Characteristics of penta- open sets in penta topological spaces

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Abstract
The aim of the presented study is to introduce and verify two new spaces called $P_\text{c}$ compactness and $P_\text{c}$ connectedness using $P_\text{o}$ open sets and some of their properties. Moreover, we study the relationship between these spaces. Another purpose of this study is to examine a new form of separation axioms, by using $P_\text{o}$ open set namely $T_{P_i}$-spaces where $(i = 0, 1, 2)$. The pertinence between them has been discussed and several features of these spaces are demonstrated as well.

Keywords: Penta Topological Space, $P_{\text{Irresolute}}$ function, $P_\text{c}$ compactness, locally $P_\text{c}$ compact, $P_\text{c}$ Connectedness, $P_\text{c}$ separated, strongly $P_\text{c}$ connected, $T_{P_i}$-space.

1. Introduction

The aims of semi-open sets and their properties were initiated by Levine \[7\] in 1963. The idea of dealing with single topological space was developed to bi-topological space, tri-topological, quad topological by researchers Kelly \[4\], Kovar \[6\] and Mukundan \[9\], lastly the notion Penta topological space $(X, \tau_1, \tau_2, \tau_3, \tau_4, \tau_5)$ was introduced by Muhammad and Khan \[5\] in 2018, where $X$ is nonempty set together with five topologies $\tau_1, \tau_2, \tau_3, \tau_4, \& \tau_5$. Many researchers verified the basic properties of connectedness and compactness powerful tools in topology. The idea of Hausdorff spaces is almost an integral part of compactness. Many authors in such as Srivastava and Bhatia \[13\] introduced some kinds of compact spaces in topological space according to the sets. Topological space is said to be compact or have the compact property, if every open cover of $X$ has a finite sub-cover \[2\]. Last years

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the generalization of compact spaces and locally generalized to bi-topological and tri-topological setting as in [1] [12] [13]. A topological space $X$ is said to be disconnected space if $X$ can be expressed as the union of two disjoint nonempty open subsets of $X$, otherwise $X$ is said to be connected space [2]. In 1965, Levine [8] introduced strongly connected in topology. In 1967 Pervin [10] studied connectedness in bi-topological spaces and in 2016 Tapi and others [16, 17] studied Tri-connectedness in Tri-topological spaces and he also introduced connectedness in Quad Topological Spaces [14]. Many papers discussed separation axioms, essentially by replacing open sets, many definitions of separation axioms according to open sets have been introduced by many researchers as El-Tantawy, Hameed, Tapi, and others [3, 15]. In this work we developed compactness and connectedness and separation axioms in Penta topological spaces. we introduced different types of Penta compact and Penta connected and Penta separation axioms in Penta topological spaces, additionally some properties of these spaces are investigated. Throughout this paper. A Penta topological space is denoted by $(X, \mathcal{P})$ and $(\mathcal{Y}, \mathcal{P})$ or simply by $X$ and $\mathcal{Y}$. The concept Penta topological space $(X, \tau_1, \tau_2, \tau_3, \tau_4, \tau_5)$ where $X$ is non empty set together with five topologies $\tau_1, \tau_2, \tau_3, \tau_4 \& \tau_5$, was introduced by Khan and Khan [3] in 2018. we write $\mathcal{P}$ for Penta Topology ($\mathcal{P}$-topology) and $(X, \mathcal{P})$ for Penta Topological Space where $\mathcal{P} = (\tau_1, \tau_2, \tau_3, \tau_4, \tau_5)$.

In the present work, we introduce $\mathcal{P}$ on five different topologies on $X$, therefore $(X, \mathcal{P})$ is called Penta Topological Space. The topologies $(X, \mathcal{P}_1)$, $(X, \mathcal{P}_2)$, $(X, \mathcal{P}_3)$, $(X, \mathcal{P}_4)$, $(X, \mathcal{P}_5)$ are independently satisfying the axioms of topology. The elements of $\mathcal{P}_1$ are called $\mathcal{P}_1$- open set and the complement of $\mathcal{P}_1$-open set is called $\mathcal{P}_1$-closed set. And the same with the elements of $\mathcal{P}_2, \mathcal{P}_3, \mathcal{P}_4, \mathcal{P}_5$.

Definition 1.1. [3] Let $(X, \mathcal{P})$ be a Penta Topological Space. Elements of $\tau_i; i = \{1, 2, 3, 4, 5\}$ are called $\tau_i$- open sets and their relative complements are called $\tau_i$- closed sets. Also a subset $\mathcal{A}$ of $X$ is called penta-open $(\mathcal{P} - \text{open})$ if $\mathcal{A} \in \bigcup \tau_i; i \in \{1, 2, 3, 4, 5\}$ and its complement is said to be penta-closed $(\mathcal{P} - \text{closed})$, $\mathcal{P}$- open sets satisfies all the axioms of topology. The set of all $\mathcal{P}$-open sets contains $\tau_1 \cup \tau_2 \cup \tau_3 \cup \tau_4 \cup \tau_5$. So; the family of all $\mathcal{P}$-open $(\mathcal{P}$-closed) sub sets of $(X, \mathcal{P})$ will be denoted by $(\mathcal{P}\mathcal{O}(X)), (\mathcal{P}\mathcal{C}(X))$.

Definition 1.2. [3] Let $\mathcal{H}$ be a subset of a Penta topological Space $(X, \mathcal{P})$, then:

1. The $(\mathcal{P}\text{-interior})$ of $\mathcal{H}$ is the union of all $(\mathcal{P}$-open) subset contained in $\mathcal{H}$ and is denoted by int$_\mathcal{P}(\mathcal{H})$. Thus int$_\mathcal{P}(\mathcal{H})$ is the largest $(\mathcal{P}$-open) subset of $\mathcal{H}$.
2. The $(\mathcal{P}\text{-closure})$ of $\mathcal{H}$ is the intersection of all $(\mathcal{P}$-closed) sets containing $\mathcal{H}$ and is denoted by cl$_\mathcal{P}(\mathcal{H})$. that is cl$_\mathcal{P}(\mathcal{H})$ is the smallest $(\mathcal{P}$-closed) set containing $\mathcal{H}$. Some properties for each $\mathcal{H} \subseteq X$ [13]
   - $\text{i. }$ (int$_\mathcal{P}(\mathcal{H}))^c = \text{cl}_\mathcal{P}(\mathcal{H}^c)$
   - $\text{ii. }$ $\mathcal{H}$ is $\mathcal{P}$-open if int$_\mathcal{P}(\mathcal{H}) = \mathcal{H}$
   - $\text{iii. }$ $\mathcal{H}$ is $\mathcal{P}$-closed if cl$_\mathcal{P}(\mathcal{H}) = \mathcal{H}$
3. The $\mathcal{P}\text{-neighborhood } (\text{in short } \mathcal{G}_\mathcal{P}) \mathcal{H}$ of a point $x \in X$ if and only if there exist a $\mathcal{P}$-open set $\mathcal{G}$ such that $x \in \mathcal{G} \subseteq \mathcal{H}$.

Definition 1.3. [3] A function $f : (X, \mathcal{P}) \rightarrow (\mathcal{Y}, \mathcal{P})$ is called

1. $\mathcal{P}$-continuous if $f^{-1}(\mathcal{H}) \in \mathcal{P}\mathcal{O}(X)$, for each $\mathcal{H} \in \mathcal{P}\mathcal{O}(\mathcal{Y})$.
2. $\mathcal{P}$-open if, $f(\mathcal{H}) \in \mathcal{P}\mathcal{O}(\mathcal{Y})$, for each $\mathcal{H} \in \mathcal{P}$.
3. $\mathcal{P}$-closed if any $\mathcal{P}$-closed set $\mathcal{D}$ then $f(\mathcal{D}) \in \mathcal{P}\mathcal{C}(\mathcal{Y})$. 

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4. \( \mathcal{P} - \text{homeomorphism if } f \text{ is bijective, } \mathcal{P} - \text{continuous and } \mathcal{P} - \text{open}. \)

**Proposition 1.4.** A function \( f : (X, \mathcal{S}_E) \to (Y, \mathcal{S}_F) \) is \( \mathcal{P} - \text{continuous if and only if the inverse image of every } \mathcal{P} - \text{closed in } Y \text{ is an } \mathcal{P} - \text{closed set in } X \).

**Proof.** Suppose that \( D \) is \( \mathcal{P} - \text{closed set in } Y \) then \( D^C \in \mathcal{P}(X) \), therefore \( J^{-1}(Y - D) = J^{-1}(Y) - J^{-1}(D) = X - J^{-1}(D) = (J^{-1}(D))^C \).

Then \( (J^{-1}(D))^C \in \mathcal{P} - \text{open in } X \) therefore \( J^{-1}(D) \) \( \mathcal{P} - \text{closed set in } X \). \( \square \)

**Example 1.5.** A Penta topology \( \mathcal{S}_P = \{X, \emptyset, \{a\}, \{b, c\}, \{a, d\}, \{a, b, c\}, \{a, b, d\}\} \)
on \( X = \{a, b, c, d\} \), when \( \mathcal{S}_1 = \{X, \emptyset, \{a\}, \{b\}, \{a, b\}\} \), \( \mathcal{S}_2 = \{X, \emptyset, \{c\}, \{d\}\} \), \( \mathcal{S}_3 = \{X, \emptyset, \{a\}, \{b, d\}\} \), \( \mathcal{S}_4 = \{X, \emptyset, \{a, d\}, \{b, c\}\} \) and \( \mathcal{S}_5 = \{X, \emptyset, \{a, b, c\}\} \).

Let \( \mathcal{S}_P = \{Y, \emptyset, \{a\}, \{b\}, \{b, c\}, \{a, b, d\}\} \), \( \mathcal{S}_3 = \{Y, \emptyset, \{c\}, \{d\}\} \), \( \mathcal{S}_2 = \{Y, \emptyset, \{b\}, \{b, c\}\} \) and \( \mathcal{S}_1 = \{Y, \emptyset, \{a\}, \{a, b, d\}\} \).

We get \( \mathcal{P}(X) = \{X, \emptyset, \{a\}, \{b, c\}, \{a, d\}, \{a, b, c\}, \{a, b, d\}\} \), \( \mathcal{P}(Y) = \{Y, \emptyset, \{a\}, \{b\}, \{b, c\}, \{a, b, d\}\} \), \( \mathcal{P}(X) = \{X, \emptyset, \{a\}, \{b\}, \{a, b, d\}\} \), \( \mathcal{P}(Y) = \{Y, \emptyset, \{a\}, \{b\}, \{a, b, d\}\} \).

Let \( J : (X, \mathcal{S}_P) \to (Y, \mathcal{S}_P) \) and \( f(b) = \dot{b}, f(c) = \dot{c}, f(d) = \dot{d}, f(a) = \dot{a} \). Then \( f \) is clearly a \( \mathcal{P} - \text{homeomorphism}. \)

**Definition 1.6.** Let \( E \) be a subset of \( (X, \mathcal{S}_P) \), then classes \( \mathcal{S}_{PE} \) of all intersections of \( E \) with \( \mathcal{P} - \text{open subsets of } X \) belong to \( \mathcal{S}_P \) is a topology on \( E \), it is called penta- subspace (relative Penta -topological space for \( E \) with respect to \( \mathcal{P} - \text{open sets} \)). The relative Penta-topological space for \( E \) is denoted by \( (E, \mathcal{S}_{PE}) \), such that \( \mathcal{S}_{PE} = \{G \cap E : G \in \mathcal{S}_P\} \), \( \mathcal{P} = \{1, 2, 3, 4, 5\} \). From example \ref{ex:1.5}, we get \( \mathcal{S}_{PE} = \{E, \emptyset, \{a\}, \{b\}, \{b, c\}, \{a, b, c\}\} \) on \( E = \{a, b, c\} \) then \( (E, \mathcal{S}_{PE}) \) is relative \( \mathcal{P} - \text{topology}. \)

**Definition 1.7.** \[ (\mathcal{H}) \] The subset \( \mathcal{H} \subseteq (X, \mathcal{S}_P) \) is said to be semi penta open (semi \( \mathcal{P} \text{O} \) set) if \( \mathcal{H} \subseteq \text{cl}_P(\text{int}_P(\mathcal{H})) \) and its complement is said to be semi penta-closed (semi \( \mathcal{P} \text{C} \) set). Therefore; the family of all semi \( \mathcal{P} \text{O}.(\text{semi} \mathcal{P} \text{C}) \) sub sets of \( (X, \mathcal{S}_P) \) will be denoted by \( (\mathcal{POS}(X)), (\mathcal{PCS}(X)) \).

Note: Every \( \mathcal{P} - \text{open set is (semi} \mathcal{P} \text{O}). \)

**Definition 1.8.** A function \( f : (X, \mathcal{S}_P) \to (Y, \mathcal{S}_P) \) is called \( \mathcal{P} - \text{irresolute function if } f^{-1}(\mathcal{H}) \in \mathcal{P}(X) \), for every \( \mathcal{H} \in \mathcal{POS}(Y) \). From example \ref{ex:1.5} we get:

- \( \mathcal{POS}(X) = \{X, \emptyset, \{a\}, \{b\}, \{a, b, c\}, \{a, b, d\}\} \)
- \( \mathcal{PCS}(X) = \{X, \emptyset, \{c\}, \{d\}, \{a, c, d\}\} \)
- \( \mathcal{POS}(Y) = \{Y, \emptyset, \{a\}, \{b\}, \{a, b, c\}, \{a, b, d\}\} \)
- \( \mathcal{PCS}(Y) = \{Y, \emptyset, \{b, c, d\}, \{a, c, d\}, \{d, a\}, \{b, c\}, \{d\}\} \)

Therefore \( \mathcal{P} - \text{irresolute}. \)
2. Penta Connectedness in Penta Topological Space

In this section, we discuss $\mathcal{P}$–connectedness and $\mathcal{P}$–disconnectedness by means of $\mathcal{P}$–separation.

**Definition 2.1.** A Penta Topological space $(X, \mathcal{S}_P)$, is called Penta separated space ($\mathcal{P}$–separated) if and only if there exist $\mathcal{P}$–open subsets $H$ and $K$ of $X$, such that $H \cap cl_P(K) = \emptyset$ and $K \cap cl_P(H) = \emptyset$. These two conditions are equivalent to $(H \cap cl_P(K)) \cup (K \cap cl_P(H)) = \emptyset$. From example 1.5 we note that $X$ is $\mathcal{P}$–separated space because $PO(X) = \{X, \emptyset, \{a\}, \{b\}, \{a,b\}, \{a,b,c\}\}$. If we take $\{b,c\}$ and $\{a,d\}$ then $(\{b,c\} \cap cl_P(\{a,d\})) \cup (\{a,d\} \cap cl_P(\{b,c\})) = \emptyset$. Therefore $\{b,c\}$ and $\{a,d\}$ are $\mathcal{P}$–separation sets. Also $\{b\} \subseteq \{b,c\}$ and $\{a\} \subseteq \{a,d\}$, then $\{b\}$ and $\{a\}$ are $\mathcal{P}$–separation.

**Remark 2.2.**
1. Every two disjoint $\mathcal{P}$–open ($\mathcal{P}$–closed) subsets of any space are ($\mathcal{P}$–separated).
2. Let $H$ and $K$ be ($\mathcal{P}$–separated) subsets of $X$, then if $D \subseteq H$ and $S \subseteq K$, then $D$ and $S$ are also ($\mathcal{P}$–separated).

**Theorem 2.3.** Every two $\mathcal{P}$–open ($\mathcal{P}$–closed) subsets of $X$ are ($\mathcal{P}$–separated), if and only if they are disjoint.

**Proof.** Any two ($\mathcal{P}$–separated) $H, K$ sets are disjoint, then two sets $\mathcal{P}$–open ($\mathcal{P}$–closed) are ($\mathcal{P}$–separated) (because if $H, K$ are both disjoint ($\mathcal{P}$–closed), then $H \cap K = \emptyset$, $cl_P(H) = H$, $cl_P(K) = K$ and so that $H \cap cl_P(K) = \emptyset, K \cap cl_P(H) = \emptyset$).

Conversely, if $H, K$ are both disjoint $\mathcal{P}$–open, then $H^c$ and $K^c$ are both $\mathcal{P}$–closed, so that; $H \cap K = \emptyset \rightarrow H \subseteq K^c$ and $K \subseteq H^c$, also $cl_P(K^c) = K^c$ and $cl_P(H^c) = H^c$. We get $cl_P(K) \subseteq cl_P(H^c) = H^c$ and $cl_P(H) \subseteq cl_P(K^c) = K^c$. Hence $H$ and $K$ are ($\mathcal{P}$–separated) so that $H \cap cl_P(K) = \emptyset$, $K \cap cl_P(H) = \emptyset$.

**Definition 2.4.** A Penta Topological Space $(X, \mathcal{S}_P)$ is said to be Penta connected ($\mathcal{P}$–connected) if there exist two $\mathcal{P}$–open subsets $H$ and $K$ of $X$, provided that $H \cap K = \emptyset$ and $K \cup H \neq X$. A Penta Topological Space is Penta disconnected ($\mathcal{P}$–disconn.) if it does not ($\mathcal{P}$–connected).

**Theorem 2.5.** Let $(X, \mathcal{S}_P)$ be ($\mathcal{P}$–connected) space, then at least one of the five topologies is connected.

**Proof.** Suppose that $\mathcal{S}_i$ is disconnected, where $\mathcal{S}_i = \{\tau_1, \tau_2, \tau_3, \tau_4, \tau_5\}$, then $X = H \cup K$ where $H$ and $K$ are disjoint non-empty open sets. Since every open set is a $\mathcal{P}$–open set, then $(X, \mathcal{S}_P)$ is not ($\mathcal{P}$–connected) space (contradiction by hypothesis). Then at least one of the five topologies is connected.

**Remark 2.6.** The converse of theorem 2.5 is not necessary true.

**Example 2.7.** From example 1.5 we note that $(X, \mathcal{S}_P)$ is not ($\mathcal{P}$–connected) space in spite of there exist at least one of five topologies is connected.

**Theorem 2.8.** Let $E$ be a subset of $(X, \mathcal{S}_P)$. Then $E$ is $\mathcal{P}$–disconnected set iff it can be expressed as the union of two non-empty ($\mathcal{P}$–separated) sub sets of $X$. 
Proof. Let \( E \) be \( \mathcal{P} \)-disconnected set, then \( E = \mathcal{H} \cup \mathcal{K} \) where \( \mathcal{H} \) and \( \mathcal{K} \) are disjoint non-empty subsets of \( \mathcal{X}_P \) - \( \mathcal{P} \) - closed sets. \( \mathcal{H}, \mathcal{K} \) are \( (\mathcal{P} \text{- separated}) \) subsets of \( \mathcal{X} \), thus \( \mathcal{H} \cap \text{cl}_P(\mathcal{K}) = (\mathcal{H} \cap \mathcal{E}) \cap \text{cl}_P(\mathcal{K}) = \mathcal{H} \cap \text{cl}_{\mathcal{P}_E}(\mathcal{K}) = \mathcal{H} \cap \mathcal{K} = \emptyset \), similarly \( \mathcal{K} \cap \text{cl}_P(\mathcal{H}) = \emptyset \). Then \( (\mathcal{H} \cap \text{cl}_P(\mathcal{K})) \cup (\mathcal{K} \cap \text{cl}_P(\mathcal{H})) = \emptyset \).

Conversely, suppose that \( E = \mathcal{H} \cup \mathcal{K} \), where \( \mathcal{H} \) and \( \mathcal{K} \) are non-empty \( \mathcal{P} \)-separated subsets of \( \mathcal{X} \). We have \( \mathcal{H} \cap \text{cl}_P(\mathcal{K}) = (\mathcal{H} \cap \mathcal{E}) \cap \text{cl}_P(\mathcal{K}) = \emptyset \) and similarly \( \mathcal{K} \cap \text{cl}_P(\mathcal{H}) = \emptyset \), so \( E \) is a union of non-empty \( (\mathcal{P} \text{- separated}) \) subsets of \( \mathcal{E} \). Thus \( E \) is \( \mathcal{P} \)-disconnected. \( \square \)

The following example constructs a \( \mathcal{P} \)-connected space.

Example 2.9. From example \([1,5]\) we note that \((\mathcal{X}, \mathcal{S}_P)\) is \((\mathcal{P} \text{- connected}), because \((\mathcal{P} \text{- open sets of } \mathcal{Y} = \{\mathcal{Y}, \emptyset, \{\mathcal{a}\}, \{\mathcal{b}\}, \{\mathcal{b}, \mathcal{a}\}, \{\mathcal{a}, \mathcal{d}\}, \{\mathcal{a}, \mathcal{b}, \mathcal{c}\}, \{\mathcal{a}, \mathcal{b}, \mathcal{d}\}\} \) on \( N = \{\mathcal{a}, \mathcal{b}, \mathcal{c}, \mathcal{d}\} \), if we take \( \mathcal{H} = \{\mathcal{a}\} \) and \( \mathcal{K} = \{\mathcal{b}, \mathcal{c}\} \). That is ascertain \( \mathcal{P} \)-connection condition.

Remark 2.10. 
1. Let \( \mathcal{X} \) be \( \mathcal{P} \)-connected space, then any of the five topologies is not necessary to connected space. As in the example \([1,5]\).
2. \( \mathcal{X} \) is \( \mathcal{P} \)-connected set, iff it is not the union of two non-empty \( \mathcal{P} \)-separated sets.
3. If \( \mathcal{E} \) is the union of two disjoint non-empty \( \mathcal{P} \)-open sub sets then \( \mathcal{X} \) is \( \mathcal{P} \)-disconnected.
4. If \( E \) is \( \mathcal{P} \)-connected set of \( \mathcal{X} \) and \( \mathcal{H}, \mathcal{K} \) are \( (\mathcal{P} \text{- separated}) \) sets of \( \mathcal{X} \) with \( E \subseteq \mathcal{H} \cup \mathcal{K} \), then either \( E \subseteq \mathcal{H} \) or \( E \subseteq \mathcal{K} \).
5. If \( E \) subset of \( \mathcal{X} \) is a \( \mathcal{P} \)-connected, then cl\(_P\)(E) is \( \mathcal{P} \)-connected.

We know that if \( \mathcal{H} \) and \( \mathcal{K} \), \( \mathcal{P} \)-connected sets then, \( \mathcal{H} \cup \mathcal{K} \) is \( \mathcal{P} \)-disconnected set but by adding some condition we can prove that \( \mathcal{P} \)-connected sets by the following theorem.

Theorem 2.11. Let \( \mathcal{H}, \mathcal{K} \) be \( \mathcal{P} \)-connected sets and \( \mathcal{H} \cap \mathcal{K} \neq \emptyset \), then \( \mathcal{H} \cup \mathcal{K} \) is \( \mathcal{P} \)-connected set.

Proof. Assume that \( \mathcal{H}, \mathcal{K} \subseteq \mathcal{X}, \mathcal{H}, \mathcal{K} \) are \( \mathcal{P} \)-connected and \( \mathcal{H} \cap \mathcal{K} \neq \emptyset \).

Suppose that \( \mathcal{H} \cup \mathcal{K} \) is \( \mathcal{P} \)-disconnected, if \( \mathcal{X}, \mathcal{Y} \) are two disjoint non-empty \( \mathcal{P} \)-open sets, \( \mathcal{X}, \mathcal{Y} \in \mathcal{S}(\mathcal{H}, \mathcal{K}) \) then \( \mathcal{H} \cup \mathcal{K} = \mathcal{X} \cup \mathcal{Y} \); so, \( \mathcal{H} \subseteq \mathcal{H} \cup \mathcal{K} \rightarrow \mathcal{H} \subseteq \mathcal{X} \cup \mathcal{Y} \rightarrow \mathcal{H} \subseteq \mathcal{X} \) or \( \mathcal{H} \subseteq \mathcal{Y} \) (because \( \mathcal{H} \) is \( \mathcal{P} \)-connected). Also \( \mathcal{K} \subseteq \mathcal{H} \cup \mathcal{K} \rightarrow \mathcal{K} \subseteq \mathcal{X} \cup \mathcal{Y} \rightarrow \mathcal{K} \subseteq \mathcal{X} \) or \( \mathcal{K} \subseteq \mathcal{Y} \) (because \( \mathcal{K} \) is \( \mathcal{P} \)-connected).

Now, either \( \mathcal{H} \subseteq \mathcal{X} \cap \mathcal{K} \subseteq \mathcal{X} \rightarrow \mathcal{H} \cup \mathcal{K} \subseteq \mathcal{X} \rightarrow \mathcal{Y} \neq \emptyset \) contradiction. 

Or \( \mathcal{H} \subseteq \mathcal{Y} \cap \mathcal{K} \subseteq \mathcal{Y} \rightarrow \mathcal{H} \cup \mathcal{K} \subseteq \mathcal{Y} \rightarrow \mathcal{X} = \emptyset \) contradiction.

Or \( \mathcal{Y} \subseteq \mathcal{X} \cap \mathcal{K} \subseteq \mathcal{X} \rightarrow \mathcal{H} \cup \mathcal{K} \subseteq \mathcal{X} \cup \mathcal{Y} = \emptyset \rightarrow \mathcal{X} \cap \mathcal{Y} = \emptyset \) contradiction. 

Or \( \mathcal{Y} \subseteq \mathcal{X} \cap \mathcal{K} \subseteq \mathcal{Y} \rightarrow \mathcal{H} \cap \mathcal{K} \subseteq \mathcal{X} \cap \mathcal{Y} = \emptyset \rightarrow \mathcal{X} \cap \mathcal{Y} = \emptyset \) contradiction. Hence \( \mathcal{H} \cup \mathcal{K} \) is \( \mathcal{P} \)-connected. \( \square \)

And by generalizing the above theorem to any family of \( \mathcal{P} \)-connected sets we obtain the following theorem.

Theorem 2.12. The union of any family of \( \mathcal{P} \)-connected sets have non-empty intersection \( \mathcal{P} \)-connected sets.

Proof. Let \( \mathcal{M}_i : i \in \mathbb{N} \) is non-empty of \( \mathcal{P} \)-connected subset of \( \mathcal{X} \) and suppose that \( \bigcup_{i \in \mathbb{N}} \mathcal{M}_i \) is \( \mathcal{P} \)-disconnected, then \( \bigcup_{i \in \mathbb{N}} \mathcal{M} = \mathcal{H} \cup \mathcal{K} \), where \( \mathcal{H} \) and \( \mathcal{K} \) are \( \mathcal{P} \)-separated sets in \( \mathcal{X} \). Since \( \bigcap_{i \in \mathbb{N}} \mathcal{M}_i \neq \emptyset \), we get \( x \in \bigcap_{i \in \mathbb{N}} \mathcal{M}_i \). Since \( x \in \bigcup_{i \in \mathbb{N}} \mathcal{M}_i \) either \( x \in \mathcal{H} \) or \( x \in \mathcal{K} \) if \( x \in \mathcal{H} \) - then \( x \in \mathcal{H} \cap \mathcal{K} \). Similarly \( \mathcal{K} \) - then \( x \in \mathcal{H} \cap \mathcal{K} \) (because \( \mathcal{M}_i \subseteq \mathcal{H} \) for all \( i \in \mathbb{Z} \)) that leads to \( \mathcal{K} \) is empty. this is a contradiction. By similar discussion \( \mathcal{H} \) is also empty and this is a contradiction. Then \( \bigcup_{i \in \mathbb{N}} \mathcal{M}_i \) is \( \mathcal{P} \)-connected sets. \( \square \)
Theorem 2.13. Let \( f : (X, \mathcal{S}_P) \to (Y, \mathcal{S}_P) \) be a surjective \( P \) – continuous function and \( R, K \) are \( P \) – separated sets in \( Y \), then \( f^{-1}(R), f^{-1}(K) \) are \( P \) – separated in \( X \).

Proof. Since \( R, K \) are \( P \) – separated sets in \( Y \), if \( f \) is surjective, then \( f^{-1}(R), f^{-1}(K) \) are non-empty in \( X \). Suppose that \( f^{-1}(R), f^{-1}(K) \) are not \( P \) – separated in \( X \), then \( f^{-1}(R) \cap f^{-1}(K) \neq \emptyset \), then we obtain that \( f^{-1}(R) \cap f^{-1}(cl_R(K)) \neq \emptyset \). Similarly, \( R \cap cl_R(K) \neq \emptyset \). So we get \( R, K \) are not \( P \) – separated sets in \( Y \), which contradict the hypothesis. Hence \( f^{-1}(R), f^{-1}(K) \) are \( P \) – separated in \( X \). \( \square \)

Theorem 2.14. Let \( f : (X, \mathcal{S}_P) \to (Y, \mathcal{S}_P) \) be a \( P \) – continuous function and \( X \) is \( P \) – connected, then \( Y \) is connected.

Proof. Suppose that \( Y \) is not connected, let \( Y = H \cup K \) where \( H \) and \( K \) are disjoint non-empty open in \( Y \). Since \( f \) is \( P \) – continuous, \( X = f^{-1}(H) \cup f^{-1}(K) \) where \( f^{-1}(H) \) and \( f^{-1}(K) \) are disjoint non-empty \( P \) – open sets in \( X \). Hence \( Y \) is connected. \( \square \)

Proposition 2.15. Let \( f : (X, \mathcal{S}_P) \to (Y, \mathcal{S}_P) \) be a bijective \( P \) – continuous function and \( E \) is \( P \) – connected in \( X \), then \( f(E) \) is \( P \) – connected in \( Y \).

Proof. Suppose that \( f(E) \) is \( P \) – disconnected in \( Y \). Then \( f(E) = H \cup K \) where \( H \) and \( K \) are disjoint non-empty \( P \) – separated in \( Y \). By theorem 2.14, we have \( f^{-1}(H), f^{-1}(K) \) are \( P \) – separated in \( X \). Since \( f \) is bijective, then \( E = f^{-1}(f(E)) = f^{-1}(H) \cup f^{-1}(K) \). Hence \( E \) is not \( P \) – connected in \( X \), which contradict the hypothesis. Thus \( f(E) \) is \( P \) – connected in \( Y \). \( \square \)

Theorem 2.16. Let \( f : (X, \mathcal{S}_P) \to (Y, \mathcal{S}_P) \) be surjective \( P \) – irresolute and \( X \) is \( P \) – connected, then \( Y \) is \( P \) – connected.

Proof. Suppose that \( Y \) is \( P \) – disconnected, let \( Y = H \cup K \) where \( H \) and \( K \) are disjoint non-empty \( P \) – open in \( Y \). Since \( f \) is \( P \) – irresolute and onto, then \( X = f^{-1}(H) \cup f^{-1}(K) \) where \( f^{-1}(H) \) and \( f^{-1}(K) \) are disjoint non-empty \( P \) – open sets in \( K \) (every \( P \) – open set is semi\( P \)O). Since \( X \) is \( P \) – connected then we get that \( Y \) is \( P \) – connected. \( \square \)

Definition 2.17. A space \((X, \mathcal{S}_P)\) is called strongly \( P \) – connected briefly \((PSC)\) iff it is not a disjoint union of countably many but more one \( P \) – closed set.

By another words, if \( X \neq \cup F_i \), where \( F_i \) are disjoint non empty \( P \) – closed sets of \( X \).

Remark 2.18. A subset \( E \) of a \((X, \mathcal{S}_P)\) is said to be strongly \( P \) – connected iff \( E \subseteq H \) or \( E \subseteq K \) whenever \( E \subseteq H \cup K \), \( H \) and \( K \) are \( P \) – open sets in \( X \).

Proposition 2.19. Let \( E \) be \( PSC \), then \( E \) is \( P \) – connected.

Proof. Suppose that \( E \) is \( P \) – disconnected, then \( \exists \ P \) – open sets \( H \) and \( K \) for which \( E = (E \cap H) \cup (E \cap K) \), such that \( E \cap H \neq \emptyset \), \( E \cap K \neq \emptyset \), \( E \cap (H \cap K) = \emptyset \), then \( E \subseteq (H \cup K) \), but by definition \( P \)SC \( E \subseteq H \) and \( E \subseteq K \) contradicts \( E \) being \( P \) – connected. \( \square \)

Remark 2.20. A set \( E \) will be weakly \( P \) – connected iff it is not \( PSC \).

By adding some condition to the \( P \) – irresolute function we obtain the following theorem.

Theorem 2.21. Any surjective \( P \) – irresolute image of a strongly \( P \) – connected space is strongly \( P \) – connected.

Proof. Let \( f : (X, \mathcal{S}_P) \to (Y, \mathcal{S}_P) \) be surjective \( P \) – irresolute function and assume that \( f(X) \) is weakly \( P \) – disconnected, since \( f \) is \( P \) – irresolute using definition 2.17, then the inverse image of \( P \) – semi open sets is \( P \) – open sets, so \( X \) is a disjoint union of \( P \) – open sets. Hence \( f(X) \) is \( PSC \). \( \square \)
3. Penta Compactness in Penta Topological Space

**Definition 3.1.** A collection \( \{G_i : i \in \Lambda\} \) of \( \mathcal{P} \) – open sets in \((X, \mathcal{S}_P)\) is called a \( \mathcal{P} \) – open cover of a subset \( E \) of \( X \) if \( E \subseteq \bigcup \{G_i : i \in \Lambda\} \).

**Definition 3.2.** A \((X, \mathcal{S}_P)\) is said to be \( \mathcal{P} \) – compact if every \( \mathcal{P} \) – open cover of \( X \) has a finite sub cover.

*Note:* For each \( \{G_i : i \in \Lambda\} \) of \( \mathcal{P} \) – open sets for which \( E \subseteq \bigcup \{G_i : i \in \Lambda\} \), \( \exists \{G_{i_1}, G_{i_2}, ..., G_{i_m}\} \) among the \( G_i \)'s, such that \( E \subseteq G_{i_1} \cup G_{i_2} \cup ... \cup G_{i_m} \). Then a space \((X, \mathcal{S}_P)\) is \( \mathcal{P} \) – compact if for each \( \{G_i : i \in \Lambda\} \) of \( \mathcal{P} \) – open sets for which \( X = \bigcup \{G_i : i \in \Lambda\} \), there exist finitely many sets \( G_{i_1}, G_{i_2}, ..., G_{i_m} \) among the \( G_i \)'s such that \( X = G_{i_1} \cup G_{i_2} \cup ... \cup G_{i_m} \).

**Theorem 3.3.** Every \( \mathcal{P} \) – compact space is compact space.

**Proof.** Let \((X, \mathcal{S}_P)\) be a \( \mathcal{P} \) – compact space. Assume that \((X, \mathcal{S}_i)\) is not compact, where \( \mathcal{S}_i = (\tau_1, \tau_2, \tau_3, \tau_4, \tau_5) \). Then every open cover of \( X \) has not finite sub cover. Since every open set is a \( \mathcal{P} \) – open set, so \((X, \mathcal{S}_P)\) is not \( \mathcal{P} \) – compact space (contradiction by hypothesis). Hence \((X, \mathcal{S}_i)\) is compact. \( \square \)

**Remark 3.4.** The converse is not true.

**Example 3.5.** Let \( \mathcal{S}_P = \{\mathbb{R}, \emptyset, \mathbb{Q}, \mathbb{I}, Z, N, (n, \mathbb{N})\} \) be a Penta topology on \( \mathbb{R} \), when co-finite topolog \( y = \{G \subseteq \mathbb{R}; G^c \text{ is finite}\} \cup \{\emptyset\} \) such that \( \mathcal{S}_1 = \{\mathbb{R}, \emptyset, \mathbb{Q}\}, \mathcal{S}_2 = \{\mathbb{R}, \emptyset, \mathbb{I}\}, \mathcal{S}_3 = \{\mathbb{R}, \emptyset, Z\}, \mathcal{S}_4 = \{\mathbb{R}, \emptyset, N\} \) and the usual topology \( \mathcal{S}_5 = \{\mathbb{R}, \emptyset, u \subseteq \mathbb{R}; u = (n, \mathbb{N})\text{isanopeninterval}\} \). We can show that \((\mathbb{R}, \mathcal{S}_P)\) is \( \mathcal{P} \) – compact space but \( \mathcal{S}_5 \) is not compact, because \( C = \{(-n, n) : n \in \mathbb{N}\} \) is open cover of \( \mathbb{R} \) but it hasn't sub cover.

**Proposition 3.6.** Every \( \mathcal{P} \) – closed subset of a \( \mathcal{P} \) – compact space is \( \mathcal{P} \) – compact space.

**Proof.** Let \((X, \mathcal{S}_P)\) be a \( \mathcal{P} \) – compact space and \( A \subseteq X \) be \( \mathcal{P} \) – closed, suppose that \( \{G_i : i \in \Lambda\} \) is \( \mathcal{P} \) – open set of A, then \( A \subseteq \bigcup \{G_i : i \in \Lambda\} \), it is clearly that \( X - A \) is \( \mathcal{P} \) – open set. This shows that the family consisting of the sets \( X - A \) and \( G_i \)'s is an \( \mathcal{P} \) – open covers of \( X \), which are known to be \( \mathcal{P} \) – compact. Hence these covers has finite subcovers, say \( X - A, G_{i_1}, G_{i_2}, ..., G_{i_m} \), we get \( X = (X - A) \cup \bigcup_{r=1}^{m} G_{i_r} \). We claim that \( A \subseteq \bigcup_{r=1}^{m} G_{ir} \) and assume that there exist \( a \in A \), such that \( a \notin \bigcup_{r=1}^{m} G_{ir} \) and \( a \notin X - A \). Now the family consisting of \( X - A, G_{i_1}, G_{i_2}, ..., G_{i_m} \) is not an \( \mathcal{P} \) – open covers of \( X \). That is a contradiction. Hence \( A \subseteq \bigcup_{r=1}^{m} G_{ir} \) hold, such that \( \{G_{ir} : r = 1, 2, ..., n\} \) are \( \mathcal{P} \) – open covers of \( A \). So \( \{G_i\} \) of A, has finite subcover. Then A is a \( \mathcal{P} \) – compact. \( \square \)

**Proposition 3.7.** Every \( \mathcal{P} \) – closed subset of a \( \mathcal{P} \) – compact space is \( \mathcal{P} \) – compact relative \( \mathcal{P} \) – topology.

**Proof.** Let \( A \) be a \( \mathcal{P} \) – closed subset of a \( \mathcal{P} \) – compact \((X, \mathcal{S}_P)\), then \( X - A \) is \( \mathcal{P} \) – open set in \((X, \mathcal{S}_P)\), so \( \{G_i : i \in \Lambda\} \) be \( \mathcal{P} \) – open cover of A such that \( A \subseteq \bigcup \{G_i : i \in \Lambda\} \) and \( (X - A) \cup \{G_i : i \in \Lambda\} = X \). Hence \( A \subseteq \bigcup \{G_i : i \in \Lambda\} \) (because \( X \) is \( \mathcal{P} \) – compact, \( \exists A \subseteq (X - A) \cup \{G_i : i \in \Lambda\} = X \)). Then A is \( \mathcal{P} \) – compact relative \( X \). \( \square \)

**Theorem 3.8.** \( \mathcal{P} \) – continuous image of a \( \mathcal{P} \) – compact space is \( \mathcal{P} \) – compact.
Figure 1: Relationship between the compactness space

\[
\begin{array}{ccc}
P_{\text{compact space}} & \rightarrow & \text{compact space} \\
\downarrow & & \downarrow \\
\text{locally } P_{\text{compact space}} & \leftarrow & \text{locally compact space}
\end{array}
\]

Proposition 3.9. Any surjective \( P \)- irresolute image function of a \( P \)- compact space is \( P \)- compact.

Proof. Let \( f : (X, \mathcal{P}) \rightarrow (Y, \mathcal{P}) \) be surjective \( P \)- irresolute function and \( X \) is \( P \)- compact space, then to prove \( Y \) \( P \)- compact space, suppose that \( \{G_i : i \in \Lambda\} \) is \( P \)- open sets of \( Y \), then \( \{f^{-1}(G_i) : i \in \Lambda\} \) is \( P \)- open sets of \( X \) (because \( f \) is \( P \)- irresolute). Since \( X \) is \( P \)- compact has a finite subcover, we get \( \{f^{-1}(G_1), f^{-1}(G_2), \ldots, f^{-1}(G_n)\} \) then it leads to \( \{G_1, \ldots, G_n\} \) is a finite subcover of \( Y \), we get \( Y \) is \( P \)- compact. \( \square \)

Definition 3.10. A Penta Topological Space \((X, \mathcal{P})\) is called locally \( P \)- compact if every point in \( X \) has at least one \( G_{PN} \) whose \( P \)- closure is \( P \)- compact. Hence \((X, \mathcal{P})\) is locally \( P \)- compact.

Theorem 3.11. Every \( P \)- compact space is locally \( P \)- compact.

Proof. Let \((X, \mathcal{P})\) be \( P \)- compact space, then \( X \) is both \( P \)- closed and \( P \)- open set and has at least one \( G_{PN} \) whose \( P \)- closure is \( P \)- compact. Hence \((X, \mathcal{P})\) is locally \( P \)- compact. \( \square \)

Remark 3.12. 1. The converse of theorem 3.11 is not true, this can be seen through the following example.

Example. Let \( X \) be infinite set with discrete penta topological space, then \( X \) is not \( P \)- compact but the collection of all singleton sets is an infinite \( P \)- open cover of \( X \) which cannot has a finite subcover, by hypotheses \( X \) is locally \( P \)- compact, hence for each point of \( x \in \{x\} \) has a \( G_{PN} \) whose \( P \)- closure is \( P \)- compact.

2. Every locally compact space is locally \( P \)- compact space, from definition locally compact space and every open set is \( P \)- open set.

By theorems 3.3, 3.11 and remarks 3.12, present the following diagram that illustrates the relationship between the compactness space. We get Figure 1.
4. Penta Separation Axioms in Penta Topological Spaces

Definition 4.1. A space \( (X, \mathcal{P}) \) is said to be

i. \( T_{P_0} \)-space if for every pair of distinct points \( v, u \) of \( X \), there exists a \( \mathcal{P} \) - open set containing one of them but not the other.

ii. \( T_{P_1} \)-space if for every pair of distinct points \( v, u \) of \( X \), there exists two \( \mathcal{P} \) - open sets containing one of the two points but not the other.

iii. \( T_{P_2} \)-space if for every pair of distinct points \( v, u \) of \( X \), there exists two distinct \( \mathcal{P} \) - open sets \( \mathcal{H}, \mathcal{K} \), such that \( v \in \mathcal{H}, u \in \mathcal{K} \).

Results 4.2. A Penta Topological Spaces \( (X, \mathcal{P}) \):

First case: Every \( T_0 \)- Topological Space is a \( \mathcal{P}_{P_0} \)- Topological Space.

Second case: Every \( T_1 \)- Topological Space is a \( \mathcal{P}_{P_1} \)- Topological Space.

Third case: If there is no one of the topologies is a \( T_0 \) – space, then the Penta Topological Space is a \( \mathcal{P}_{P_0} \)- Topological Space.

Fourth case: If there is no one of the topologies is a \( T_1 \) – space, then the Penta Topological Space is a \( \mathcal{P}_{P_1} \)- Topological Space.

Fifth case: If at least one of the topologies is a \( T_i \) – space \( \forall i = 0, 1 \), then the Penta Topological Space is a \( \mathcal{P}_{P_i} \)- Topological Space.

Let us discuss the following examples for above cases:

Example 4.3. Let \( X = \{a, b, c, d\} \).

1. For First and second cases.
   \[ \mathcal{P}_1 = \{X, \emptyset, \{b\}, \{b, d\}, \{b, c, d\}, \{a, d\}, \{a, b, d\}, \{a, b, c, d\}\} \]
   \[ \mathcal{P}_2 = \{X, \emptyset, \{a\}, \{a, b\}, \{a, b, d\}, \{a, b, c\}\} \]
   \[ \mathcal{P}_3 = \{X, \emptyset, \{a\}, \{a, b\}, \{a, b, d\}, \{a, b, c\}\} \]

2. For the third and Fourth cases.
   \[ \mathcal{P}_1 = \{X, \emptyset, \{a\}, \{a, b\}, \{a, b, c\}\} \]
   \[ \mathcal{P}_2 = \{X, \emptyset, \{a\}, \{a, b\}, \{a, b, d\}, \{a, b, c\}\} \]
   \[ \mathcal{P}_3 = \{X, \emptyset, \{a\}, \{a, b\}, \{a, b, c\}\} \]

3. For the fifth case.
   \[ \mathcal{P}_1 = \{X, \emptyset, \{a\}, \{a, b\}\} \]
   \[ \mathcal{P}_2 = \{X, \emptyset, \{a\}, \{a, b\}, \{a, b, d\}\} \]
   \[ \mathcal{P}_3 = \{X, \emptyset, \{a\}, \{a, b\}, \{a, b, c\}\} \]

Results 4.4. A Penta Topological Spaces \( (X, \mathcal{P}) \):
∃ T_2 − space → T_1 − space → T_0 − space

\[ T_{P_2} - \text{space} \rightarrow T_{P_1} - \text{space} \rightarrow T_{P_0} - \text{space} \]

Figure 2:

**First case:** If there is no one of the topologies is a T_2 − space then the Penta Topological Space is a \( \mathcal{P}_{P_2} \)- Topological Space.

**Second case:** If at least one of the topologies is a T_2 − space then the Penta Topological Space is a \( \mathcal{P}_{P_2} \)- Topological Space.

**Example 4.5.** A Penta topology
\[ \mathcal{P} = \{ \mathcal{X}, \emptyset, \{a\}, \{b\}, \{c\}, \{d\}, \{b, d\}, \{b, c\}, \{a, d\}, \{a, b\}, \{a, c\}, \{a, b, d\}, \{b, c, d\}, \{a, c, d\} \} \]
on X = \{a, b, c, d\}, when

I. \( \mathcal{S}_1 = \{ \mathcal{X}, \emptyset, \{a\}, \{b\}, \{d\}, \{b, c\}, \{a, c\}, \{a, b\}, \{a, c\}, \{a, b, d\}, \{c, d\}, \{a, c, d\} \} \)
\( \mathcal{S}_2 = \{ \mathcal{X}, \emptyset, \{a\}, \{c\}, \{d\}, \{a, b\}, \{a, c\}, \{a, b, d\}, \{b, c, d\} \} \)
\( \mathcal{S}_3 = \{ \mathcal{X}, \emptyset, \{c\}, \{d\}, \{a, b\}, \{a, c\}, \{a, b, d\}, \{b, c, d\} \} \)
\( \mathcal{S}_4 = \{ \mathcal{X}, \emptyset, \{a\}, \{b\}, \{a, b\}, \{a, c\}, \{a, b, c\}, \{a, b, d\}, \{b, c, d\} \} \) and
\( \mathcal{S}_5 = \{ \mathcal{X}, \emptyset, \{a\}, \{b\}, \{a, c\}, \{a, b, c\} \} \) are five Topological Spaces, then \( (\mathcal{X}, \mathcal{S}_1) \) is \( T_{P_2} - \text{space} \) but not necessary that one of the five topologies \( T_2 - \text{space} \). It is clear that \( (\mathcal{X}, \mathcal{S}_1) \) is \( T_2 - \text{space} \).

II. \( \mathcal{S}_1 = \{ \mathcal{X}, \emptyset, \{a\}, \{b\}, \{c\}, \{d\}, \{b, c\}, \{a, b, d\}, \{c, d\}, \{a, b, c\} \} \)
\( \mathcal{S}_2 = \{ \mathcal{X}, \emptyset, \{a\}, \{c\}, \{b\}, \{a, c\}, \{a, b\}, \{a, b, c\} \} \)
\( \mathcal{S}_3 = \{ \mathcal{X}, \emptyset, \{a\}, \{b\}, \{c\}, \{d\}, \{a, b\} \} \)
\( \mathcal{S}_4 = \{ \mathcal{X}, \emptyset, \{a\}, \{b\}, \{a, b\}, \{a, c\}, \{a, b, c\} \} \) and
\( \mathcal{S}_5 = \{ \mathcal{X}, \emptyset, \{a\}, \{b\}, \{a, c\}, \{a, b, c\} \} \) are five Topological Spaces.
Then \( (\mathcal{X}, \mathcal{S}_1) \) is \( T_{P_2} - \text{space} \) but the five topologies are \( T_2 - \text{space} \).

By results \[4.2\] and definition \[4.1\] we obtain the following diagram which illustrates the relationship between the types of separation axioms. By using case (1,5) we get Figure 2.

**Proposition 4.6.** A space \( (\mathcal{X}, \mathcal{S}) \) is a \( T_{P_0} - \text{space} \) if and only if for every distinct points \( v, u \) of \( \mathcal{X} \), \( cl_{\mathcal{P}} \{v\} \neq cl_{\mathcal{P}} \{u\} \).

**Proof.** For every \( v, u \) of \( \mathcal{X} \) and \( v \neq u \), whenever \( \mathcal{X} \) is a \( T_{P_0} - \text{space} \) there exist a \( \mathcal{P} - \text{open} \) set such that \( v \in \mathcal{H}, u \in \mathcal{X}\setminus \mathcal{H} \), hence \( \{u\} \subseteq \mathcal{X}\setminus \mathcal{H} \) is a \( \mathcal{P} - \text{closed} \) set, \( cl_{\mathcal{P}} \{u\} \subseteq \mathcal{X}\setminus \mathcal{H} \), so \( v \notin cl_{\mathcal{P}} \{u\} \). Then \( cl_{\mathcal{P}} \{v\} \neq cl_{\mathcal{P}} \{u\} \).

Conversely, assume that \( v \neq u \), then \( cl_{\mathcal{P}} \{v\} \) and \( cl_{\mathcal{P}} \{u\} \) are distinct sets, \( \exists \rho \in \mathcal{X} \) belong to one sets \( \rho \in cl_{\mathcal{P}} \{v\} \) and \( \rho \notin cl_{\mathcal{P}} \{u\} \). Now \( v \notin cl_{\mathcal{P}} \{u\} \) (because \( v \in cl_{\mathcal{P}} \{u\} \) then \( cl_{\mathcal{P}} \{v\} \subseteq cl_{\mathcal{P}}(cl_{\mathcal{P}} \{u\}) = cl_{\mathcal{P}} \{u\} \)). Also \( \rho \in cl_{\mathcal{P}} \{v\} \) is a contradiction, therefore \( v \in (cl_{\mathcal{P}} \{u\})^{c} \), so \( (cl_{\mathcal{P}} \{u\})^{c} \) is \( \mathcal{P} - \text{open} \) set contained one but not the other. Then \( \mathcal{X} \) is a \( T_{P_0} - \text{space} \). □

**Theorem 4.7.** A Penta topological space \( \mathcal{X} \) is a \( T_{P_1} - \text{space} \), iff every singleton is \( \mathcal{P} - \text{closed} \) sets.

**Proof.** Obvious. □

**Theorem 4.8.** A strongly \( \mathcal{P} - \text{connected} \) \( T_{P_1} - \text{space} \) has at most one point in uncountable space.
Proof. By theorem \[\text{L7}\] we have a singleton set in a \(T_{P_1} - \text{space}\) is \(P - \text{closed set}\). Therefore, we get \(T_{P_1} - \text{space}\) cannot have countably many but more than one point. \(\square\)

By adding some conditions to the function, we get the following theorems.

Theorem 4.9. Let \(f : (X, \mathcal{S}_P) \to (Y, \mathcal{S}_P)\) be a bijective \(P - \text{open function}\) and \(X\) is a \(T_1 - \text{space}\) then \(Y\) is \(T_{P_1} - \text{space}\), where \(i = 0, 1, 2\).

Proof. We prove the case \(i = 2\).
Let \(v_2, u_2\) be two points in \(Y\) and \(v_2 \neq u_2\), since \(f\) is bijective, then \(\exists v_1, u_1 \in X\) and \(f(v_1) = v_2, f(u_1) = u_2\). But \(X\) is a \(T_2\), then \(\exists\) two disjoint \(P - \text{open sets}\) \(H, K \in X\), whenever \(v_1 \in H, u_1 \in K\). Then \(f(H), f(K)\) are \(P - \text{open sets}\) in \(Y\) (because every \(P - \text{open}\) is \(\text{semi}_P 0\)). and \(f\) is \(P - \text{open}\) we get \(v_2 \in f(H), u_2 \in f(K)\) and \(f(H) \cap f(K) = \emptyset\). Hence \(Y\) is \(T_{P_2} - \text{space}\). \(\square\)

Theorem 4.10. Let \(f : (X, \mathcal{S}_P) \to (Y, \mathcal{S}_P)\) be an injective \(P - \text{continuous function}\) and \(Y\) is \(T_1 - \text{space}\), then \(X\), is \(T_{P_1} - \text{space}\), where \(i = 0, 1, 2\).

Proof. We prove the case \(i = 1\)
Since \(Y\) is \(T_1\) and let \(v, u\) of \(X\) and \(v \neq u\), there exist two disjoint \(P - \text{open sets}\) \(H, K \in Y\) (because every \(P - \text{open}\) is \(\text{semi}_P 0\)) such that \(f(v) \in H, f(u) \in K, f(v) \neq f(u)\), since \(f\) is \(P - \text{continuous}\), then \(f^{-1}(H)\) and \(f^{-1}(K)\) are \(P - \text{open sets of mathbb X}\), we get \(v \in f^{-1}(H), u \in f^{-1}(K)\). Hence \(X\) is \(T_{P_1} - \text{space}\). \(\square\)

Theorem 4.11. Let \(f : (X, \mathcal{S}_P) \to (Y, \mathcal{S}_P)\) be an injective \(P - \text{continuous function}\) and \(Y\) is \(T_{P_1} - \text{space}\), then \(X\), is \(T_{P_1} - \text{space}\), where \(i = 0, 1, 2\).

Proof. We prove the case \(i = 2\)
Let \(v, u\) of \(X\) and \(v \neq u\), since \(f\) is one to one, then \(f(v) \neq f(u)\) in \(Y\). But \(Y\) is \(T_{P_2} - \text{space}\), then there exist two disjoint \(P - \text{open sets}\) \(H, K \in Y\), whenever \(f(v) \in H, f(u) \in K\). Then \(f^{-1}(H), f^{-1}(K)\) a \(P - \text{open (because f is P – continuous)}\), we get \(v \in f^{-1}(H), u \in f^{-1}(K)\) and \(f^{-1}(H) \cap f^{-1}(K) = \emptyset\). So \(X\) is \(T_{P_2} - \text{space}\) \(\square\)

Theorem 4.12. Let \(f : (X, \mathcal{S}_P) \to (Y, \mathcal{S}_P)\) be an injective \(P - \text{irresolute function}\) and \(Y\), is \(T_{P_1} - \text{space}\). Then \(X\), is \(T_{P_1} - \text{space}\), where \(i = 0, 1, 2\).

Proof. We prove the case \(i = 0\)
Let \(v, u\) in \(X\) and \(v \neq u\), since \(f\) is one to one, then \(f(v) \neq f(u)\) in \(Y\), \(Y\) is \(T_{P_0} - \text{space}\), then \(\exists\) a \(P - \text{open set}\) \(H \in Y\), whenever \(f(v) \in H, f(u) \notin H\). Then \(f^{-1}(H)\) is \(\text{semi}_P 0\). Set (because \(f\) is \(P - \text{irresolute}\) and every \(P - \text{open is semi}_P 0\). set), we get \(v \in f^{-1}(H), u \notin f^{-1}(H)\). So \(X\) is \(T_{P_0} - \text{space}\). \(\square\)

Proposition 4.13. Let \(f : (X, \mathcal{S}_P) \to (Y, \mathcal{S}_P)\) be \(P - \text{homeomorphim and Y is T_{P_2} – space}, then X \text{ T_{P_2} – space}.\)

Proof. We prove the case \(i = 0\)
Suppose that \(v_1, v_2 \in X\), with \(v_1 \neq v_2\). We get \(f(v_1) \neq f(v_2)\) and \(f(v_1), f(v_2) \in Y\), since \(Y\) is \(T_{P_2} - \text{space}\) there exist two \(P - \text{open sets}\) \(H, K \in Y\) such that \(f(v_1) \in H, f(v_2) \in K\) and \(H \cap K = \emptyset\). Now \(v_1 \in f^{-1}(H), v_2 \in f^{-1}(K)\) and \(f^{-1}(H) \cap f^{-1}(K) = f^{-1}(H \cap K) = f^{-1}(\emptyset) = \emptyset\). Hence \(X\) is \(T_{P_2} - \text{space}\). \(\square\)

Theorem 4.14. Every \(P - \text{compact subset of T_{P_2} – space is P – closed set}.\)
Proof. Let $E$ be a $\mathcal{P}$–compact subset of $T_{P2}$ – space $(\mathcal{X}, \mathcal{S}_P)$ and suppose that $u \notin E$, then there exist distinct $\mathcal{P}$–open set contains $u$ and $v \in E$. We obtain $\mathcal{H}, \mathcal{K} \mathcal{P}$–open sets contains $u$ and $v$ respectively. $\exists \{G_v : v \in E\}$ is $\mathcal{P}$–cover of $E$ by $\mathcal{P}$–open sets in $\mathcal{X}$, we get $\exists$ finitely many of them, $G_{v1}, ..., G_{vn}$ is $\mathcal{P}$–cover of $E$, thus $G = \bigcup^n_{i=1} G_{vi}$ contains $E$ and disjoint from $\mathcal{P}$–open set $C = \bigcap^n_{i=1} C_{vi}$. Taking the intersection of $\mathcal{P}$–open sets contains $u$, if $w \in G$, then $w \in G_{vi}$, then $w \notin C_{vi}$ and $w \notin C$. Then $C$ is $\mathcal{P}$–open set and $u \in C$ disjoint from $E$. □

Example 4.15. From example 1.5, since $\mathcal{X}$ is finite, then $(\mathcal{X}, \mathcal{S}_P) \mathcal{P}$–compact but $(\mathcal{X}, \mathcal{S}_P)$ is not $T_{P2}$ – space.

Results 4.16. All discrete spaces are locally $\mathcal{P}$–compact and $T_{P2}$ – space these are $\mathcal{P}$–compact if and only if they are finite.

5. Conclusions

The main results of this paper are stated as below:

1. the concepts of connectedness, compactness and separation axioms on Penta topological space developed with some theorems and the relationship between them.
2. If $f : (\mathcal{X}, \mathcal{S}_P) \to (\mathcal{Y}, \mathcal{S}_P)$ be bijective $\mathcal{P}$–open function and $\mathcal{X}$ is a $T_i$ – space, then $\mathcal{Y}$ is $T_{P_i}$ – space, where $i = 0, 1, 2$.
3. If $f : (\mathcal{X}, \mathcal{S}_P) \to (\mathcal{Y}, \mathcal{S}_P)$ be bijective $\mathcal{P}$–continuous function and $\mathcal{Y}$ is a $T_i$ – space, then $\mathcal{X}$ is $T_{P_i}$ – space, where $i = 0, 1, 2$.
4. If $f : (\mathcal{X}, \mathcal{S}_P) \to (\mathcal{Y}, \mathcal{S}_P)$ be bijective $\mathcal{P}$–continuous function and $\mathcal{Y}$ is a $T_{P_i}$ – space, then $\mathcal{X}$ is $T_{P_{i+1}}$ – space, where $i = 0, 1, 2$.
5. If $f : (\mathcal{X}, \mathcal{S}_P) \to (\mathcal{Y}, \mathcal{S}_P)$ be bijective $\mathcal{P}$–irresolute function and $\mathcal{Y}$ is a $T_{P_i}$ – space, then $\mathcal{X}$ is $T_{P_{i+1}}$ – space, where $i = 0, 1, 2$.
6. If $\mathcal{P}$–connected space then there exist at least one of the five topologies is connected.
7. Every $\mathcal{P}$–connected space is locally $\mathcal{P}$–compact space.
8. Every locally compact space is locally $\mathcal{P}$–compact space.

References


