the prey species is considered. We have studied the local stability of its equilibria .The dynamical behavior of the proposed model is studied analytically. The constant harvesting does not give the optimal gain at all, so that the model is extended to optimal control strategy to get optimal management policy . It can be also persevered the population far from the collapse. The optimal problem is solved through the discrete of Pontryagin's maximum principle. Numerical simulations are used to confirm the theoretical results as well as to solve the optimality problem.

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