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# Set-valued nonlinear contractive operators in $PM$-spaces

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**Abstract**

In this paper, we consider some nonlinear contraction for set-valued operators and prove some fixed point results in the case of set-valued operators are ordered-close and not ordered-close, and in the case of set-valued operators are UCAV (LCAV) in quasi-ordered $PM$-spaces. Moreover, we present two examples and an application to show the validity of the main theorems.

*Keywords:* Fixed point, $PM$-spaces, quasi-ordered $PM$-spaces, ordered-close operator.
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## 1. Introduction and preliminaries

The investigation of contractivity on $PM$-spaces was initiated in 1966 by Sehgal [14]. The fixed point theorems in $PM$-spaces are important because they suggest a significant instrumentation to solve random equations. Hence, some of probabilistic contractions have been defined in [6, 7, 8], and references contained therein.

In 2009, Ćirić et al. [5] obtained several fixed point theorems in $PM$-spaces equipped with a quasi-order. After that, several fixed point results in quasi-ordered $PM$-spaces were investigated in [3, 4, 9, 10, 12, 13] and references contained therein. In 2014, Wu [16] proved the following theorem for the single-valued operator in a $PM$-space.

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**Theorem 1.1.** *[16] Let $(\mathcal{X}, \mathcal{F}, \mathcal{T}, \preceq)$ be a complete PM-space provided with a partial order " $\preceq$ " on $\mathcal{X}$, $G : \mathcal{X} \longrightarrow \mathcal{X}$ be a single-valued operator and $\Phi$ be all of the functions $\varphi : [0, +\infty) \longrightarrow [0, +\infty)$ such that $\varphi(i) < i$ and $lim_{n\to\infty}\varphi^n(i) = 0$ for all $i > 0$. Assume that the following properties are held:*

*$H_1$) There exists $x \in \mathcal{X}$ provided that $x \preceq Gx$;*

*$H_2$) For every $x, y \in \mathcal{X}$ with $x \preceq y$, $\varphi \in \Phi$: $\mathcal{F}_{Gx,Gy}(\varphi(t)) \geq min\{\mathcal{F}_{Gx,x}(t), \mathcal{F}_{x,y}(t), \mathcal{F}_{Gy,y}(t)\}$;*

*$H_3$) If $\{x_n\}$ a non-decreasing monotone convergent sequence to $x^*$, then $x_n \preceq x^*$ for each $n$.*

*Then $G$ have a fixed point $x^*$ in $\mathcal{X}$.*

In this paper, some nonlinear contractions for set-valued operators are considered and several fixed point results in the case of are ordered-close and are not ordered-close in $PM$-spaces provided with a quasi-order are proven. Moreover, we prepare several fixed point results for UCAV (LCAV) operators satisfying some set-valued contractions in quasi-ordered $PM$-spaces. To motivate this study, two illustrative examples and an application are considered.

In the following, we give some preliminary definitions which are needed. Throughout this paper, we denote $CB(\mathcal{X})$ for the collection of every nonempty closed and bounded subsets of $\mathcal{X}$ and denote $N(\mathcal{X})$ for the collection of every nonempty subsets of $\mathcal{X}$.

**Definition 1.2.** *[11] Consider a quasi-ordered set $(\mathcal{X}, \preceq)$ with two nonempty subsets $A$ and $B$ of $N(\mathcal{X})$. The relation between $A$ and $B$ is considered as follow:*

($r_1$) *If for all $a \in A$, there is $b \in B$ provided that $a \preceq b$, then $A \sqsubseteq_1 B$.*

($r_2$) *If for all $b \in B$, there is $a \in A$ provided that $a \preceq b$, then $A \sqsubseteq_2 B$.*

($r_3$) *If $A \sqsubseteq_1 B$ and $A \sqsubseteq_2 B$, then $A \sqsubseteq B$.*

**Definition 1.3.** *[2] Consider a ordered set $(\mathcal{X}, \preceq)$ with $\{x_n\} \subset \mathcal{X}$ provided that $\cdots \preceq x_n \preceq \cdots \preceq x_2 \preceq x_1$ or $x_1 \preceq x_2 \preceq \cdots \preceq x_n \preceq \cdots$. Then $\{x_n\}$ is called a monotone sequence.*

About definitions such as distribution function, triangular norm (abbreviated, $t$-norm), H-type (Hadzić type)$t$-norm and etc, one can refer to [5, 8] and their references therein. Also, $D^+$ is considered for the set of every Menger distance distribution functions.

**Definition 1.4.** *[8] Let $\mathcal{X}$ be a nonempty set, $\mathcal{T}$ be a continuous $t$-norm and $\mathcal{F} : \mathcal{X} \times \mathcal{X} \to D^+$ be a mapping such that*

*(PM1) $\mathcal{F}_{x,y}(t) = 1$ for all $t > 0$ iff $x = y$,*

*(PM2) $\mathcal{F}_{x,y}(t) = \mathcal{F}_{y,x}(t)$ for every $x, y \in \mathcal{X}$ and $t > 0$,*

*(PM3) $\mathcal{F}_{x,z}(t + s) \geq \mathcal{T}(\mathcal{F}_{x,y}(t), \mathcal{F}_{y,z}(s))$ for every $x, y, z \in \mathcal{X}$ and $t, s \geq 0$.*

*Then $(\mathcal{X}, \mathcal{F}, \mathcal{T})$ is a Menger PM-space.*

**Definition 1.5.** *[8] Consider a PM-space $(\mathcal{X}, \mathcal{F}, \mathcal{T})$. For every $A \subset \mathcal{X}$ and $x \in \mathcal{X}$, $\mathcal{F}_{x,A}(t) = sup_{p\in A}\mathcal{F}_{p,x}(t)$ is the distance between a point and a set in PM-space.*

About the definitions of convergent, Cauchy sequence, and etc, one can refer to [8].

**Lemma 1.6.** *[16] Assume $n \geq 1$, $\mathcal{F} \in D^+$, $\varphi \in \Phi$, $g_1, g_2, ..., g_n : \mathbb{R} \longrightarrow [0,1]$ and*

$$\mathcal{F}(\varphi(t)) \geq min\{g_1(t), g_2(t), ..., g_n(t), \mathcal{F}(t)\}.$$

*Then $\mathcal{F}(\varphi(t)) \geq min\{g_1(t), g_2(t), ..., g_n(t)\}$ for every $t \geq 0$.*

**Lemma 1.7.** *[16] Consider a PM-space $(\mathcal{X}, \mathcal{F}, \mathcal{T})$. If $\mathcal{F}_{p,q}(\varphi(t)) = \mathcal{F}_{p,q}(t)$ for every $t > 0$, where $\varphi \in \Phi$. Then $p = q$.*

**Definition 1.8.** *The set-valued operator $G : \mathcal{X} \to CB(\mathcal{X})$ is called ordered-close if for two monotone sequences $\{x_n\}, \{y_n\} \subset \mathcal{X}$ and $x_0, y_0 \in \mathcal{X}$; $x_n \to x_0$, $y_n \to y_0$ and $y_n \in G(x_n)$ imply $y_0 \in G(x_0)$.*

## 2. Results on ordered-close and non ordered-close set-valued operators in $PM$-spaces

Here, we first establish several fixed point results for ordered-close and non ordered-close set-valued operators.

**Theorem 2.1.** *Suppose that $(\mathcal{X}, \mathcal{F}, \mathcal{T})$ is a complete PM-space provided with a quasi-order " $\preceq$ ". Also, suppose that $\mathcal{T}$ is a Hadzić-type t-norm, $\varphi \in \Phi$ and $G : \mathcal{X} \longrightarrow CB(\mathcal{X})$ is an ordered-close set-valued operator. Further, assume that*

*($H_1$) for every $x \in \mathcal{X}$, there is $y \in Gx$ provided that $x \preceq y$;*

*($H_2$) for every $x, y \in \mathcal{X}$ with $x \preceq y$, $u \in Gx$ and $v \in Gy$:*

$$\mathcal{F}_{u,v}(\varphi(t)) \geq min\{\mathcal{F}_{u,x}(t), \mathcal{F}_{x,y}(t), \mathcal{F}_{v,y}(t)\}.$$

*Then $G$ have a fixed point $x^*$ in $\mathcal{X}$.*

**Proof .** Select $x_0 \in \mathcal{X}$. If $x_0 \in Gx_0$, then the proof is complete. Otherwise, there is $x_1 \in Gx_0$ with $x_0 \neq x_1$ and $x_0 \preceq x_1$. Also, for $x_1 \in \mathcal{X}$, there exists $x_2 \in Gx_1$ with $x_1 \neq x_2$ and $x_1 \preceq x_2$. Continue this process, we obtain a non-decreasing sequence $\{x_n\}$, where $x_n \in Gx_{n-1}$ and $x_{n-1} \preceq x_n$. By the condition $(H_2)$, we have

$$\mathcal{F}_{x_n,x_{n+1}}(\varphi(t)) \geq min\{\mathcal{F}_{x_n,x_{n-1}}(t), \mathcal{F}_{x_{n-1},x_n}(t), \mathcal{F}_{x_{n+1},x_n}(t)\}.$$

Then $\mathcal{F}_{x_n,x_{n+1}}(\varphi(t)) \geq \mathcal{F}_{x_{n-1},x_n}(t)$ for all $t > 0$. Hence, we obtain

$$\mathcal{F}_{x_n,x_{n+1}}(\varphi^2(t)) \geq \mathcal{F}_{x_{n-1},x_n}(\varphi(t)) \geq \mathcal{F}_{x_{n-2},x_{n-1}}(t)$$

for every $t > 0$. Thus, $\mathcal{F}_{x_n,x_{n+1}}(\varphi^n(t)) \geq \mathcal{F}_{x_1,x_0}(t)$ for all $t > 0$. Since $lim_{t \longrightarrow \infty} \mathcal{F}_{x_1,x_0}(t) = 1$, there is a $t_0$ so that $\mathcal{F}_{x_1,x_0}(t_0) > 1 - \varepsilon$ for $\delta > 0$ and $\varepsilon \in (0,1)$. Also, by $lim_{n \to \infty} \varphi^n(t) = 0$, there is a $N_0 \in \mathbb{N}$ provided that $\varphi^n(t_0) < \delta$ for $n > N_0$. Thus, we obtain

$$\mathcal{F}_{x_n,x_{n+1}}(\delta) \geq \mathcal{F}_{x_n,x_{n+1}}(\varphi^n(t_0)) \geq \mathcal{F}_{x_1,x_0}(t_0) > 1 - \varepsilon$$

for $n > N_0$, that's mean $lim_{n \to \infty} \mathcal{F}_{x_{n+1},x_n}(t) = 1$. Next, we should show that for every $\delta > 0$ and $\varepsilon \in (0,1)$, there exists $N(\varepsilon, \delta)$ provided that for all $m > n > N(\delta, \varepsilon)$, we have $\mathcal{F}_{x_n,x_m}(\delta) > 1 - \varepsilon$. Firstly, we show the inequality

$$\mathcal{F}_{x_{n+k},x_n}(\delta) \geq \mathcal{T}^k(\mathcal{F}_{x_{n+1},x_n}(\delta - \varphi(\delta))) \tag{2.1}$$

for every $k \geq 1$ is established by induction. For $k = 1$, we have

$$\begin{aligned}\mathcal{F}_{x_n,x_{n+1}}(\delta) &\geq (\mathcal{F}_{x_{n+1},x_n}(\delta - \varphi(\delta)))\\ &= \mathcal{T}(\mathcal{F}_{x_{n+1},x_n}(\delta - \varphi(\delta)), 1)\\ &\geq \mathcal{T}\Big(\mathcal{F}_{x_{n+1},x_n}(\delta - \varphi(\delta)), \mathcal{F}_{x_{n+1},x_n}(\delta - \varphi(\delta))\Big)\\ &= \mathcal{T}^1(\mathcal{F}_{x_{n+1},x_n}(\delta - \varphi(\delta))).\end{aligned}$$

Hence, (2.1) holds for $k = 1$. Assume that (2.1) is held for $1 \leq k \leq p$. If $k = p + 1$, then

$$\mathcal{F}_{x_{n+p+1},x_n}(\delta) \geq \mathcal{T}\Big(\mathcal{F}_{x_{n+1},x_n}(\delta - \varphi(\delta)), \mathcal{F}_{x_{n+1},x_{n+p+1}}(\varphi(\delta))\Big). \tag{2.2}$$

By $(H_2)$ and by contradiction we show that $\mathcal{F}_{x_{n+1},x_{n+2}}(\delta) \geq \mathcal{F}_{x_n,x_{n+1}}(\delta)$. Thus, $\mathcal{F}_{x_{n+p+1},x_{n+p}}(\delta) \geq \mathcal{F}_{x_n,x_{n+1}}(\delta)$. Therefore,

$$\begin{aligned}\mathcal{F}_{x_{n+1},x_{n+p+1}}\Big(\varphi(\delta)\Big) &\geq min\{\mathcal{F}_{x_n,x_{n+1}}(\delta), \mathcal{F}_{x_n,x_{n+p}}(\delta), \mathcal{F}_{x_{n+p},x_{n+p+1}}(\delta)\}\\ &= min\{\mathcal{F}_{x_n,x_{n+1}}(\delta), \mathcal{F}_{x_n,x_{n+p}}(\delta)\}\\ &\geq min\Big\{\mathcal{F}_{x_n,x_{n+1}}(\delta - \varphi(\delta)), \mathcal{T}^p(\mathcal{F}_{x_n,x_{n+1}}(\delta - \varphi(\delta)))\Big\}\\ &= \mathcal{T}^p\Big(\mathcal{F}_{x_n,x_{n+1}}(\delta - \phi(\delta))\Big).\end{aligned} \tag{2.3}$$

By (2.2) and (2.3), we obtain

$$\begin{aligned}\mathcal{F}_{x_{n+p+1},x_n}(\delta) &\geq \mathcal{T}\Big(\mathcal{F}_{x_n,x_{n+1}}(\delta - \varphi(\delta)), \mathcal{T}^p(\mathcal{F}_{x_n,x_{n+1}}(\delta - \varphi(\delta)))\Big)\\ &= \mathcal{T}^{p+1}\Big(\mathcal{F}_{x_n,x_{n+1}}(\delta - \varphi(\delta))\Big).\end{aligned}$$

Thus, (2.1) is held. Also, since $t$-norm $\mathcal{T}$ is H-type, for a selective $\varepsilon \in (0, 1)$, there is $\lambda \in (0, 1)$ so that for all $n \geq 1$, $\mathcal{T}^n(t) > 1 - \varepsilon$ when $t > 1 - \lambda$. By $lim_{n\to\infty}\mathcal{F}_{x_n,x_{n+1}}(\delta - \varphi(\delta)) = 1$, there exists a $N_1(\varepsilon, \delta)$ so that $\mathcal{F}_{x_n,x_{n+1}}(\delta - \varphi(\delta)) > 1 - \lambda$ for each $n > N_1(\varepsilon, \delta)$. Therefore,

$$\mathcal{F}_{x_{n+k},x_n}(\delta) \geq \mathcal{T}^k(\mathcal{F}_{x_{n+1},x_n}(\delta - \varphi(\delta))) \geq \mathcal{T}^k(1 - \lambda) \geq 1 - \varepsilon$$

for all $k \geq 1$. So the sequence $\{x_n\}$ is a Cauchy sequence in $\mathcal{X}$. Due to the completeness of $\mathcal{X}$, there is $x^* \in \mathcal{X}$ provided that $lim_{n\to\infty}x_n = x^*$. Since $G$ is ordered-closed, $\{x_n\}$ is monotone and $x_{n+1} \in Gx_n$, we deduce $x^* \in Gx^*$ and $x^*$ is a fixed point of $G$. □

**Theorem 2.2.** *Suppose that $(\mathcal{X}, \mathcal{F}, \mathcal{T})$ is a complete PM-space provided with a quasi-order " $\preceq$ ". Also, suppose that $\mathcal{T}$ is a Hadzić-type t-norm, $\varphi \in \Phi$ and $G : \mathcal{X} \longrightarrow CB(\mathcal{X})$ is an ordered-close set-valued operator. Further, assume that*

*$(H_1)$ for every $x \in \mathcal{X}$, there is $y \in G(x)$ provided that $y \preceq x$;*

*$(H_2)$ for every $x, y \in \mathcal{X}$ with $y \preceq x$, $u \in G(x)$ and $v \in G(y)$:*

$$\mathcal{F}_{u,v}(\varphi(t)) \geq min\{\mathcal{F}_{u,x}(t), \mathcal{F}_{x,y}(t), \mathcal{F}_{v,y}(t)\}.$$

*Then $G$ have a fixed point $x^*$ in $\mathcal{X}$.*

**Proof .** Select $x_0 \in \mathcal{X}$. If $x_0 \in Gx_0$, then the proof is complete. Otherwise, there is $x_1 \in Gx_0$ with $x_0 \neq x_1$ and $x_1 \preceq x_0$. Also, for $x_1 \in \mathcal{X}$, there exists $x_2 \in Gx_1$ with $x_1 \neq x_2$ and $x_2 \preceq x_1$. Continue this process, we have a non-increasing sequence $\{x_n\}$, where $x_n \in Gx_{n-1}$ and $x_n \preceq x_{n-1}$. The continue of the proof as is the same similar to the proof of previous theorem. □

**Theorem 2.3.** *Suppose that $(\mathcal{X}, \mathcal{F}, \mathcal{T})$ is a complete PM-space provided with a quasi-order " $\preceq$ ". Also, suppose that $\mathcal{T}$ is a Hadzić-type t-norm, $\varphi \in \Phi$ and $G : \mathcal{X} \longrightarrow CB(\mathcal{X})$ is a set-valued operator. Further, assume that*

*($H_1$) for every $x \in \mathcal{X}$, there is $y \in G(x)$ provided that $x \preceq y$;*

*($H_2$) for every $x, y \in \mathcal{X}$ with $x \preceq y$, $u \in G(x)$ and $v \in G(y)$:*

$$\mathcal{F}_{u,v}(\varphi(t)) \geq min\{\mathcal{F}_{u,x}(t), \mathcal{F}_{x,y}(t), \mathcal{F}_{v,y}(t)\};$$

*($H_3$) if $\{x_n\}$ is a non-decreasing monotone sequence convergent to $x^*$, then $x_n \preceq x^*$ for each $n$.*

*Then $G$ have a fixed point $x^*$ in $\mathcal{X}$.*

**Proof .** Select $x_0 \in \mathcal{X}$. If $x_0 \in Gx_0$, then the proof is complete. Otherwise, there is $x_1 \in Gx_0$ with $x_0 \neq x_1$ and $x_0 \preceq x_1$. Also, for $x_1 \in \mathcal{X}$, there exists $x_2 \in Gx_1$ with $x_1 \neq x_2$ and $x_1 \preceq x_2$. Continue this process, we obtain a non-decreasing sequence $\{x_n\}$, where $x_n \in Gx_{n-1}$ and $x_{n-1} \preceq x_n$. As in the proof of Theorem 2.1 one can show $\{x_n\}$ is convergent to $x^*$. Then, by $(H_3)$, $x_n \preceq x^*$ for all $n$. By $(H_2)$, for each $y \in Gx^*$, we have

$$\mathcal{F}_{x_{n+1},y}(\varphi(t)) \geq min\{\mathcal{F}_{x_n,x_{n+1}}(t), \mathcal{F}_{x^*,y}(t), \mathcal{F}_{x_n,x^*}(t)\}.$$

Letting $n \to \infty$, we obtain $\mathcal{F}_{x^*,y}(\varphi(t)) \geq \mathcal{F}_{x^*,y}(t)$. From Lemma 1.7, we get $y = x^*$ and $x^* \in Gx^*$. Therefore, $G$ have a fixed point in $\mathcal{X}$. □

**Corollary 2.4.** *Suppose that $(\mathcal{X}, \mathcal{F}, \mathcal{T})$ is a complete PM-space provided with a quasi-order " $\preceq$ ". Also, suppose that $\mathcal{T}$ is a Hadzić-type t-norm, $\varphi \in \Phi$ and $G : \mathcal{X} \longrightarrow CB(\mathcal{X})$ is a set-valued operator. Further, assume that*

*($H_1$) for every $x \in \mathcal{X}$ and for each $y \in G(x)$, we have $x \preceq y$;*

*($H_2$) for every $x, y \in \mathcal{X}$ with $x \preceq y$, $u \in G(x)$ and $v \in G(y)$:*

$$\mathcal{F}_{u,v}(\varphi(t)) \geq min\{\mathcal{F}_{u,x}(t), \mathcal{F}_{x,y}(t), \mathcal{F}_{v,y}(t)\};$$

*($H_3$) if $\{x_n\}$ is a non-decreasing monotone sequence convergent to $x^*$, then $x_n \preceq x^*$ for all $n$.*

*Then there is $x^* \in \mathcal{X}$ so that $Gx^* = \{x^*\}$.*

**Theorem 2.5.** *Suppose that $(\mathcal{X}, \mathcal{F}, \mathcal{T})$ is a complete PM-space provided with a quasi-order " $\preceq$ ". Also, suppose that $\mathcal{T}$ is a Hadzić-type t-norm, $\varphi \in \Phi$ and $G : \mathcal{X} \longrightarrow CB(\mathcal{X})$ is a set-valued operator. Further, assume that*

*($H_1$) for each $x \in \mathcal{X}$, there is $y \in G(x)$ provided that $y \preceq x$;*

*($H_2$) for every $x, y \in \mathcal{X}$ with $y \preceq x$, $u \in G(x)$ and $v \in G(y)$:*

$$\mathcal{F}_{u,v}(\varphi(t)) \geq min\{\mathcal{F}_{u,x}(t), \mathcal{F}_{x,y}(t), \mathcal{F}_{v,y}(t)\};$$

*($H_3$) If $\{x_n\}$ is a non-increasing monotone sequence convergent to $x^*$, then $x^* \preceq x_n$ for all $n$.*

*Then $G$ have a fixed point $x^*$ in $\mathcal{X}$.*

**Proof .** Consider $x_0 \in \mathcal{X}$. If $x_0 \in Gx_0$, then the proof is complete. Otherwise, there is $x_1 \in Gx_0$ with $x_0 \neq x_1$ and $x_0 \succeq x_1$. Also, for $x_1 \in \mathcal{X}$, there exists $x_2 \in Gx_1$ with $x_1 \neq x_2$ and $x_1 \succeq x_2$. Continue this process, we have a non-increasing sequence $\{x_n\}$, where $x_n \in Gx_{n-1}$ and $x_{n-1} \succeq x_n$. Similar to proof of Theorem 2.2 one can show $\{x_n\}$ is convergent to $x^*$. Then, by ($H_3$), $x_n \succeq x^*$ for all $n$. By ($H_2$), for each $y \in Gx^*$, we have

$$\mathcal{F}_{x_{n+1},y}(\varphi(t)) \geq min\{\mathcal{F}_{x_n,x_{n+1}}(t), \mathcal{F}_{x^*,y}(t), \mathcal{F}_{x_n,x^*}(t)\}.$$

Letting $n \to \infty$, we have $\mathcal{F}_{x^*,y}(\varphi(t)) \geq \mathcal{F}_{x^*,y}(t)$. From Lemma 1.7 we have $y = x^*$ and $x^* \in Gx^*$. Therefore, $G$ has a fixed point in $\mathcal{X}$. □

**Corollary 2.6.** *Suppose that $(\mathcal{X}, \mathcal{F}, \mathcal{T})$ is a complete PM-space provided with a quasi-order " $\preceq$ ". Also, suppose that $\mathcal{T}$ is a Hadzić-type t-norm, $\varphi \in \Phi$ and $G : \mathcal{X} \longrightarrow CB(\mathcal{X})$ is a set-valued operator. Further, assume that*

*($H_1$) for every $x \in \mathcal{X}$ and for each $y \in G(x)$, we have $y \preceq x$;*

*($H_2$) for every $x, y \in \mathcal{X}$ with $y \preceq x$, $u \in G(x)$ and $v \in G(y)$:*

$$\mathcal{F}_{u,v}(\varphi(t)) \geq min\{\mathcal{F}_{u,x}(t), \mathcal{F}_{x,y}(t), \mathcal{F}_{v,y}(t)\};$$

*($H_3$) If $\{x_n\}$ is a non-increasing monotone sequence in $\mathcal{X}$ and convergent to $x^*$, then $x^* \preceq x_n$ for each $n$.*

*Then there is $x^* \in \mathcal{X}$ so that $Gx^* = \{x^*\}$.*

**Theorem 2.7.** *Suppose that $(\mathcal{X}, \mathcal{F}, \mathcal{T})$ is a complete PM-space provided with a quasi-order " $\preceq$ ". Also, suppose that $\mathcal{T}$ is a Hadzić-type t-norm, $\varphi \in \Phi$ and $G : \mathcal{X} \longrightarrow CB(\mathcal{X})$ is an ordered-close set-valued operator. Further, assume that*

*($H_1$) for each $x, y \in \mathcal{X}$ with $x \preceq y$, we have $Gx \sqsubseteq_1 Gy$;*

*($H_2$) there is $x_0 \in \mathcal{X}$ such that $\{x_0\} \sqsubseteq_1 Gx_0$;*

*($H_3$) for every $x, y \in \mathcal{X}$ with $x \preceq y$, $u \in G(x)$ and $v \in G(y)$:*

$$\mathcal{F}_{u,v}(\varphi(t)) \geq min\{\mathcal{F}_{u,x}(t), \mathcal{F}_{x,y}(t), \mathcal{F}_{v,y}(t)\}.$$

*Then $G$ have a fixed point $x^*$ in $\mathcal{X}$.*

**Proof .** From ($H_2$), there exists $x_1 \in Gx_0$ such that $x_0 \preceq x_1$. This implies that $Gx_0 \sqsubseteq_1 Gx_1$ by ($H_1$). By definition 1.2, there is $x_2 \in Gx_1$ provided that $x_1 \preceq x_2$. Continue this procedure, we have a non-decreasing sequence $\{x_n\}$ so that $x_n \in Gx_{n-1}$. Similar to the proof of Theorem 2.1, $x^*$ is a fixed point of $G$. □

**Theorem 2.8.** *Suppose that $(\mathcal{X},\mathcal{F},\mathcal{T})$ is a complete $PM$-space provided with a quasi-order " $\preceq$ ". Also, suppose that $\mathcal{T}$ is a Hadzić-type t-norm, $\varphi\in\Phi$ and $G:\mathcal{X}\longrightarrow CB(\mathcal{X})$ is an ordered-close set-valued operator. Further, assume that*

*($H_1$) for every $x,y\in\mathcal{X}$ with $x\preceq y$, we have $Gx\sqsubseteq_2 Gy$;*

*($H_2$) there is $x_0\in\mathcal{X}$ so that $Gx_0\sqsubseteq_2\{x_0\}$;*

*($H_3$) for each $x,y\in\mathcal{X}$ with $y\preceq x$, $u\in G(x)$ and $v\in G(y)$:*

$$\mathcal{F}_{u,v}(\varphi(t))\geq min\{\mathcal{F}_{u,x}(t),\mathcal{F}_{x,y}(t),\mathcal{F}_{v,y}(t)\}.$$

*Then $G$ have a fixed point $x^*$ in $\mathcal{X}$.*

**Proof .** The proof is analogously the proof of Theorem 2.7. □

**Theorem 2.9.** *Suppose that $(\mathcal{X},\mathcal{F},\mathcal{T})$ is a complete $PM$-space provided with a quasi-order " $\preceq$ ". Also, suppose that $\mathcal{T}$ is a Hadzić-type t-norm, $\varphi\in\Phi$ and $G:\mathcal{X}\longrightarrow CB(\mathcal{X})$ is a set-valued operator. Further, assume that*

*($H_1$) for all $x,y\in\mathcal{X}$ with $x\preceq y$, we have $Gx\sqsubseteq_1 Gy$;*

*($H_2$) there is $x_0\in\mathcal{X}$ such that $\{x_0\}\sqsubseteq_1 Gx_0$;*

*($H_3$) for every $x,y\in\mathcal{X}$ with $x\preceq y$, $u\in G(x)$ and $v\in G(y)$:*

$$\mathcal{F}_{u,v}(\varphi(t))\geq min\{\mathcal{F}_{u,x}(t),\mathcal{F}_{x,y}(t),\mathcal{F}_{v,y}(t)\};$$

*($H_4$) if $\{x_n\}$ is a non-decreasing monotone sequence convergent to $x^*$, then $x_n\preceq x^*$ for each $n$.*

*Then $G$ have a fixed point $x^*$ in $\mathcal{X}$.*

**Proof .** From ($H_2$), there is $x_1\in Gx_0$ such that $x_0\preceq x_1$. This implies that $Gx_0\sqsubseteq_1 Gx_1$ by ($H_1$). By Definition 1.2, there is $x_2\in Gx_1$ such that $x_1\preceq x_2$. Continue this procedure, we have a non-decreasing sequence $\{x_n\}$ such that $x_n\in Gx_{n-1}$. The rest of the proof is in the like manner given in Theorem 2.3. □

**Theorem 2.10.** *Suppose that $(\mathcal{X},\mathcal{F},\mathcal{T})$ is a complete $PM$-space provided with a quasi-order " $\preceq$ ". Also, suppose that $\mathcal{T}$ is a Hadzić-type t-norm, $\varphi\in\Phi$ and $G:\mathcal{X}\longrightarrow CB(\mathcal{X})$ is a set-valued operator. Further, assume that*

*($H_1$) for every $x,y\in\mathcal{X}$ with $x\preceq y$, we have $Gx\sqsubseteq_2 Gy$;*

*($H_2$) there is $x_0\in\mathcal{X}$ such that $Gx_0\sqsubseteq_2\{x_0\}$;*

*($H_3$) for every $x,y\in\mathcal{X}$ with $x\preceq y$, $u\in G(x)$ and $v\in G(y)$:*

$$\mathcal{F}_{u,v}(\varphi(t))\geq min\{\mathcal{F}_{u,x}(t),\mathcal{F}_{x,y}(t),\mathcal{F}_{v,y}(t)\};$$

*($H_4$) if $\{x_n\}$ is a non-increasing monotone sequence in $\mathcal{X}$ and convergent to $x^*$, then $x^*\preceq x_n$ for each $n$.*

*Then $G$ have a fixed point $x^*$ in $\mathcal{X}$.*

**Proof .** The proof is like the proof of Theorem 2.9. □

**Example 2.11.** *Suppose $\mathcal{X} = \mathbb{R}^+$, " $\preceq$ " is a quasi-order on $\mathcal{X}$, $\mathcal{T}(a,b) = min\{a,b\}$ for every $a,b \in [0,1]$ and*

$$\mathcal{F}_{x,y}(t) = \begin{cases} 1, & x = y \\ \frac{1}{\exp(t)}, & x \neq y \end{cases}$$

*Then $(\mathcal{X},\mathcal{F},\mathcal{T})$ is a quasi-ordered complete PM-space provided with the quasi-order " $\preceq$ ". Also, consider $\varphi(t) = \frac{t}{2}$ for $t \geq 0$ and define $G : \mathcal{X} \to CB(\mathcal{X})$ by $G(x) = [0, 3x]$. Thus $G$ is satisfied in condition $(H_1)$ of Theorem 2.1. Now, for each $x, y \in \mathcal{X}$ with $x \neq y$ and for each $u \in Gx$ and $v \in Gy$ with $u \neq v$, we have*

$$\mathcal{F}_{u,v}(\varphi(t)) \geq \min\{\mathcal{F}_{u,x}(t), \mathcal{F}_{x,y}(t), \mathcal{F}_{v,y}(t)\}.$$

*Therefore, Theorem 2.1 implies that $G$ have a fixed point in $\mathbb{R}^+$.*

## 3. Results on set-valued operators in quasi-ordered *PM*-spaces

In this section, we give several fixed point results for set-valued contractive operators in quasi-ordered $PM$-space, where quasi-order is a reflexive and transitive relation.

**Definition 3.1.** *Consider a PM-space $(\mathcal{X},\mathcal{F},\mathcal{T})$ provided with a quasi-order " $\preceq$ " on $\mathcal{X}$.*

*(i) A subset $D \subset \mathcal{X}$ is approximative, when the set-valued operator $P_{x,D}(t) = \{p \in D : \mathcal{F}_{x,D}(t) = \mathcal{F}_{p,x}(t)\}$ for every $x \in \mathcal{X}$ has nonempty value.*

*(ii) The operator $G : \mathcal{X} \longrightarrow CB(\mathcal{X})$ is approximative valued (briefly, AV), if $Gx$ is approximative for every $x \in \mathcal{X}$.*

*(iii) The operator $G : \mathcal{X} \longrightarrow CB(\mathcal{X})$ is comparable approximative valued (briefly, CAV), if $Gx$ has approximative values for every $x \in \mathcal{X}$, and there is $y \in P_{x,Gz}(t)$ provided that $y$ is comparable to $z$ for every $z \in \mathcal{X}$.*

*(iv) The operator $G : \mathcal{X} \longrightarrow CB(\mathcal{X})$ is upper comparable approximative valued (briefly, UCAV) [resp. lower comparable approximative values, (briefly, LCAV)] if $Gx$ has approximative values, and there is $y \in P_{x,Gz}(t)$ provided that $y \succeq z$ (resp. $y \preceq z$) for every $z \in \mathcal{X}$.*

**Theorem 3.2.** *Suppose that $(\mathcal{X},\mathcal{F},\mathcal{T})$ is a complete PM-space provided with a quasi-order " $\preceq$ ". Also, suppose that $\mathcal{T}$ is a Hadzić-type t-norm, $\varphi \in \Phi$ and $G : \mathcal{X} \longrightarrow CB(\mathcal{X})$ is a set-valued operator, where has UCAV. Further, assume that*

$$\mathcal{F}_{u,v}(\varphi(t)) \geq min\{\mathcal{F}_{u,x}(t), \mathcal{F}_{x,y}(t), \mathcal{F}_{v,y}(t)\}$$

*for every $x, y \in \mathcal{X}$ with $x \preceq y$, $u \in Gx$ and $v \in Gy$. Then $G$ have a fixed point $x^*$ in $\mathcal{X}$.*

**Proof .** Consider $x_0 \in \mathcal{X}$. If $x_0 \in Gx_0$, then the proof is complete. Otherwise, since $Gx_0$ has UCAV, there is $x_1 \in Gx_0$ with $x_0 \neq x_1$ and $x_0 \preceq x_1$ so that $\mathcal{F}_{x_0,x_1}(t) = \mathcal{F}_{x_0,Gx_0}(t)$ and there is $x_2 \in Gx_1$ with $x_2 \neq x_1$ and $x_1 \preceq x_2$ so that $\mathcal{F}_{x_1,x_2}(t) = \mathcal{F}_{x_1,Gx_1}(t)$. Continue this procedure, we obtain a non-decreasing sequence $\{x_n\}$, where $x_n \in Gx_{n-1}$ and $x_{n-1} \preceq x_n$. Analogous the proof of the Theorem 2.1, one can show $G$ have a fixed point $x^*$ in $\mathcal{X}$. □

**Theorem 3.3.** *Suppose that $(\mathcal{X},\mathcal{F},\mathcal{T})$ is a complete PM-space provided with a quasi-order " $\preceq$ ". Also, suppose that $\mathcal{T}$ is a Hadzić-type t-norm, $\varphi \in \Phi$ and $G : \mathcal{X} \longrightarrow CB(\mathcal{X})$ is a set-valued operator, where has LCAV. Further, assume that*

$$\mathcal{F}_{u,v}(\varphi(t)) \geq min\{\mathcal{F}_{u,x}(t), \mathcal{F}_{x,y}(t), \mathcal{F}_{v,y}(t)\}$$

*for every $x, y \in \mathcal{X}$ with $x \preceq y$, $u \in Gx$ and $v \in Gy$. Then $G$ have a fixed point $x^*$ in $\mathcal{X}$.*

**Proof .** Consider $x_0 \in \mathcal{X}$. If $x_0 \in Gx_0$, then the proof is complete. Otherwise, since $Gx_0$ has LCAV, there is $x_1 \in Gx_0$ with $x_0 \neq x_1$ and $x_1 \preceq x_0$ so that $\mathcal{F}_{x_0,x_1}(t) = \mathcal{F}_{x_0,Gx_0}(t)$ and there is $x_2 \in Gx_1$ with $x_2 \neq x_1$ and $x_2 \preceq x_1$ so that $\mathcal{F}_{x_1,x_2}(t) = \mathcal{F}_{x_1,Gx_1}(t)$. Continue this procedure, we obtain a non-increasing sequence $\{x_n\}$, where $x_n \in Gx_{n-1}$ and $x_n \preceq x_{n-1}$. Analogous the proof of the Theorem 2.2, one can show $G$ have a fixed point $x^*$ in $\mathcal{X}$. □

**Theorem 3.4.** *Suppose that $(\mathcal{X}, \mathcal{F}, \mathcal{T})$ is a complete PM-space provided with a quasi-order " $\preceq$ ". Also, suppose that $\mathcal{T}$ is a Hadzić-type t-norm, $\varphi \in \Phi$ and $G : \mathcal{X} \longrightarrow CB(\mathcal{X})$ is a set-valued operator, where has AV. Further, assume that $x_0 \in \mathcal{X}$ so that $\{x_0\} \sqsubseteq Gx_0$ and*

$$\mathcal{F}_{u,v}(\varphi(t)) \geq min\{\mathcal{F}_{u,x}(t), \mathcal{F}_{x,y}(t), \mathcal{F}_{v,y}(t)\}$$

*for every $x, y \in \mathcal{X}$ with $x \preceq y$, $u \in Gx$ and $v \in Gy$. Then $G$ have a fixed point $x^*$ in $\mathcal{X}$.*

**Proof .** If $x_0 \in Gx_0$, then the proof ends. Otherwise, for any $x \in Gx_0$ one has that $x \succeq x_0$. Since $G$ has approximative values, there exists $x_1 \in Gx_0$ with $x_0 \preceq x_1$ and $\mathcal{F}_{x_0,x_1}(t) = \mathcal{F}_{x_0,Gx_0}(t)$. Continue the procedure of constructing $x_n$ inductively. Then there is $x_n \in Gx_{n-1}$ with $x_n \neq x_{n-1}$ and $x_{n-1} \preceq x_n$. The rest of the proof is like manner given in Theorem 3.2. □

**Theorem 3.5.** *Suppose that $(\mathcal{X}, \mathcal{F}, \mathcal{T})$ is a complete PM-space provided with a quasi-order " $\preceq$ ". Also, suppose that $\mathcal{T}$ is a Hadzić-type t-norm, $\varphi \in \Phi$ and $G : \mathcal{X} \longrightarrow CB(\mathcal{X})$ is a set-valued operator, where has AV. Further, assume that $x_0 \in \mathcal{X}$ so that $Gx_0 \sqsubseteq \{x_0\}$ and*

$$\mathcal{F}_{u,v}(\varphi(t)) \geq min\{\mathcal{F}_{u,x}(t), \mathcal{F}_{x,y}(t), \mathcal{F}_{v,y}(t)\}$$

*for every $x, y \in \mathcal{X}$ with $x \preceq y$, $u \in Gx$ and $v \in Gy$. Then $G$ have a fixed point $x^*$ in $\mathcal{X}$.*

**Proof .** If $x_0 \in Gx_0$, then the proof ends. Otherwise, for any $x \in Gx_0$ one has that $x_0 \succeq x$. Since $G$ has approximative values, there is $x_1 \in Gx_0$ with $x_1 \preceq x_0$ and $\mathcal{F}_{x_0,x_1}(t) = \mathcal{F}_{x_0,Gx_0}(t)$. Continue the procedure of constructing $x_n$ inductively. Then there exists $x_n \in Gx_{n-1}$ with $x_n \neq x_{n-1}$ and $x_n \preceq x_{n-1}$. The rest of the proof is like manner given in Theorem 3.3. □

**Example 3.6.** *Suppose $\mathcal{X} = \mathbb{R}^+$, " $\preceq$ " is a quasi-order on $\mathcal{X}$, $\mathcal{T}(a, b) = min\{a, b\}$ for all $a, b \in [0, 1]$ and*

$$\mathcal{F}_{x,y}(t) = \begin{cases} 1, & if\ d(x, y) < t\ and\ t > 0 \\ 0, & if\ d(x, y) \geq t\ or\ t \leq 0 \end{cases}$$

*Then $(\mathcal{X}, \mathcal{F}, \mathcal{T})$ is a complete PM-space provided with the quasi-order " $\preceq$ ". Also, consider $\varphi(t) = \frac{t}{2}$ for $t \geq 0$ and define $G : \mathcal{X} \to CB(\mathcal{X})$ by $G(x) = [0, 3x]$. Thus $G$ is satisfied in condition $(H_1)$ of Theorem 2.1. Now, for each $x, y \in \mathcal{X}$ with $x \neq y$ and for each $u \in Gx$ and $v \in Gy$ with $u \neq v$, we have*

$$\mathcal{F}_{u,v}(\varphi(t)) \geq \min\{\mathcal{F}_{u,x}(t), \mathcal{F}_{x,y}(t), \mathcal{F}_{v,y}(t)\}.$$

*Therefore, Theorem 2.1 implies that $G$ have a fixed point in $\mathbb{R}^+$.*

*Also, $\varphi(t) = \frac{t}{2}$ for $t \geq 0$ and $G : \mathcal{X} \to CB(\mathcal{X})$ be defined by $G(x) = [0, 3x]$. Since for each $x_0 \in \mathcal{X}$ there exist $x_1 \in [0, 3x_0]$ provided that $\mathcal{F}_{x_0,x_1}(t) = \mathcal{F}_{x_0,Gx_0}(t)$ and $x_0 \preceq x_1$, then $G$ is UCAV. Now, for each $x, y \in \mathcal{X}$ with $x \neq y$ and for each $u \in Gx$ and $v \in Gy$ with $u \neq v$, we have*

$$\mathcal{F}_{u,v}(\varphi(t)) \geq \min\{\mathcal{F}_{u,x}(t), \mathcal{F}_{x,y}(t), \mathcal{F}_{v,y}(t)\}.$$

*Therefore, Theorem 3.2 implies that $G$ have a fixed point in $\mathbb{R}^+$.*

## 4. Application

Here, by Theorem 2.1, we establish the existence of a solution for the following Volterra type integral equation.

$$y(u) = \int_0^u M(u, v, y(v))dv + g(u) \quad (u \in I) \tag{4.1}$$

for all $u \in I = [0, a]$, where $a > 0$ and $[0, a]$ is a real interval.

First, we introduce the mathematical background (see also [1, 13]). Suppose $C(I, \mathbb{R})$ is the Banach space of whole real continuous functions considered on $I$ with two norms
$(i)$ maximum norm $||y||_\infty$ for $y \in C(I, \mathbb{R})$
$(ii)$ Bielecki norm $||y||_B = \max_{u \in I}(|y(u)|e^{-Lu})$ for all $y \in C(I, \mathbb{R})$ and $L > 0$ which induces a metric $d_B(x, y) = ||x - y||_B$ for all $x, y \in C(I, \mathbb{R})$ (see [1]).

Also, assume $C(I \times I \times C(I, \mathbb{R}), \mathbb{R})$ is the space of whole continuous functions considered on $I \times I \times C(I, \mathbb{R})$. Now, consider the mapping $\mathcal{F} : C(I, \mathbb{R}) \times C(I, \mathbb{R}) \to D^+$ with $\mathcal{F}_{x,y}(t) = \chi(t - d_B(x, y))$ for every $x, y \in C(I, \mathbb{R})$ and $t > 0$, where

$$\chi(t) = \begin{cases} 0 & if \quad t \leq 0, \\ 1 & if \quad t > 0. \end{cases}$$

The space $(C(I, \mathbb{R}), \mathcal{F}, \mathcal{T})$ with $\mathcal{T}(a, b) = min\{a, b\}$ is a complete $PM$-space ([15, Theorem 3]). Note that the convergence in both norms $||.||_\infty$ and $||.||_B$ are equivalent in this spaces. Also, we define quasi-order " $\preceq$ " on $C(I, \mathbb{R})$ by $x \preceq y$ iff $||x||_\infty \leq ||y||_\infty$ for all $x, y \in C(I, \mathbb{R})$. Now, $(C(I, \mathbb{R}), \mathcal{F}, \mathcal{T}, )$ is a complete $PM$-space provided with the quasi-order " $\preceq$ ".

Define $f : C(I, \mathbb{R}) \to C(I, \mathbb{R})$ by

$$fy(u) = \int_0^u M(u, v, y(v))dv + g(u) \quad g \in C(I, \mathbb{R}).$$

**Theorem 4.1.** *Let $(C(I, \mathbb{R}), \mathcal{F}, \mathcal{T}, \preceq)$ be the quasi-ordered complete $PM$-space, $G : C(I, \mathbb{R}) \to CB(C(I, \mathbb{R}))$ be a set-valued operator such that $G(y) = \{fy(u)\}$ and $M \in C(I \times I \times \mathbb{R}, \mathbb{R})$ be an operator. Suppose that*

*(i)* $||M||_\infty = \sup_{u,v \in I,\, y \in C(I,\mathbb{R})} |M(u, v, y(v))| < \infty$*;*

*(ii) for every $x, y \in C(I, \mathbb{R})$ and every $u, v \in I$, there is $L > 0$ provided that*

$$||M(u, v, fx(v)) - M(u, v, fy(v))|| \leq L \max\{|x(v) - y(v)|, |x(v) - fx(v)|, |y(v) - fy(v)|\}.$$

*Then 4.1 have a solution in $C(I, \mathbb{R})$.*

**Proof .** consider $d_B(x, y) = \max_{u \in I}(|x(u) - y(u)|e^{-Lu})$ for $x, y \in C(I, \mathbb{R})$, where $L$ satisfies condition $(ii)$. Also, by definition of $G$, we get $t = fx$ and $s = fy$ for all $t \in Gx$ and $s \in Gy$. Thus, we conclude

$$\begin{aligned} d_B(fx, fy) &\leq \max_{u \in I} \int_0^u |M(u, v, x(u)) - M(u, v, y(v))|e^{L(v-u)}e^{-Lv}dv \\ &\leq L \max\{d_B(x, y), d_B(x, fx), d_B(y, fy)\} \max_{u \in I} \int_0^u e^{L(v-u)}dv \\ &\leq (1 - e^{-aL}) \max\{d_B(x, y), d_B(x, fx), d_B(y, fy)\}. \end{aligned}$$

for every $x, y \in C(I, \mathbb{R})$. Now, set $k = (1 - e^{-La})$ and $\varphi(r) = kr$ that $\varphi \in \Phi$. Now, for each $r > 0$, we obtain

$$\begin{aligned}
\mathcal{F}_{t,s}\varphi(r) &= \mathcal{F}_{fx,fy}\varphi(r) = \chi(kr - d_B(fx, fy)) = \chi(r - \frac{1}{k} d_B(fx, fy)) \\
&\geq \chi(r - max\{d_B(x, y), d_B(x, fx), d_B(y, fy)\}) \\
&= \min\{\chi(r - d_B(x, y)), \chi(r - d_B(x, fx)), \chi(r - d_B(y, fy))\} \\
&= \min\{\mathcal{F}_{x,y}(r), \mathcal{F}_{x,fx}(r), \mathcal{F}_{y,fy}(r)\} \\
&= \min\{\mathcal{F}_{x,y}(r), \mathcal{F}_{x,t}(r), \mathcal{F}_{y,s}(r)\}.
\end{aligned}$$

So the condition $(H_2)$ of Theorem 2.1 is established. Also, by definition of $G$ and " $\preceq$ ", the condition $(H_1)$ of Theorem 2.1 is established. Therefore, Theorem 2.1 assures that $G$ has a fixed point. □

**Remark 4.2.** *Note that the existence of a solution of integral equation 4.1 was proved by Sadeghi and Vaezpour (see [13, Theorem 4.4]). In particular, they considered very conditions in Theorem* 3.2 *and Theorem* 4.4 *of their work to obtain this solution. But, in Theorem 4.1, we obtain the same result by only two conditions.*

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