Some results on fuzzy soft sesquilinear functional

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Abstract

In this paper, we study and discussion new kinds of Sesquilinear functional which is fuzzy soft Sesquilinear functional and given some properties with characterization and also theories related on fuzzy soft Sesquilinear functional have been given. Additionally, we present the relationship between this kind and other kinds.

Keywords: Fuzzy soft set, fuzzy soft Hilbert space, fuzzy soft adjoint operator

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1. Introduction

Functional analysis is a branch of pure mathematics. It was first developed in about century ago. It aims to solve many problems in pure mathematics. Therefore it provides an indispensable tool for solving those problems. It also provides us with techniques for estimating error in the solutions of infinite and finite dimensional problems.

In our everyday life, we often faced with uncertainty that arises from the ambiguity of the phenomenon under study. This type of problems arises in areas like economics, medical science, business and engineering. Our classical mathematical methods often fail to tackle such problems.

The mathematical models related to real-world is too problematical and we cannot usually to find the exact solutions [4]. Then may be interested to use the concept of approximate approach to compute their solutions by using some mathematical tools in Hilbert space like, fuzzy, soft, or fuzzy

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Thus, in 1965, a generalization of set theory was introduced, by Zadeh [14]. The resulting theory was called fuzzy set theory. Fuzzy set theory soon became an excellent tool to deal with problems that associate with uncertainty. In classical set theory, a set $X$ is defined with its characteristic function from $X$ to set $\{0, 1\}$. On the other hand, in fuzzy set theory, a set is defined with its membership function form $X$ to the closed interval $[0, 1]$.

Also, in 1999, a yet another generalization was introduced by Molodsov [9] to deal with uncertainty. The resulting theory was called soft set theory. Since then, it was used to solve complicated problems in subjects like computer science, medicine, engineering, etc. A soft set is a parametrized collection of a universal set. The concept soft set was then applied on various mathematical concepts in functional analysis resulting in concepts like soft point [1], Soft normed Spaces [13], Soft Inner Product Spaces [3], Soft Hilbert space [12] and Projection operators on soft inner product spaces [11].

In 2001, Maji et al. [8], was first to introduce the concept of a fuzzy soft set. The concept resulted from combining the concept of a fuzzy and a soft set. The necessity of combining the two concepts was to provide more accurate and general results. The other concepts consequently followed this structure resulting in the introduction fuzzy soft point [10] and fuzzy soft normed spaces [2].

In 2020, the fuzzy soft Hilbert space [4] were introduce by Faried et al. In addition Fuzzy soft linear operators [6]. and finally fuzzy soft self−adjoint operators [5] and studied its properties.

In this paper, introduce a new kind is said to be fuzzy soft Sesquilinear functional, and given some theorems relating to this functional with properties.

2. BASIC CONCEPTS

**Definition 2.1.** [14] If $X$ universe set and $\hat{A}$ is a set characterized by a membership function $\mu_{\hat{A}} : X \rightarrow \overline{\mathbb{T}}$, such as $\overline{\mathbb{T}} = [0, 1]$ then $\hat{A}$ is said to be fuzzy set over $X$, and $\hat{A} = \left\{ \frac{\mu_{\hat{A}}(x)}{x} : x \in X \right\}$.

And $\overline{\mathbb{T}} = \{ \hat{A} : \hat{A} \text{ is a function from } X \text{ into } \overline{\mathbb{T}} \}$

**Definition 2.2.** [9] Let $\mathcal{P}(X)$ be the power set of universe set $X$ and $E$ be set of parameters and $A \subseteq E$. The mapping $\mathcal{G} : A \rightarrow \mathcal{P}(X)$, where $(\mathcal{G}, A) = \{ \mathcal{G}(a) \in \mathcal{P}(X) : a \in A \}$. The pair $(\mathcal{G}, A)$ is said to be soft set.

**Definition 2.3.** [8] The soft set $(\mathcal{G}, A)$ is called fuzzy soft set ($\mathcal{FS}$ − set) over $X$, whenever $\mathcal{G} : A \rightarrow \overline{\mathbb{T}}^X$ and $\{ \mathcal{G}(a) \in \overline{\mathbb{T}}^X : a \in A \}$.

The collection of every $\mathcal{FS}$− sets, denoted by $\mathcal{FSS}(\hat{X})$

**Definition 2.4.** [10] If $(\mathcal{G}, A) \in \mathcal{FSS}(\hat{X})$ is called $\mathcal{FS}$−point over $X$, symbolized by $\hat{a}_{\mu_{\mathcal{G}(e)}}$, if $e \in A$ and $a \in X$,

$$\mu_{\mathcal{G}(e)}(x) = \begin{cases} \delta, & \text{if } a = a_0 \in X \text{ and } e = e_0 \in A \\ 0, & \text{if } a \in X - \{a_0\} \text{ or } e \in A - \{e_0\} \end{cases} ; \text{ where } \delta \in (0, 1]$$

**Remark 2.5.** [10] The collection of every $\mathcal{FS}$− Complex numbers denoted by $\hat{\mathcal{C}}(A)$ and the collection of every $\mathcal{FS}$− Real numbers denoted by $\hat{\mathcal{R}}(A)$.
Definition 2.6. [2] The mapping \( \| \cdot \| : \tilde{X} \to \mathcal{R}(A) \) where \( \tilde{X} \) is FS – vector space is said to be FS – norm on \( \tilde{X} \) if \( \| \cdot \| \) satisfies:

1. \( \| \tilde{a}_{\mu}(\cdot) \| \geq \tilde{0}, \) for all \( \tilde{a}_{\mu}(\cdot) \in \tilde{X}, \) and \( \| \tilde{a}_{\mu}(\cdot) \| = \tilde{0} \) if and only if \( \tilde{a}_{\mu}(\cdot) = \tilde{0} \)

2. \( \| \tilde{r} \tilde{a}_{\mu}(\cdot) \| = |\tilde{r}| \| \tilde{a}_{\mu}(\cdot) \|, \) for all \( \tilde{a}_{\mu}(\cdot) \in \tilde{X}, \tilde{r} \in \mathcal{C}(A) \)

3. \( \| \tilde{a}_{\mu}(\cdot) + \tilde{b}_{\mu}(\cdot) \| \leq \| \tilde{a}_{\mu}(\cdot) \| + \| \tilde{b}_{\mu}(\cdot) \|, \forall \tilde{a}_{\mu}(\cdot), \tilde{b}_{\mu}(\cdot) \in \tilde{X} \)

Then \( (\tilde{X}, \| \cdot \|) \) is called FS – normed vector space (FSN – space )

Definition 2.7. [4] The mapping \( \langle \cdot, \cdot \rangle : \tilde{X} \times \tilde{X} \to (\mathcal{C}(A) \) or \( \mathcal{R}(A)) \) where \( \tilde{X} \) is FSV – space is called FS – inner product on \( \tilde{X} (\tilde{FS}) \) if \( \langle \cdot, \cdot \rangle \) satisfies:

1. \( \langle \tilde{a}_{\mu}(\cdot), \tilde{a}_{\mu}(\cdot) \rangle \geq \tilde{0}, \) for all \( \tilde{a}_{\mu}(\cdot) \in \tilde{X} \) and \( \langle \tilde{a}_{\mu}(\cdot), \tilde{a}_{\mu}(\cdot) \rangle = \tilde{0} \) if and only if \( \tilde{a}_{\mu}(\cdot) = \tilde{0} \)

2. \( \langle \tilde{a}_{\mu}(\cdot), \tilde{b}_{\mu}(\cdot) \rangle = \langle \tilde{b}_{\mu}(\cdot), \tilde{a}_{\mu}(\cdot) \rangle \), for all \( \tilde{a}_{\mu}(\cdot), \tilde{b}_{\mu}(\cdot) \in \tilde{X} \)

3. \( \langle \tilde{a}_{\mu}(\cdot), \tilde{b}_{\mu}(\cdot) + \tilde{c}_{\mu}(\cdot) \rangle = \langle \tilde{a}_{\mu}(\cdot), \tilde{b}_{\mu}(\cdot) \rangle + \langle \tilde{a}_{\mu}(\cdot), \tilde{c}_{\mu}(\cdot) \rangle \), for all \( \tilde{a}_{\mu}(\cdot), \tilde{b}_{\mu}(\cdot), \tilde{c}_{\mu}(\cdot) \in \tilde{X} \), for all \( \tilde{a}, \tilde{b} \in \tilde{C}(A) \)

4. \( \langle \tilde{a}_{\mu}(\cdot) + \tilde{b}_{\mu}(\cdot), \tilde{a}_{\mu}(\cdot) \rangle = \langle \tilde{a}_{\mu}(\cdot), \tilde{a}_{\mu}(\cdot) \rangle + \langle \tilde{a}_{\mu}(\cdot), \tilde{b}_{\mu}(\cdot) \rangle \), for all \( \tilde{a}_{\mu}(\cdot), \tilde{b}_{\mu}(\cdot) \in \tilde{X} \)

For all \( \tilde{a}_{\mu}(\cdot), \tilde{b}_{\mu}(\cdot), \tilde{c}_{\mu}(\cdot) \in \tilde{X} \)

Then \( (\tilde{X}, \langle \cdot, \cdot \rangle) \) is called FS – inner product space (FSI – space )

Definition 2.8. [7] The FSN – space \( (\tilde{X}, \| \cdot \|) \) is called FS – complete if all FS – Cauchy sequence in \( X \) is FS – convergence in \( X \).

Definition 2.9. [4] The FSI – complete inner product space \( (\tilde{X}, \langle \cdot, \cdot \rangle) \) is called FS – Hilbert space \( (FSH) – space \), and symbolized by \( (\tilde{X}, \langle \cdot, \cdot \rangle) \).

Definition 2.10. If \( \tilde{H} \) be FSH – space and \( \tilde{S} : \tilde{H} \to \tilde{H} \) be FS – operator . Then \( \tilde{S} \) is called FS – linear operator \( (FSL) – operator \) if:

\[ \tilde{S} \left( \alpha \tilde{a}_{\mu}(\cdot) + \beta \tilde{b}_{\mu}(\cdot) \right) = \alpha \tilde{S} \left( \tilde{a}_{\mu}(\cdot) \right) + \beta \tilde{S} \left( \tilde{b}_{\mu}(\cdot) \right) \]

for all \( \alpha \tilde{a}_{\mu}(\cdot), \beta \tilde{b}_{\mu}(\cdot) \in \tilde{H} \) and \( \alpha, \beta \in \tilde{C}(A) \)

Definition 2.11. If \( \tilde{H} \) be FSH – space and \( \tilde{S} : \tilde{H} \to \tilde{H} \) be FS – operator is called FS – bounded operator, if \( \exists \tilde{m} \in \mathcal{R}(A) \) such that \( \| \tilde{S} \left( \tilde{a}_{\mu}(\cdot) \right) \| \leq \tilde{m} \| \tilde{a}_{\mu}(\cdot) \| \)

for all \( \tilde{a}_{\mu}(\cdot) \in \tilde{H} \).

Now, the family of all FS – bounded linear operators denoted by \( \tilde{B}(\tilde{H}) \).

Example 2.12. [6] The FS – operator \( \tilde{I} : \tilde{H} \to \tilde{H} \) defined by \( \tilde{I} \left( \tilde{a}_{\mu}(\cdot) \right) = \tilde{a}_{\mu}(\cdot), \forall \tilde{a}_{\mu}(\cdot) \in \tilde{H}, \)

it is called FS – identity operator.
Definition 2.13. [6] If $\tilde{H}$ be $FSH-$ space and $\tilde{S} : \tilde{H} \rightarrow \tilde{H}$ be $FSB-$ operator, then The $FS-$ adjoint operator $\tilde{S}^*$ is defined by
\[
\langle \tilde{S}a_{\mu_1g(e_1)}, \tilde{b}_{\mu_2g(e_2)} \rangle = \langle \tilde{a}_{\mu_1g(e_1)}, \tilde{\tilde{S}}^* \tilde{b}_{\mu_2g(e_2)} \rangle
\]

Theorem 2.14. [6] If $\tilde{S}, \tilde{\varphi} \in \tilde{B}(\tilde{H})$, where $\tilde{H}$ is $FSH-$ space and $\tilde{\beta} \in C(A)$, then $\tilde{S}^* = \tilde{S}, (\tilde{\beta} \tilde{S})^* = \tilde{\beta}^{\tilde{S}}^*, (\tilde{S} + \tilde{\varphi})^* = \tilde{S}^* + \tilde{\varphi}^*$

Definition 2.15. [5] The $FS-$ operator $\tilde{S}$ of $FSH-$ space $\tilde{H}$ is called $FS-$ self adjoint operator if $\tilde{S} = \tilde{S}^*$.

3. MAIN RESULTS

Definition 3.1. Let $\tilde{H}$ be $FSH-$ space. A mapping $\tilde{a}(\ldots) : \tilde{H} \times \tilde{H} \rightarrow C(A)$ is called a fuzzy soft sesquilinear functional ($FS-$ sesquilinear functional) if the following conditions are satisfied:

1. $\tilde{a}\left(\tilde{x}_{\mu_1g(e_1)} + \tilde{y}_{\mu_2g(e_2)}, \tilde{z}_{\mu_3g(e_3)}\right) = \tilde{a}\left(\tilde{x}_{\mu_1g(e_1)}, \tilde{z}_{\mu_3g(e_3)}\right) + \tilde{a}\left(\tilde{y}_{\mu_2g(e_2)}, \tilde{z}_{\mu_3g(e_3)}\right)$
2. $\tilde{a}\left(\tilde{\tilde{\beta}}\tilde{x}_{\mu_1g(e_1)}, \tilde{y}_{\mu_2g(e_2)}\right) = \tilde{\tilde{\beta}}\tilde{a}\left(\tilde{x}_{\mu_1g(e_1)}, \tilde{y}_{\mu_2g(e_2)}\right)$
3. $\tilde{a}\left(\tilde{x}_{\mu_1g(e_1)}, \tilde{y}_{\mu_2g(e_2)} + \tilde{z}_{\mu_3g(e_3)}\right) = \tilde{a}\left(\tilde{x}_{\mu_1g(e_1)}, \tilde{y}_{\mu_2g(e_2)}\right) + \tilde{a}\left(\tilde{x}_{\mu_1g(e_1)}, \tilde{z}_{\mu_3g(e_3)}\right)$
4. $\tilde{a}\left(\tilde{x}_{\mu_1g(e_1)}, \tilde{\tilde{\beta}}\tilde{y}_{\mu_2g(e_2)}\right) = \tilde{\tilde{\beta}}\tilde{a}\left(\tilde{x}_{\mu_1g(e_1)}, \tilde{y}_{\mu_2g(e_2)}\right)$

Remark 3.2. 1. The $FS-$ sesquilinear functional is $FS-$ Linearity in the first variable, but not in the second variable. $FS-$ sesquilinear functional Which is also $FS-$ linear in second variable is said to be $FS-$ bilinear form or a $FS-$ bilinear functional. Thus, $FS-$ bilinear form $\tilde{a}(\ldots)$ is a mapping defined on $\tilde{H} \times \tilde{H} \rightarrow C(A)$ which satisfies condition (1) through (3) of Definition 3.1 and (4) $\tilde{a}\left(\tilde{x}_{\mu_1g(e_1)}, \tilde{\tilde{\beta}}\tilde{y}_{\mu_2g(e_2)}\right) = \tilde{\tilde{\beta}}\tilde{a}\left(\tilde{x}_{\mu_1g(e_1)}, \tilde{y}_{\mu_2g(e_2)}\right)$

2. If $\tilde{H}$ is a real $FSH-$ space, then the concepts of $FS-$ sesquilinear functional and $FS-$ bilinear forms coincide.
3. If $\tilde{a}(\ldots)$ is $FS-$ sesquilinear function, so $\tilde{g}(x, y) = \tilde{a}(\ldots)$ is $FS-$ sesquilinear functional

Definition 3.3. Suppose that $\tilde{a}(\ldots)$ is $FS-$ bilinear forms. Then

1. $\tilde{a}(\ldots)$ is said to be $FS-$ symmetric if $\tilde{a}\left(\tilde{x}_{\mu_1g(e_1)}, \tilde{\tilde{y}}_{\mu_2g(e_2)}\right) = \tilde{a}\left(\tilde{\tilde{y}}_{\mu_2g(e_2)}, \tilde{x}_{\mu_1g(e_1)}\right)$
\[\forall (\tilde{x}_{\mu_1g(e_1)}, \tilde{\tilde{y}}_{\mu_2g(e_2)}) \in \tilde{H} \times \tilde{H}\]
2. $\tilde{a}(\ldots)$ is called $FS-$ positive if $\tilde{a}\left(\tilde{x}_{\mu_1g(e_1)}, \tilde{x}_{\mu_1g(e_1)}\right) \geq 0 \forall \tilde{x}_{\mu_1g(e_1)} \in \tilde{H}$.
3. $\tilde{a}(\ldots)$ is called $FS-$ positive definite if
\[\tilde{a}\left(\tilde{x}_{\mu_1g(e_1)}, \tilde{x}_{\mu_1g(e_1)}\right) \geq 0 \forall \tilde{x}_{\mu_1g(e_1)} \in \tilde{H} \text{ and } \tilde{a}\left(\tilde{x}_{\mu_1g(e_1)}, \tilde{0}_{\mu_1g(e_1)}\right) = \tilde{0} \text{ implies that } \tilde{x}_{\mu_1g(e_1)} = \tilde{0}\]
4. \( \widetilde{\mathfrak{S}}(\widetilde{x}_{\mu_1G(e_1)}) = \widetilde{a}(\widetilde{x}_{\mu_1G(e_1)}; \widetilde{x}_{\mu_1G(e_1)}) \) is called FS - quadratic form.

5. \( \widetilde{a}(\cdot, \cdot) \) is called FS - bounded or FS - continuous if there exists \( \widetilde{M} \in \mathcal{R}(A) \) such that
\[
\left| \widetilde{a}(\widetilde{x}_{\mu_1G(e_1)}; \widetilde{y}_{\mu_2G(e_2)}) \right| \leq \widetilde{M} \left\| \widetilde{x}_{\mu_1G(e_1)} \right\| \left\| \widetilde{y}_{\mu_2G(e_2)} \right\|
\]

6. \( \widetilde{a}(\cdot, \cdot) \) is said to be FS - coercive (\( \mathcal{H} \) - coercive ) if there exists \( \lambda \in \mathcal{R}(A) \) such that
\[
\widetilde{a}(\widetilde{x}_{\mu_1G(e_1)}; \widetilde{x}_{\mu_1G(e_1)}) \geq \lambda \left\| \widetilde{x}_{\mu_1G(e_1)} \right\|^2 \forall \widetilde{x}_{\mu_1G(e_1)} \in \mathcal{H}
\]

7. A FS - quadratic form \( \widetilde{\mathfrak{S}} \) is called FS - real if \( \widetilde{\mathfrak{S}}(\widetilde{x}_{\mu_1G(e_1)}) \) is FS - real for all \( \widetilde{x}_{\mu_1G(e_1)} \in \mathcal{H} \)

Remark 3.4.

1. If \( \widetilde{a}(\cdot, \cdot): \mathcal{H} \times \mathcal{H} \rightarrow \mathcal{R}(A) \), then the FS - bilinear form \( \widetilde{a}(\cdot, \cdot) \) is FS - symmetric if
\[
\widetilde{a}(\widetilde{x}_{\mu_1G(e_1)}; \widetilde{y}_{\mu_2G(e_2)}) = \widetilde{a}(\widetilde{y}_{\mu_2G(e_2)}; \widetilde{x}_{\mu_1G(e_1)})
\]

2. \[
\left\| \widetilde{a} \right\| = \sup_{\widetilde{x}_{\mu_1G(e_1)} = \widetilde{0}, \widetilde{y}_{\mu_2G(e_2)} = \widetilde{0}} \left| \widetilde{a}(\widetilde{x}_{\mu_1G(e_1)}; \widetilde{y}_{\mu_2G(e_2)}) \right| = \sup_{\left\| \widetilde{x}_{\mu_1G(e_1)} \right\| = 1, \left\| \widetilde{y}_{\mu_2G(e_2)} \right\| = 1} \left| \widetilde{a}(\widetilde{x}_{\mu_1G(e_1)}; \widetilde{y}_{\mu_2G(e_2)}) \right|
\]

It is clear that \( \left| \widetilde{a}(\widetilde{x}_{\mu_1G(e_1)}; \widetilde{y}_{\mu_2G(e_2)}) \right| \leq \left\| \widetilde{a} \right\| \left\| \widetilde{x}_{\mu_1G(e_1)} \right\| \left\| \widetilde{y}_{\mu_2G(e_2)} \right\|
\]

3. \( \left\| \widetilde{\mathfrak{S}} \right\| = \sup_{\widetilde{x}_{\mu_1G(e_1)} = \widetilde{1}} \left| \widetilde{\mathfrak{S}}(\widetilde{x}_{\mu_1G(e_1)}) \right| \)

4. If \( \widetilde{a}(\cdot, \cdot) \) is any fixed FS - sesquilinear form and \( \widetilde{\mathfrak{S}}(\widetilde{x}_{\mu_1G(e_1)}) \) be associated FS - quadratic form on \( \mathcal{FSH} \) - space \( \mathcal{H} \), then
\[
\widetilde{a}(\widetilde{x}_{\mu_1G(e_1)}; \widetilde{y}_{\mu_2G(e_2)}) = \frac{1}{4} \left[ \widetilde{\mathfrak{S}}(\widetilde{x}_{\mu_1G(e_1)}; \widetilde{y}_{\mu_2G(e_2)}) - \widetilde{\mathfrak{S}}(\widetilde{x}_{\mu_1G(e_1)}; \widetilde{y}_{\mu_2G(e_2)}) + i\widetilde{\mathfrak{S}}(\widetilde{x}_{\mu_1G(e_1)}; \widetilde{y}_{\mu_2G(e_2)}) - i\widetilde{\mathfrak{S}}(\widetilde{x}_{\mu_1G(e_1)}; \widetilde{y}_{\mu_2G(e_2)}) \right]
\]

Verification: By using linearity of the FS - bilinear form \( \widetilde{a} \), we have
\[
\widetilde{\mathfrak{S}}(\widetilde{x}_{\mu_1G(e_1)} + \widetilde{y}_{\mu_2G(e_2)}) = \widetilde{a}(\widetilde{x}_{\mu_1G(e_1)} + \widetilde{y}_{\mu_2G(e_2)}; \widetilde{x}_{\mu_1G(e_1)} + \widetilde{y}_{\mu_2G(e_2)})
\]
\[
= \widetilde{a}(\widetilde{x}_{\mu_1G(e_1)}; \widetilde{x}_{\mu_1G(e_1)}) + \widetilde{a}(\widetilde{x}_{\mu_1G(e_1)}; \widetilde{y}_{\mu_2G(e_2)}) + \widetilde{a}(\widetilde{y}_{\mu_2G(e_2)}; \widetilde{x}_{\mu_1G(e_1)}) + \widetilde{a}(\widetilde{y}_{\mu_2G(e_2)}; \widetilde{y}_{\mu_2G(e_2)})
\]

And
\[
\widetilde{\mathfrak{S}}(\widetilde{x}_{\mu_1G(e_1)} - \widetilde{y}_{\mu_2G(e_2)}) = \widetilde{a}(\widetilde{x}_{\mu_1G(e_1)} - \widetilde{y}_{\mu_2G(e_2)}; \widetilde{x}_{\mu_1G(e_1)} - \widetilde{y}_{\mu_2G(e_2)})
\]
\[
= \widetilde{a}(\widetilde{x}_{\mu_1G(e_1)}; \widetilde{x}_{\mu_1G(e_1)}) - \widetilde{a}(\widetilde{x}_{\mu_1G(e_1)}; \widetilde{y}_{\mu_2G(e_2)}) - \widetilde{a}(\widetilde{y}_{\mu_2G(e_2)}; \widetilde{x}_{\mu_1G(e_1)}) + \widetilde{a}(\widetilde{y}_{\mu_2G(e_2)}; \widetilde{y}_{\mu_2G(e_2)})
\]
By subtracting the second from the above equation from the first, and we get
\[
\tilde{\mathcal{F}}(\tilde{x}_{\mu_1G(e_1)} + \tilde{y}_{\mu_2G(e_2)}) - \tilde{\mathcal{F}}(\tilde{x}_{\mu_1G(e_1)} - \tilde{y}_{\mu_2G(e_2)})
= 2\tilde{a}(\tilde{x}_{\mu_1G(e_1)}, \tilde{y}_{\mu_2G(e_2)}) + 2\tilde{a}(\tilde{y}_{\mu_2G(e_2)} - \tilde{x}_{\mu_1G(e_1)})
\] (3.1)

Replacing \(\tilde{y}_{\mu_2G(e_2)}\) by \(i\tilde{y}_{\mu_2G(e_2)}\) in Eq. (3.1), we obtain
\[
\tilde{\mathcal{F}}(\tilde{x}_{\mu_1G(e_1)} + i\tilde{y}_{\mu_2G(e_2)}) - \tilde{\mathcal{F}}(\tilde{x}_{\mu_1G(e_1)} - i\tilde{y}_{\mu_2G(e_2)})
= i2\tilde{a}(\tilde{x}_{\mu_1G(e_1)}, \tilde{y}_{\mu_2G(e_2)}) + i2\tilde{a}(\tilde{y}_{\mu_2G(e_2)} - \tilde{x}_{\mu_1G(e_1)})
\] (3.2)

Multiply Eq. (3.2) by \(i\) and adding it to Eq. (3.2), we obtain a result.

Lemma 3.5. A \(\mathcal{FS}-\) bilinear form \(\tilde{a}(\tilde{x}_{\mu_1G(e_1)}, \tilde{y}_{\mu_2G(e_2)})\) is \(\mathcal{FS}-\) symmetric if and only if associated \(\mathcal{FS}\) - quadratic functional \(\tilde{\mathcal{F}}(\tilde{x}_{\mu_1G(e_1)})\) is \(\mathcal{FS}-\) real.

Lemma 3.6. The \(\mathcal{FS}-\) bilinear form \(\tilde{a}(\tilde{x}_{\mu_1G(e_1)}, \tilde{y}_{\mu_2G(e_2)})\) is \(\mathcal{FS}\) - bounded if and only if the associated \(\mathcal{FS}\)-sesquilinear form \(\tilde{\mathcal{F}}\) is \(\mathcal{FS}\)- bounded. If \(\tilde{a}(.,.)\) is \(\mathcal{FS}\)- bounded, then \(\|\tilde{\mathcal{F}}\| \geq \|\tilde{a}\| \geq \tilde{2}\|\tilde{\mathcal{F}}\|\).

Theorem 3.7. If \(\mathcal{FS}\)- bilinear form \(\tilde{a}(.,.)\) is \(\mathcal{FS}\)- bounded and \(\mathcal{FS}\)- symmetric, then \(\|\tilde{a}\| = \|\tilde{\mathcal{F}}\|\), such that \(\tilde{\mathcal{F}}\) is the associated \(\mathcal{FS}\)- quadratic functional.

Proof. By Lemma 3.5, \(\tilde{\mathcal{F}}\) is \(\mathcal{FS}\)- real. In view of Lemma 3.6,

To show \(\|\tilde{a}\| \leq \|\tilde{\mathcal{F}}\|\)

Let \(\tilde{a}(\tilde{x}_{\mu_1G(e_1)}, \tilde{y}_{\mu_2G(e_2)}) = \tilde{\gamma}e^{i\tilde{\beta}}\), where \(\tilde{\gamma}, \tilde{\beta} \in \mathcal{R}(A)\)

So by using the Remark 3.4(4) and bearing in mind that purely imaginary terms are \(\tilde{0}\). we have
\[
\tilde{a}(\tilde{x}_{\mu_1G(e_1)}, \tilde{y}_{\mu_2G(e_2)}) = \tilde{\gamma} = \tilde{\mathcal{F}}(\tilde{x}_{\mu_1G(e_1)}, \tilde{y}_{\mu_2G(e_2)})
\]

By using \(\mathcal{F}\)-parallelogram law, we have
\[
\tilde{a}(\tilde{x}_{\mu_1G(e_1)}, \tilde{y}_{\mu_2G(e_2)}) = \frac{1}{2} \|\tilde{\mathcal{F}}\| \left( \|\tilde{x}_{\mu_1G(e_1)}\|^2 + \|\tilde{y}_{\mu_2G(e_2)}\|^2 \right)
\]

Then \(\|\tilde{a}\| \leq \|\tilde{\mathcal{F}}\|\). □
Theorem 3.8. Let $\tilde{T} \in \tilde{B}(\tilde{H})$. Then, the $\tilde{a}(\cdot,\cdot): \tilde{H} \times \tilde{H} \to \mathcal{C}(A)$ defined by

$$\tilde{a}(\tilde{x}_\mu G(e_1); \tilde{y}_\mu G(e_2)) = \left\langle \tilde{x}_\mu G(e_1); \tilde{T}(\tilde{z}_\mu G(e_3)) \right\rangle \forall (\tilde{x}_\mu G(e_1); \tilde{z}_\mu G(e_3)) \in \tilde{H} \times \tilde{H}$$

is $\mathcal{FS}$- bounded bilinear form on $\tilde{H}$, and $\|\tilde{a}\| = \|\tilde{T}\|$. Conversely, let $\tilde{a}(\cdot,\cdot)$ be $\mathcal{FS}$- bounded bilinear form on $\tilde{H}$. Then, there exists a unique $\mathcal{FS}$- bounded linear operator $\tilde{T}$ on $\tilde{H}$ such that

$$\tilde{a}(\tilde{x}_\mu G(e_1); \tilde{y}_\mu G(e_2)) = \left\langle \tilde{x}_\mu G(e_1); \tilde{T}(\tilde{z}_\mu G(e_3)) \right\rangle \forall (\tilde{x}_\mu G(e_1); \tilde{z}_\mu G(e_3)) \in \tilde{H} \times \tilde{H}$$

**Proof.**

1. Let $\tilde{T} \in \tilde{B}(\tilde{H})$. Then, $\tilde{a}(\tilde{x}_\mu G(e_1); \tilde{z}_\mu G(e_3)) = \left\langle \tilde{x}_\mu G(e_1); \tilde{T}(\tilde{z}_\mu G(e_3)) \right\rangle$ satisfies the following condition:

(a) $$\left\langle \tilde{x}_\mu G(e_1) + \tilde{y}_\mu G(e_2); \tilde{z}_\mu G(e_3) \right\rangle = \left\langle \tilde{x}_\mu G(e_1); \tilde{T}(\tilde{z}_\mu G(e_3)) \right\rangle + \left\langle \tilde{y}_\mu G(e_2); \tilde{T}(\tilde{z}_\mu G(e_3)) \right\rangle$$

(b) $$\tilde{a}(\tilde{\beta} \tilde{x}_\mu G(e_1); \tilde{z}_\mu G(e_3)) = \tilde{\beta} \tilde{a}(\tilde{x}_\mu G(e_1); \tilde{z}_\mu G(e_3))$$

(c) $$\left\| \tilde{a}(\tilde{x}_\mu G(e_1); \tilde{z}_\mu G(e_3)) \right\| \leq \|\tilde{T}\| \left\| \tilde{x}_\mu G(e_1) \right\| \left\| \tilde{z}_\mu G(e_3) \right\|$$

This implies that

$$\sup_{\|\tilde{x}_\mu G(e_1)\| \leq \left\| \tilde{z}_\mu G(e_3) \right\| = 1} \left| \tilde{a}(\tilde{x}_\mu G(e_1); \tilde{z}_\mu G(e_3)) \right| \leq \|\tilde{T}\|$$

Then $\|\tilde{a}\| \leq \|\tilde{T}\|$. In fact $\|\tilde{a}\| = \|\tilde{T}\|$

2. For the converse, let $\tilde{a}(\cdot,\cdot)$ be $\mathcal{FS}$- bounded bilinear form on $\tilde{H}$. For any $\tilde{y}_\mu G(e_2) \in \tilde{H}$, we define $\tilde{f}_\mu G(e_2)$ on $\tilde{H}$ as follows:

$$\tilde{f}_\mu G(e_2)(\tilde{x}_\mu G(e_1)) = \tilde{a}(\tilde{x}_\mu G(e_1); \tilde{z}_\mu G(e_3))$$

(3.3)

We have

$$\tilde{f}_\mu G(e_2)(\tilde{x}_\mu G(e_1) + \tilde{y}_\mu G(e_2)) = \tilde{a}(\tilde{x}_\mu G(e_1) + \tilde{y}_\mu G(e_2); \tilde{z}_\mu G(e_3))$$

$$\tilde{f}_\mu G(e_2)(\tilde{x}_\mu G(e_1)) + \tilde{f}_\mu G(e_2)(\tilde{y}_\mu G(e_2)) = \tilde{a}(\tilde{x}_\mu G(e_1); \tilde{z}_\mu G(e_3)) + \tilde{a}(\tilde{y}_\mu G(e_2); \tilde{z}_\mu G(e_3))$$
\[ \tilde{f}_{\bar{\mu}_3 G(e_3)} \left( \tilde{\beta} \bar{x}_{\mu_1 G(e_1)} \right) = \tilde{a} \left( \tilde{\beta} \bar{x}_{\mu_1 G(e_1)}, \bar{z}_{\mu_3 G(e_3)} \right) = \tilde{\beta} \tilde{a} \left( \bar{x}_{\mu_1 G(e_1)}, \bar{z}_{\mu_3 G(e_3)} \right) \]

\[ \left\| \tilde{f}_{\bar{\mu}_3 G(e_3)} \left( \bar{x}_{\mu_1 G(e_1)} \right) \right\| \leq \left\| \tilde{a} \right\| \left\| \bar{x}_{\mu_1 G(e_1)} \right\| \left\| \bar{z}_{\mu_3 G(e_3)} \right\| \]

\[ \Rightarrow \left\| \tilde{f}_{\bar{\mu}_3 G(e_3)} \right\| \leq \left\| \tilde{a} \right\| \left\| \bar{z}_{\mu_3 G(e_3)} \right\| \]

Thus, \( \tilde{f}_{\bar{\mu}_3 G(e_3)} \) is \( \mathcal{F} \mathcal{S} \)- bounded bilinear form on \( \bar{\mathcal{H}} \). By \( \mathcal{F} \mathcal{S} \)- Riesz representation theorem, there exists a unique \( \tilde{T} \bar{z}_{\mu_3 G(e_3)} \in \bar{\mathcal{H}} \) such that

\[ \tilde{f}_{\bar{\mu}_3 G(e_3)} \left( \bar{x}_{\mu_1 G(e_1)} \right) = \left\langle \bar{x}_{\mu_1 G(e_1)}, \tilde{T} \bar{z}_{\mu_3 G(e_3)} \right\rangle \]

\[ \text{And } \left\| \tilde{T} \bar{z}_{\mu_3 G(e_3)} \right\| = \left\| \tilde{f}_{\bar{\mu}_3 G(e_3)} \right\| \leq \left\| \tilde{a} \right\| \left\| \bar{z}_{\mu_3 G(e_3)} \right\| \]

The \( \tilde{T} : \bar{z}_{\mu_3 G(e_3)} \rightarrow \tilde{T} \bar{z}_{\mu_3 G(e_3)} \) defined by \( \tilde{T} \bar{z}_{\mu_3 G(e_3)} \left\langle \bar{x}_{\mu_1 G(e_1)}, \tilde{T} \bar{z}_{\mu_3 G(e_3)} \right\rangle \) is \( \mathcal{F} \mathcal{S} \)- linear in view of the following relations:

\[ \left\langle \bar{x}_{\mu_1 G(e_1)}, \tilde{T} \left( \tilde{\beta} \bar{z}_{\mu_3 G(e_3)} \right) \right\rangle = \tilde{f}_{\tilde{\beta} \bar{z}_{\mu_3 G(e_3)}} \left( \bar{x}_{\mu_1 G(e_1)} \right) = \left\langle \bar{x}_{\mu_1 G(e_1)}, \tilde{\beta} \bar{z}_{\mu_3 G(e_3)} \right\rangle \]

\[ = \tilde{\beta} \left\langle \bar{x}_{\mu_1 G(e_1)}, \bar{z}_{\mu_3 G(e_3)} \right\rangle = \tilde{\beta} \tilde{f}_{\bar{\mu}_3 G(e_3)} \left( \bar{x}_{\mu_1 G(e_1)} \right) = \tilde{\beta} \tilde{T} \bar{z}_{\mu_3 G(e_3)} \left( \bar{x}_{\mu_1 G(e_1)} \right) \]

\[ = \left\langle \bar{x}_{\mu_1 G(e_1)}, \tilde{T} \left( \tilde{\beta} \bar{z}_{\mu_3 G(e_3)} \right) \right\rangle = \left\langle \bar{x}_{\mu_1 G(e_1)}, \tilde{\beta} \tilde{T} \bar{z}_{\mu_3 G(e_3)} \right\rangle \]

\[ \Rightarrow \tilde{T} \left( \tilde{\beta} \bar{z}_{\mu_3 G(e_3)} \right) = \tilde{\beta} \tilde{T} \bar{z}_{\mu_3 G(e_3)} \]

Now

\[ \left\langle \bar{x}_{\mu_1 G(e_1)}, \tilde{T} \left( \bar{y}_{\mu_3 G(e_2)} + \bar{z}_{\mu_3 G(e_3)} \right) \right\rangle = \tilde{f}_{\bar{y}_{\mu_3 G(e_2)} + \bar{z}_{\mu_3 G(e_3)}} \left( \bar{x}_{\mu_1 G(e_1)} \right) = \left\langle \bar{x}_{\mu_1 G(e_1)}, \bar{y}_{\mu_3 G(e_2)} + \bar{z}_{\mu_3 G(e_3)} \right\rangle \]

\[ = \left\langle \bar{x}_{\mu_1 G(e_1)}, \bar{z}_{\mu_3 G(e_3)} \right\rangle + \left\langle \bar{x}_{\mu_1 G(e_1)}, \bar{y}_{\mu_2 G(e_2)} \right\rangle = \tilde{f}_{\bar{y}_{\mu_2 G(e_2)}} \left( \bar{x}_{\mu_1 G(e_1)} \right) = \tilde{T} \bar{y}_{\mu_2 G(e_2)} \left( \bar{x}_{\mu_1 G(e_1)} \right) \]

\[ = \left\langle \bar{x}_{\mu_1 G(e_1)}, \tilde{T} \left( \bar{y}_{\mu_2 G(e_2)} \right) \right\rangle + \tilde{T} \left( \bar{y}_{\mu_2 G(e_2)} + \tilde{T} \bar{z}_{\mu_3 G(e_3)} \right) \]

\[ \Rightarrow \left\langle \bar{x}_{\mu_1 G(e_1)}, \tilde{T} \left( \bar{y}_{\mu_2 G(e_2)} + \bar{z}_{\mu_3 G(e_3)} \right) \right\rangle = \tilde{T} \left( \bar{y}_{\mu_2 G(e_2)} \right) + \tilde{T} \left( \bar{z}_{\mu_3 G(e_3)} \right) \]

Which gives \( \tilde{T} \left( \bar{y}_{\mu_2 G(e_2)} + \bar{z}_{\mu_3 G(e_3)} \right) = \tilde{T} \left( \bar{y}_{\mu_2 G(e_2)} \right) + \tilde{T} \left( \bar{z}_{\mu_3 G(e_3)} \right) \)

Equation (3.5) implies that \( \left\| \tilde{T} \right\| \leq \left\| \tilde{a} \right\| \). By Eqs. (3.3) and (3.4), we have

\[ \tilde{a} \left( \bar{x}_{\mu_1 G(e_1)}, \bar{z}_{\mu_3 G(e_3)} \right) = \tilde{f}_{\bar{z}_{\mu_3 G(e_3)}} \left( \bar{x}_{\mu_1 G(e_1)} \right) = \left\langle \bar{x}_{\mu_1 G(e_1)}, \tilde{T} \bar{z}_{\mu_3 G(e_3)} \right\rangle \forall \left( \bar{x}_{\mu_1 G(e_1)}, \bar{z}_{\mu_3 G(e_3)} \right) \in \bar{\mathcal{H}} \times \bar{\mathcal{H}} \]
Then, for every fixed \( \tilde{z}_{\mu_3 G(e_3)} \in \tilde{\mathcal{H}} \), we get
\[
\langle \tilde{x}_{\mu_1 G(e_1)}, \tilde{T}(\tilde{z}_{\mu_3 G(e_3)}) \rangle = \langle \tilde{x}_{\mu_1 G(e_1)}, \tilde{S}(\tilde{z}_{\mu_3 G(e_3)}) \rangle
\]
\[\mapsto \langle \tilde{x}_{\mu_1 G(e_1)}, \tilde{T} - \tilde{S} \rangle (\tilde{z}_{\mu_3 G(e_3)}) = \tilde{0} \]
This implies that \( \langle \tilde{T} - \tilde{S} \rangle (\tilde{z}_{\mu_3 G(e_3)}) = \tilde{0} \) \( \forall \tilde{z}_{\mu_3 G(e_3)} \in \tilde{\mathcal{H}} \), i.e., \( \tilde{T} = \tilde{S} \).
This proves that there exists a unique \( \tilde{T} \in \tilde{B}(\tilde{\mathcal{H}}) \) such that
\[
\tilde{a}(\tilde{x}_{\mu_1 G(e_1)}, \tilde{z}_{\mu_3 G(e_3)}) = \langle \tilde{x}_{\mu_1 G(e_1)}, \tilde{T}(\tilde{z}_{\mu_3 G(e_3)}) \rangle
\]
\[\square\]

**Corollary 3.9.** Let \( \tilde{T} \in \tilde{B}(\tilde{\mathcal{H}}) \). Then, the \( \tilde{b}(\langle \cdot \rangle) : \tilde{\mathcal{H}} \times \tilde{\mathcal{H}} \rightarrow \mathcal{C}(A) \) defined by \( \tilde{b}(\tilde{x}_{\mu_1 G(e_1)}, \tilde{z}_{\mu_3 G(e_3)}) = \langle \tilde{T} \tilde{x}_{\mu_1 G(e_1)}, \tilde{z}_{\mu_3 G(e_3)} \rangle \) is \( \mathcal{F} \mathcal{S} \)-bounded bilinear form on \( \tilde{\mathcal{H}} \) and \( \| \tilde{b} \| = \| \tilde{T} \| \). Conversely, let \( \tilde{b}(\langle \cdot \rangle) \) be \( \mathcal{F} \mathcal{S} \)-bounded bilinear form on \( \tilde{\mathcal{H}} \). Then, there exists a unique \( \mathcal{F} \mathcal{S} \)-bounded linear operator \( \tilde{T} \) on \( \tilde{\mathcal{H}} \) such that
\[
\tilde{b}(\tilde{x}_{\mu_1 G(e_1)}, \tilde{z}_{\mu_3 G(e_3)}) = \langle \tilde{x}_{\mu_1 G(e_1)}, \tilde{T}(\tilde{z}_{\mu_3 G(e_3)}) \rangle, \quad \forall (\tilde{x}_{\mu_1 G(e_1)}, \tilde{y}_{\mu_2 G(e_2)}) \in \tilde{\mathcal{H}} \times \tilde{\mathcal{H}}
\]

**Corollary 3.10.** If \( \tilde{T} \in \tilde{B}(\tilde{\mathcal{H}}) \), then
\[
\| \tilde{T} \| = \sup_{\| \tilde{x}_{\mu_1 G(e_1)} \| = \| \tilde{y}_{\mu_2 G(e_2)} \| = 1} \left| \langle \tilde{x}_{\mu_1 G(e_1)}, \tilde{T}(\tilde{z}_{\mu_3 G(e_3)}) \rangle \right|
\]
\[= \sup_{\| \tilde{x}_{\mu_1 G(e_1)} \| = \| \tilde{y}_{\mu_2 G(e_2)} \| = 1} \left| \langle \tilde{T} \tilde{x}_{\mu_1 G(e_1)}, \tilde{z}_{\mu_3 G(e_3)} \rangle \right|
\]

**Theorem 3.11.** Let \( \tilde{T} \in \tilde{B}(\tilde{\mathcal{H}}) \). Then, the following statements are equivalent:

1. \( \tilde{T} \) is \( \mathcal{F} \mathcal{S} \)-self–adjoint.

2. The \( \mathcal{F} \mathcal{S} \)-bilinear from \( \tilde{\mathcal{H}} \) defined by \( \tilde{a}(\tilde{x}_{\mu_1 G(e_1)}, \tilde{y}_{\mu_2 G(e_2)}) = \langle \tilde{T} \tilde{x}_{\mu_1 G(e_1)}, \tilde{y}_{\mu_2 G(e_2)} \rangle \) is \( \mathcal{F} \mathcal{S} \)-symmetric.

3. The \( \mathcal{F} \mathcal{S} \)-bilinear from \( \tilde{\mathcal{H}} \) defined by \( \tilde{\mathcal{H}}(\tilde{x}_{\mu_1 G(e_1)}) \) on \( \tilde{\mathcal{H}} \) defined by \( \tilde{\mathcal{H}}(\tilde{x}_{\mu_1 G(e_1)}) = \langle \tilde{T} \tilde{x}_{\mu_1 G(e_1)}, \tilde{x}_{\mu_1 G(e_1)} \rangle \) is \( \mathcal{F} \mathcal{S} \)-real.

**Proof.** (1) \( \Rightarrow \) (2) : \( \tilde{\mathcal{H}}(\tilde{x}_{\mu_1 G(e_1)}) = \langle \tilde{T} \tilde{x}_{\mu_1 G(e_1)}, \tilde{x}_{\mu_1 G(e_1)} \rangle = \langle \tilde{\mathcal{H}}(\tilde{x}_{\mu_1 G(e_1)}), \tilde{x}_{\mu_1 G(e_1)} \rangle \)
\[= \langle \tilde{T} \tilde{x}_{\mu_1 G(e_1)}, \tilde{x}_{\mu_1 G(e_1)} \rangle = \tilde{\mathcal{H}}(\tilde{x}_{\mu_1 G(e_1)}) \]
In view of Lemma 3.7, we obtain the result
\[3) \Rightarrow (2) : \text{By Lemma (3.7)} \tilde{\mathcal{H}}(\tilde{x}_{\mu_1 G(e_1)}) = \langle \tilde{T} \tilde{x}_{\mu_1 G(e_1)}, \tilde{x}_{\mu_1 G(e_1)} \rangle \text{ is } \mathcal{F} \mathcal{S} \text{-real if and only if the } \mathcal{F} \mathcal{S} \text{-bilinear from } \tilde{\mathcal{H}} \text{ is } \mathcal{F} \mathcal{S} \text{-symmetric}
\]
(2) \( \Rightarrow \) (1) : \( \langle \tilde{T} \tilde{x}_{\mu_1 G(e_1)}, \tilde{y}_{\mu_2 G(e_2)} \rangle \tilde{a}(\tilde{x}_{\mu_1 G(e_1)}, \tilde{y}_{\mu_2 G(e_2)}) = \tilde{a}(\tilde{x}_{\mu_1 G(e_1)}, \tilde{y}_{\mu_2 G(e_2)})
\[= \langle \tilde{T} \tilde{x}_{\mu_1 G(e_1)}, \tilde{x}_{\mu_1 G(e_1)} \rangle = \langle \tilde{x}_{\mu_1 G(e_1)}, \tilde{T} \tilde{x}_{\mu_1 G(e_1)} \rangle \]
This shows that \( \tilde{T} = \tilde{T}^* \) that \( \tilde{T} \) is \( \mathcal{F} \mathcal{S} \)-self-adjoint \( \square \)
Corollary 3.12. If $\tilde{T}$ is $\mathcal{FS}$– bounded self – adjoint on $\tilde{H}$, then

$$
\|\tilde{T}\| = \sup_{\|\tilde{x}_{\mu_1 G(e_1)}\|=1} \left| \langle \tilde{T}\tilde{x}_{\mu_1 G(e_1)}', \tilde{x}_{\mu_1 G(e_1)} \rangle \right|
$$

4. Conclusions

The necessity of combining the two concepts fuzzy and soft sets was to provide more accurate and general results. The other concepts consequently followed this structure resulting like fuzzy soft normed spaces, fuzzy soft inner product space, fuzzy soft Hilbert space and fuzzy soft bounded linear operators. We studied and discussion new kind of sesquilinear functional which is fuzzy soft sesquilinear functional and given some properties with characterization, also theorems related on fuzzy soft sesquilinear functional have been given. Additionally we presented the relationship between this kind and other kinds. types.

5. Open problem

There are other issues that can be studied using the fuzzy soft notice, including fuzzy soft quasi normal, fuzzy soft N-normal, fuzzy soft b-matric space etc.

References