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Some Mean Square Integral Inequalities For Preinvexity Involving The Beta Function

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Abstract

In the present research, we will deal with mean square integral inequalities for preinvex stochastic process and η-convex stochastic process in setting of beta function. Further, we will present some novel results for improved Hölder integral inequality. The results given in this present paper are generalizations of already existing results in the literature.

Keywords: Mean square integral inequalities, Convex stochastic process, η-convex stochastic process, Preinvex stochastic process, Beta function
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1. Introduction

In the last few years, a lot of attention has been given to the study of stochastic processes due to a wide range of applications in finance, probabilistic model, economics, information theory, noise in the physical system, operation research, weather forecasting, astronomy, and so on. The study of stochastic processes makes use of mathematical expertise and strategies for probability, calculus, set theory, functional analysis, linear algebra, real analysis, topology, and Fourier analysis. Stochastic processes is a branch of mathematics, beginning with the axioms of probability and containing a wealthy and captivating set of results following from one's axioms [12]. Yet the results are useful to several areas [11], they're best understood at the start in respect of their mathematical structure and correlation [8].

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The study on convex stochastic processes was originated by B. Nagy in 1974 (see [20]). In [21, 22] , K. Nikodem et al. investigated some more properties of convex stochastic stochastic processes. Later on, at the end of the twentieth century, Skowronski established some more results on the convex stochastic process which generalized several well-known characterizations [27, 28] . In [24] , Set et al. considered the Hermite-Hadamard integral type inequalities in the second sense for convex stochastic processes within the year 2014. Recently, in [17, 31] numerous Hermite-Hadamard type inequalities connected with fractional integrals have been considered. D. Kotrys described results on convex and strongly convex stochastic processes, together with, a Hermite-Hadamard type inequality for convex stochastic processes (see [14, 15, 16]).
For convex stochastic process, Hermite-Hadamard integral inequality is expressed as follows: Assume $\xi : I \times \Omega \to \mathbb{R}$ be a convex and mean square continuous in the interval $T \times \Omega$, then

$$\xi\left(\frac{\mu+\nu}{2}, \cdot\right) \leq \frac{1}{\nu-\mu}\int_{\nu}^{\mu} \xi(\alpha, \cdot)d\alpha \leq \frac{\xi(\mu, \cdot)+\xi(\nu, \cdot)}{2} \quad (a.e.) \tag{1.1}$$

for all $\mu, \nu \in I$. For more on these inequalities, we allude the reader (see [1, 2**?** , 29, 30]).

Fractional calculus attained quick improvement in both applied and pure mathematics because of its giant use in system learning, physics, image processing, networking, and other branches of science. For more on these, we allude the reader to see [19, 23, 5] . The fractional derivative acquires speedy interest together with experts from disparate branches of science. The vast majority of applied issues can't be modeled by classical derivations. The difficulty in real-world issues is addressed via way of means fractional differential equations.

The present paper is organized as follows. In second section we will give some preliminary material. However, in third section we will present our main results and at last we give concluding remarks to our present research.

2. Preliminaries

Definition 2.1. *[6] A stochastic process is a collection of random variables $\xi(\mu)$ parametrized by $\mu \in I$, where $I \subset \mathbb{R}$. When $I = \{1, 2, ...\}$, then $\xi(\mu)$ is known as a stochastic process in discrete time (a sequence of random variables). When I is an interval in $\mathbb{R}$ ($I = [0, \infty)$), then $\xi(\mu)$ be a stochastic process in continuous time.*

For every $\omega \in \Omega$ the function

$$I \ni \mu \longmapsto \xi(\mu, \omega)$$

is called a path or sample path of $\xi(\mu)$.

Definition 2.2. *[6] A family F_μ of α-fields on Ω parametrized by $\mu \in I$, where $I \subset \mathbb{R}$, is known as a filtration if*

$$F_\nu \subset F_\mu \subset F$$

for any $\nu, \mu \in I$ such that $\nu \leq \mu$.

Definition 2.3. *[6] A sochastic process $\xi(\mu)$ parametrized by $\mu \in T$ is known as a martingale (supermartingale, submartingale) with respect to a filtration F_μ if*

1) $\xi(\mu)$ is integrable for each $\mu \in I$;

2) $\xi(\mu)$ is F_μ-measurable for each $\mu \in I$;

3) $\xi(\nu) = E(X(\mu)|F_\nu)$ (respectively, $\leq$ or $\geq$) for every $\nu, \mu \in I$ such that $\nu \leq \mu$.

Definition 2.4. *[14] Let (Ω, A, P) be an arbitrary probability space and $I \subset \mathbb{R}$. A stochastic process $\xi : \Omega \to \mathbb{R}$ is termed as*

(1) *Stochastically continuous in I, if $\forall\ \mu_\circ \in I$*

$$P - \lim_{\mu\to\mu_\circ} \xi(\mu, \cdot) = \xi(\mu_\circ, \cdot),$$

where $P - \lim$ denotes the limit in probability.

(2) *Mean-square continuous in I, if $\forall\ \mu_\circ \in I$*

$$P - \lim_{\mu\to\mu_\circ} \mathbb{E}(\xi(\mu, \cdot) - \xi(\mu_\circ, \cdot)) = 0,$$

where $\mathbb{E}(\xi(\mu, \cdot))$ represents the expectation value of the random variable $\xi(\mu, \cdot)$.

(3) *If there exist a random variable $\xi'(\mu, \cdot) : I \times \Omega \to \mathbb{R}$ then it is differentiable at a point $\mu \in I$, such that*

$$\xi'(\mu, \cdot) = P - \lim_{\mu\to\mu_\circ} \frac{\xi(\mu, \cdot) - \xi(\mu_\circ, \cdot)}{\mu - \mu_\circ}.$$

A stochastic process $\xi : I \times \Omega \to \mathbb{R}$ is termed to be continuous (differentiable) if it is continuous (differentiable) at every point of I.

Definition 2.5. *[14, 30] Assume (Ω, A, P) be a probability space and $I \subset \mathbb{R}$ be an interval with $E(\xi(\mu)^2) < \infty\ \forall\ \alpha \in I$. If $[a, b] \subset I, a = \mu_0 < \mu_1 < \mu_2 < ... < \mu_n = b$ be a partition of $[a, b]$ and $\Theta \in [\mu_{\kappa-1}, \mu_\kappa]$ for $\kappa = 1, 2, ..., n$. A random variable $Z : \Omega \to \mathbb{R}$ is termed as mean-square integral of the process $\xi(\mu)$ on $[a, b]$ if*

$$\lim_{n\to\infty} E\left[\sum_{\kappa=1}^{\infty} \xi(\Theta_\kappa, \cdot)(\mu_\kappa, \mu_{\kappa-1}) - Z(\cdot)\right]^2 = 0,$$

then we have

$$\int_a^b \xi(\mu, \cdot)d\mu = Z(\cdot) \quad (a.e.).$$

Also, mean square integral operator is increasing, then,

$$\int_a^b \xi(\mu, \cdot)d\mu \leq \int_a^b Y(\mu, \cdot) \quad (a.e.)$$

where $\xi(\mu, \cdot) \leq Y(\mu, \cdot)$ in $[a, b]$.

For more details on stochastic processes (see [4, 7, 18, 25, 26]).
Now, we give some definitions related to our present work:

Definition 2.6. *[22] Let (Ω, A, P) be a probability space and $I \subseteq R$ be an interval. A stochastic process $\xi : I \times \Omega \to R$ is termed as a convex stochastic process, then*

$$\xi(\alpha\mu + (1 - \alpha)\nu, \cdot) \leq \alpha\xi(\mu, \cdot) + (1 - \alpha)\xi(\nu, \cdot) \quad (a.e.) \tag{2.1}$$

$\forall\ \mu, \nu \in I$ and $\alpha \in [0, 1]$.

Definition 2.7. *[3] Let $\xi : I \times \Omega \to \mathbb{R}$ (not necessarily mean-square differentiable) on η-invex index set I. $\xi(\mu, \cdot)$ is said to be preinvex with respect to η if*

$$\xi(\mu + \alpha\eta(\nu, \mu), \cdot) \leq (1 - \alpha)\xi(\mu, \cdot) + \alpha\xi(\nu, \cdot) \quad (a.e.) \tag{2.2}$$

holds $\forall\ \mu, \nu \in I$ and $\alpha \in [0, 1]$.

Definition 2.8. *[13] Let (Ω, A, P) be a probability space and $I \subseteq \mathbb{R}$ be an interval, then $\xi : I \times \Omega \to \mathbb{R}$ is called η-convex stochastic process, if*

$$\xi(\alpha\mu + (1 - \alpha)\nu, \cdot) \leq \xi(\nu, \cdot) + \alpha\eta(\xi(\mu, \cdot), \xi(\nu, \cdot)) \quad (a.e.) \tag{2.3}$$

holds $\forall\ \mu, \nu \in I$ and $\alpha \in [0, 1]$.

Also, we will use the Beta function in this present paper which is defined as follows:

$$\beta(\mu, \nu) = \int_0^1 \alpha^{\mu-1}(1 - \alpha)^{\nu-1} d\alpha, \qquad Re(\mu) > 0,\ Re(\nu) > 0.$$

Theorem 2.9. *(**Hölder-İşcan integral inequality**) [9]*

Let $\xi_1, \xi_2 : [\mu, \nu] \times \Omega \to \mathbb{R}$ be a real stochastic processes and $|\xi_1|^q, |\xi_2|^q$ be a mean square integrable on $[\mu, \nu]$. If $p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$, then we have almost everywhere

$$\begin{aligned}\int_\mu^\nu |\xi_1(z, \cdot)\xi_2(z, \cdot)|\, dz \leq \frac{1}{\nu - \mu}\Bigg\{ &\left(\int_\mu^\nu (\nu - z)\, |\xi_1(z, \cdot)|^p\, dz\right)^{\frac{1}{p}} \left(\int_\mu^\nu (\nu - z)\, |\xi_2(z, \cdot)|^q\, dz\right)^{\frac{1}{q}} \\ &+ \left(\int_\mu^\nu (z - \mu)\, |\xi_1(z, \cdot)|^p\, dz\right)^{\frac{1}{p}} \left(\int_\mu^\nu (z - \mu)\, |\xi_2(z, \cdot)|^q\, dz\right)^{\frac{1}{q}} \Bigg\}.\end{aligned}$$

Theorem 2.10. *(**Improved power-mean integral inequality**) [9]*

Let $\xi_1, \xi_2 : [\mu, \nu] \times \Omega \to \mathbb{R}$ be a real stochastic processes and $|\xi_1|, |\xi_1||\xi_2|^q$ be a mean square integrable on $[\mu, \nu]$. If $q \geq 1$, then we have almost everywhere

$$\begin{aligned}\int_\mu^\nu |\xi_1(z, \cdot)\xi_2(z, \cdot)|\, dz \leq \frac{1}{\nu - \mu}\Bigg\{ &\left(\int_\mu^\nu (\nu - z)\, |\xi_1(z, \cdot)|\, dz\right)^{1-\frac{1}{q}} \left(\int_\mu^\nu (\nu - z)\, |\xi_1(z, \cdot)|\, |\xi_2(z)|^q\, dz\right)^{\frac{1}{q}} \\ &+ \left(\int_\mu^\nu (z - \mu)\, |\xi_1(z, \cdot)|\, dz\right)^{1-\frac{1}{q}} \left(\int_\mu^\nu (z - \mu)\, |\xi_1(z, \cdot)|\, |\xi_2(z, \cdot)|^q\, dz\right)^{\frac{1}{q}} \Bigg\}.\end{aligned}$$

3. Main Results

In this section, we will establish some new lemmas and results for η-convex stochastic process and preinvex stochastic process under beta function.

Lemma 3.1. *[10] Assume that $\xi : I \times \Omega \to \mathbb{R}$ be a mean square continuous and mean square integrable stochastic process. Then the equality holds almost everywhere*

$$\int_{\mu}^{\nu} (z-\mu)^{\theta}(\nu-z)^{\vartheta}\xi(z,\cdot)dz$$
$$= (\nu-\mu)^{\theta+\vartheta+1}\int_{0}^{1}(1-\alpha)^{\theta}\alpha^{\vartheta}\xi(\alpha\mu+(1-\alpha)\nu,\cdot)d\alpha,$$

for some fixed $\theta, \vartheta > 0$.

Lemma 3.2. *Let $\xi : I = [\mu, \mu+\eta(\nu,\mu)] \times \Omega \to \mathbb{R}$ be a mean square integrable and mean square continuous stochastic process with $\mu < \mu+\eta(\nu,\mu)$. Then the equality holds almost everywhere*

$$\int_{\mu}^{\mu+\eta(\nu,\mu)} (z-\mu)^{\theta}(\mu+\eta(\nu,\mu)-z)^{\vartheta}\xi(z,\cdot)dz$$
$$= \eta(\nu,\mu)^{\theta+\vartheta+1}\int_{0}^{1}\alpha^{\theta}(1-\alpha)^{\vartheta}\xi(\mu+\alpha\eta(\nu,\mu),\cdot)d\alpha$$

for some fixed $\theta, \vartheta > 0$.

Proof . *Note that*

$$\int_{\mu}^{\mu+\eta(\nu,\mu)} (z-\mu)^{\theta}(\mu+\eta(\nu,\mu)-z)^{\vartheta}\xi(z,\cdot)dz$$
$$= \int_{0}^{1}(\mu+\alpha\eta(\nu,\mu)-\mu)^{\theta}$$
$$\times(\mu+\eta(\nu,\mu)-\mu-\alpha\eta(\nu,\mu)^{\vartheta}\xi(\mu+\alpha\eta(\nu,\mu),\cdot)d\alpha$$
$$= \eta(\nu,\mu)^{\theta+\vartheta+1}\int_{0}^{1}\alpha^{\theta}(1-\alpha)^{\vartheta}\xi(\mu+\alpha\eta(\nu,\mu),\cdot)d\alpha.$$

This completes the proof. □

We will derive the next results for η-convex stochastic process.

Theorem 3.3. *Suppose $\xi : I \times \Omega \to \mathbb{R}$ be a mean square continuous and mean square integrable stochastic process. Take $\theta, \vartheta > 0$, if $|\xi|$ is a η-convex on $[\mu,\nu]$, where $\mu,\nu \in I$ and $\mu < \nu$, then the inequality holds almost everywhere*

$$\int_{\mu}^{\nu} (z-\mu)^{\theta}(\nu-z)^{\vartheta}\xi(z,\cdot)dz$$
$$\leq (\nu-\mu)^{\theta+\vartheta+1}\Big(\beta(\theta+1,\vartheta+1)\left|\xi(\nu,\cdot)\right| + \beta(\theta+1,\vartheta+2)\eta\left(\left|\xi(\mu,\cdot)\right|,\left|\xi(\nu,\cdot)\right|\right)\Big).$$

Proof . *From Lemma 3.1 and take the definition of η-convex stochastic process of $|\xi|$ yields that*

$$\begin{aligned}\int_\mu^\nu &(z-\mu)^\theta(\nu-z)^\vartheta\xi(z,\cdot)dz\\&\le(\nu-\mu)^{\theta+\vartheta+1}\int_0^1(1-\alpha)^\theta\alpha^\vartheta|\xi(\alpha\mu+(1-\alpha)\nu,\cdot)|d\alpha\\&\le(\nu-\mu)^{\theta+\vartheta+1}\int_0^1(1-\alpha)^\theta\alpha^\vartheta\Big(|\xi(\nu,\cdot)|+\alpha\eta\left(|\xi(\mu,\cdot)|,|\xi(\nu,\cdot)|\right)\Big)d\alpha \quad (a.e.)\end{aligned}$$

and by the definition of beta function, we have

$$\le(\nu-\mu)^{\theta+\vartheta+1}\Big(\beta(\theta+1,\vartheta+1)\left|\xi(\nu,\cdot)\right|+\beta(\theta+1,\vartheta+2)\eta\left(|\xi(\mu,\cdot)|,|\xi(\nu,\cdot)|\right)\Big) \quad (a.e.)$$

This completes the proof. □

Theorem 3.4. *Suppose $\xi : I\times\Omega\to\mathbb{R}$ be a mean square continuous and mean square integrable stochastic process. Take $\theta,\vartheta>0$, if $|\xi|^q$ is a η-convex on $[\mu,\nu]$ for $q>1$ with $\frac{1}{p}+\frac{1}{q}=1$, where $\mu,\nu\in I$ and $\mu<\nu$, then the inequality holds almost everywhere*

$$\begin{aligned}\int_\mu^\nu &(z-\mu)^\theta(\nu-z)^\vartheta\xi(z,\cdot)dz\\&\le(\nu-\mu)^{\theta+\vartheta+1}\Big(\beta(p\theta+1,p\vartheta+1)\Big)^{\frac{1}{p}}\Big(|\xi(\nu,\cdot)|^q+\frac{1}{2}\eta\left(|\xi(\mu,\cdot)|^q,|\xi(\nu,\cdot)|^q\right)\Big)^{\frac{1}{q}}.\end{aligned} \tag{3.1}$$

Proof . *By using Lemma 3.1 and from Hölder integral inequality, we have* (a.e.)

$$\begin{aligned}\int_\mu^\nu &(z-\mu)^\theta(\nu-z)^\vartheta\xi(z,\cdot)dz\\&\le(\nu-\mu)^{\theta+\vartheta+1}\int_0^1(1-\alpha)^\theta\alpha^\vartheta|\xi(\alpha\mu+(1-\alpha)\nu,\cdot)|d\alpha\\&\le(\nu-\mu)^{\theta+\vartheta+1}\left(\int_0^1(1-\alpha)^{p\theta}\alpha^{p\vartheta}d\alpha\right)^{\frac{1}{p}}\left(\int_0^1|\xi(\alpha\mu+(1-\alpha)\nu,\cdot)|^qd\alpha\right)^{\frac{1}{q}}\end{aligned} \tag{3.2}$$

Since $|\xi|^q$ is a η-convex stochastic process, we have (a.e.)

$$\begin{aligned}\int_0^1|\xi(\alpha\mu+(1-\alpha)\nu,\cdot)|^qd\alpha&\le\int_0^1\Big(|\xi(\nu,\cdot)|^q+\alpha\eta\left(|\xi(\mu,\cdot)|^q,|\xi(\nu,\cdot)|^q\right)\Big)d\alpha\\&\le|\xi(\nu,\cdot)|^q+\frac{1}{2}\eta|\xi(\mu,\cdot)|^q,|\xi(\nu,\cdot)|^q,\end{aligned} \tag{3.3}$$

Now, by the definition of beta function, we can deduce

$$\int_0^1(1-\alpha)^{p\theta}\alpha^{p\vartheta}d\alpha=\beta(p\theta+1,p\vartheta+1). \tag{3.4}$$

So, by inserting (3.3) *and* (3.4) *in* (3.2) *yields the required inequality* (3.1) . □

Theorem 3.5. *Let $\xi : I\times\Omega \to \mathbb{R}$ be a mean square continuous and mean square integrable stochastic process. Take $\theta,\vartheta > 0$, if $|\xi|^q$ is a η-convex on $[\mu,\nu]$ for $q > 1$, where $\mu,\nu \in I$ and $\mu < \nu$, then the inequality holds almost everywhere*

$$\begin{aligned}
&\int_\mu^\nu (z-\mu)^\theta(\nu-z)^\vartheta \xi(z,\cdot)dz \\
&\quad \le (\nu-\mu)^{\theta+\vartheta+1}\Big(\beta\big(\theta+1,\vartheta+1\big)\Big)^{1-\frac{1}{q}} \\
&\qquad \times\left(\beta(\theta+1,\vartheta+1)|\xi(\nu,\cdot)|^q+\beta(\theta+1,\vartheta+2)\eta\left(|\xi(\mu,\cdot)|^q,|\xi(\nu,\cdot)|^q\right)\right)^{\frac{1}{q}}.
\end{aligned}$$

Proof . *By using Lemma 3.1 and power-mean integral inequality for $q \ge 1$, we can write*

$$\begin{aligned}
&\int_\mu^\nu (z-\mu)^\theta(\nu-z)^\vartheta \xi(z,\cdot)dz \\
&\quad \le (\nu-\mu)^{\theta+\vartheta+1}\int_0^1 (1-\alpha)^\theta\alpha^\vartheta|\xi(\alpha\mu+(1-\alpha)\nu,\cdot)|d\alpha \\
&\quad \le (\nu-\mu)^{\theta+\vartheta+1}\left(\int_0^1(1-\alpha)^\theta\alpha^\vartheta d\alpha\right)^{1-\frac{1}{q}} \\
&\qquad \times\left(\int_0^1(1-\alpha)^\theta\alpha^\vartheta|\xi(\alpha\mu+(1-\alpha)\nu,\cdot)|^q d\alpha\right)^{\frac{1}{q}} \quad (a.e.).
\end{aligned}$$

Take the definition of η-convex stochastic process of $|\xi|^q$ and empolying the beta function yields that

$$\begin{aligned}
&\int_\mu^\nu (z-\mu)^\theta(\nu-z)^\vartheta \xi(z,\cdot)dz \\
&\quad \le (\nu-\mu)^{\theta+\vartheta+1}\left(\int_0^1(1-\alpha)^\theta\alpha^\vartheta d\alpha\right)^{1-\frac{1}{q}} \\
&\qquad \times\left(\int_0^1(1-\alpha)^\theta\alpha^\vartheta\Big(|\xi(\nu,\cdot)|^q+\alpha\eta\left(|\xi(\mu,\cdot)|^q,|\xi(\nu,\cdot)|^q\right)\Big)d\alpha\right)^{\frac{1}{q}} \\
&\quad \le (\nu-\mu)^{\theta+\vartheta+1}\left(\beta(\theta+1,\vartheta+1)\right)^{1-\frac{1}{q}} \\
&\qquad \times\left(\beta(\theta+1,\vartheta+1)|\xi(\nu,\cdot)|^q+\beta(\theta+1,\vartheta+2)\eta\left(|\xi(\mu,\cdot)|^q,|\xi(\nu,\cdot)|^q\right)\right)^{\frac{1}{q}} \quad (a.e.)
\end{aligned}$$

which completes the proof. □

Theorem 3.6. *Let $\xi : I\times\Omega \to \mathbb{R}$ be a mean square continuous and mean square integrable stochastic process. Take $\theta,\vartheta > 0$, if $|\xi|^q$ is a η-convex on $[\mu,\nu]$ for $q > 1$ with $\frac{1}{p}+\frac{1}{q}=1$, where $\mu,\nu \in I$ and $\mu < \nu$, then the inequality holds almost everywhere*

$$\int_{\mu}^{\nu}(z-\mu)^{\theta}(\nu-z)^{\vartheta}\xi(z,\cdot)dz$$
$$\leq(\nu-\mu)^{\theta+\vartheta+1}\Bigg[\Big(\beta\big(p\theta+2,p\vartheta+1\big)\Big)^{\frac{1}{p}}\left(\frac{1}{2}|\xi(\nu,\cdot)|^{q}+\frac{1}{6}\eta\left(|\xi(\mu,\cdot)|^{q},|\xi(\nu,\cdot)|^{q}\right)\right)^{\frac{1}{q}}$$
$$+\Big(\beta\big(p\theta+1,p\vartheta+2\big)\Big)^{\frac{1}{p}}\left(\frac{1}{2}|\xi(\nu,\cdot)|^{q}+\frac{1}{3}\eta\left(|\xi(\mu,\cdot)|^{q},|\xi(\nu,\cdot)|^{q}\right)\right)^{\frac{1}{q}}\Bigg]. \tag{3.6}$$

Proof . *By Lemma 3.1 and using Hölder-İscan integral inequality, we have (a.e.)*

$$\int_{\mu}^{\nu}(z-\mu)^{\theta}(\nu-z)^{\vartheta}\xi(z,\cdot)dz$$
$$\leq(\nu-\mu)^{\theta+\vartheta+1}\int_{0}^{1}(1-\alpha)^{\theta}\alpha^{\vartheta}|\xi(\alpha\mu+(1-\alpha)\nu,\cdot)|d\alpha$$
$$\leq(\nu-\mu)^{\theta+\vartheta+1}\Bigg[\left(\int_{0}^{1}(1-\alpha)(1-\alpha)^{p\theta}\alpha^{p\vartheta}d\alpha\right)^{\frac{1}{p}}$$
$$\times\left(\int_{0}^{1}(1-\alpha)|\xi(\alpha\mu+(1-\alpha)\nu,\cdot)|^{q}d\alpha\right)^{\frac{1}{q}}$$
$$+\left(\int_{0}^{1}\alpha(1-\alpha)^{p\theta}\alpha^{p\vartheta}d\alpha\right)^{\frac{1}{p}}\left(\int_{0}^{1}\alpha|\xi(\alpha\mu+(1-\alpha)\nu,\cdot)|^{q}d\alpha\right)^{\frac{1}{q}}\Bigg].$$

Since $|\xi|^{q}$ is a η-convex stochastic process and by using the definition of beta function, we have (a.e.)

$$\int_{\mu}^{\nu}(z-\mu)^{\theta}(\nu-z)^{\vartheta}\xi(z,\cdot)dz$$
$$\leq(\nu-\mu)^{\theta+\vartheta+1}\Bigg[\left(\int_{0}^{1}(1-\alpha)^{p\theta+1}\alpha^{p\vartheta}d\alpha\right)^{\frac{1}{p}}$$
$$\times\left(\int_{0}^{1}(1-\alpha)\Big(|\xi(\nu,\cdot)|^{q}+\alpha\eta\left(|\xi(\mu,\cdot)|^{q},|\xi(\nu,\cdot)|^{q}\right)\Big)d\alpha\right)^{\frac{1}{q}}\Bigg]$$
$$+\left(\int_{0}^{1}(1-\alpha)^{p\theta}\alpha^{p\vartheta+1}d\alpha\right)^{\frac{1}{p}}$$
$$\times\left(\int_{0}^{1}\alpha\Big(|\xi(\nu,\cdot)|^{q}+\alpha\eta\left(|\xi(\mu,\cdot)|^{q},|\xi(\nu,\cdot)|^{q}\right)\Big)d\alpha\right)^{\frac{1}{q}}\Bigg]$$
$$\leq(\nu-\mu)^{\theta+\vartheta+1}\Bigg[\Big(\beta\big(p\theta+2,p\vartheta+1\big)\Big)^{\frac{1}{p}}\left(\frac{1}{2}|\xi(\nu,\cdot)|^{q}+\frac{1}{6}\eta\left(|\xi(\mu,\cdot)|^{q},|\xi(\nu,\cdot)|^{q}\right)\right)^{\frac{1}{q}}$$
$$+\Big(\beta\big(p\theta+1,p\vartheta+2\big)\Big)^{\frac{1}{p}}\left(\frac{1}{2}|\xi(\nu,\cdot)|^{q}+\frac{1}{3}\eta\left(|\xi(\mu,\cdot)|^{q},|\xi(\nu,\cdot)|^{q}\right)\right)^{\frac{1}{q}}\Bigg]$$

which completes the proof. □

Theorem 3.7. *Suppose that $\xi : I \times \Omega \to \mathbb{R}$ be a mean square continuous and mean square integrable stochastic process. Take $\theta, \vartheta > 0$, if $|\xi|^q$ is a η-convex on $[\mu, \nu]$ for $q > 1$, where $\mu, \nu \in I$ and $\mu < \nu$, then the inequality holds almost everywhere*

$$\begin{aligned}
&\int_{\mu}^{\nu} (z-\mu)^{\theta}(\nu - z)^{\vartheta}\xi(z,\cdot)dz \\
&\quad \le (\nu-\mu)^{\theta+\vartheta+1}\Big(\beta(\theta+2,\vartheta+1)\Big)^{1-\frac{1}{q}} \\
&\qquad \times \Big(\beta(\theta+2,\vartheta+1)|\xi(\nu,\cdot)|^q + \beta(\theta+2,\vartheta+2)\eta\left(|\xi(\mu,\cdot)|^q, |\xi(\nu,\cdot)|^q\right)\Big)^{\frac{1}{q}} \\
&\qquad + \Big(\beta(\theta+1,\vartheta+2)\Big)^{1-\frac{1}{q}} \\
&\qquad \times \Big(\beta(\theta+1,\vartheta+2)|\xi(\nu,\cdot)|^q + \beta(\theta+1,\vartheta+3)\eta\left(|\xi(\mu,\cdot)|^q, |\xi(\nu,\cdot)|^q\right)\Big)^{\frac{1}{q}}.
\end{aligned}$$

Proof . *By using Lemma 3.1 and improved power-mean integral inequality for $q \ge 1$ yields that*

$$\begin{aligned}
&\int_{\mu}^{\nu} (z-\mu)^{\theta}(\nu - z)^{\vartheta}\xi(z,\cdot)dz \\
&\quad \le (\nu-\mu)^{\theta+\vartheta+1}\int_0^1 (1-\alpha)^{\theta}\alpha^{\vartheta}|\xi(\alpha\mu+(1-\alpha)\nu,\cdot)|d\alpha \\
&\quad \le (\nu-\mu)^{\theta+\vartheta+1}\left(\int_0^1 (1-\alpha)(1-\alpha)^{\theta}\alpha^{\vartheta}d\alpha\right)^{1-\frac{1}{q}} \\
&\qquad \times \left(\int_0^1 (1-\alpha)(1-\alpha)^{\theta}\alpha^{\vartheta}|\xi(\alpha\mu+(1-\alpha)\nu,\cdot)|^q d\alpha\right)^{\frac{1}{q}} \\
&\qquad + \left(\int_0^1 \alpha(1-\alpha)^{\theta}\alpha^{\vartheta}d\alpha\right)^{1-\frac{1}{q}} \\
&\qquad \times \left(\int_0^1 \alpha(1-\alpha)^{\theta}\alpha^{\vartheta}|\xi(\alpha\mu+(1-\alpha)\nu,\cdot)|^q d\alpha\right)^{\frac{1}{q}} \quad (a.e.).
\end{aligned}$$

Take the definition of η-convex stochastic process of $|\xi|^q$ and beta function yields that

$$\begin{aligned}
&\int_\mu^\nu (z-\mu)^\theta(\nu-z)^\vartheta\xi(z,\cdot)dz \\
&\quad\le (\nu-\mu)^{\theta+\vartheta+1}\left(\int_0^1(1-\alpha)^{\theta+1}\alpha^\vartheta d\alpha\right)^{1-\frac{1}{q}} \\
&\qquad\times\left(\int_0^1(1-\alpha)^{\theta+1}\alpha^\vartheta\Big(|\xi(\nu,\cdot)|^q+\alpha\eta\left(|\xi(\mu,\cdot)|^q,|\xi(\nu,\cdot)|^q\right)\Big)d\alpha\right)^{\frac{1}{q}} \\
&\qquad+\left(\int_0^1(1-\alpha)^{\theta}\alpha^{\vartheta+1} d\alpha\right)^{1-\frac{1}{q}} \\
&\qquad\times\left(\int_0^1(1-\alpha)^{\theta}\alpha^{\vartheta+1}\Big(|\xi(\nu,\cdot)|^q+\alpha\eta\left(|\xi(\mu,\cdot)|^q,|\xi(\nu,\cdot)|^q\right)\Big)d\alpha\right)^{\frac{1}{q}} \\
&\quad\le (\nu-\mu)^{\theta+\vartheta+1}\Big(\beta(\theta+2,\vartheta+1)\Big)^{1-\frac{1}{q}} \\
&\qquad\times\left(\beta(\theta+2,\vartheta+1)|\xi(\nu,\cdot)|^q+\beta(\theta+2,\vartheta+2)\eta\left(|\xi(\mu,\cdot)|^q,|\xi(\nu,\cdot)|^q\right)\right)^{\frac{1}{q}} \\
&\qquad+\Big(\beta(\theta+1,\vartheta+2)\Big)^{1-\frac{1}{q}} \\
&\qquad\times\left(\beta(\theta+1,\vartheta+2)|\xi(\nu,\cdot)|^q+\beta(\theta+1,\vartheta+3)\eta\left(|\xi(\mu,\cdot)|^q,|\xi(\nu,\cdot)|^q\right)\right)^{\frac{1}{q}}
\end{aligned}$$

which completes the proof. □

We will derive the following results for preinvex stochastic process.

Theorem 3.8. *Suppose $\xi : I\times\Omega\to\mathbb{R}$ be a mean square continuous and mean square integrable stochastic process. If $|\xi|$ is preinvex on $[\mu,\mu+\eta(\nu,\mu)]$, where $\mu,\mu+\eta(\nu,\mu)\in I$ with $\mu<\mu+\eta(\nu,\mu)$ and take $\theta,\vartheta>0$, then the inequality holds almost everywhere*

$$\begin{aligned}
&\int_\mu^{\mu+\eta(\nu,\mu)}(z-\mu)^\theta(\mu+\eta(\nu,\mu)-z)^\vartheta\xi(z,\cdot)dz \\
&\quad\le\eta(\nu,\mu)^{\theta+\vartheta+1}\Big(\beta(\theta+1,\vartheta+2)\,|\xi(\mu,\cdot)|+\beta(\theta+2,\vartheta+1)\,|\xi(\nu,\cdot)|\Big)
\end{aligned}\tag{3.7}$$

for some fixed $\theta,\vartheta>0$.

Proof . *From Lemma 3.2 and take the preinvex stochastic process of $|\xi|$ yields that*

$$\begin{aligned}
&\int_\mu^{\mu+\eta(\nu,\mu)}(z-\mu)^\theta(\mu+\eta(\nu,\mu)-z)^\vartheta\xi(z,\cdot)dz \\
&\quad\le\eta(\nu,\mu)^{\theta+\vartheta+1}\int_0^1\alpha^\theta(1-\alpha)^\vartheta\,|\xi(\mu+\alpha\eta(\nu,\mu),\cdot)|\,d\alpha \\
&\quad\le\eta(\nu,\mu)^{\theta+\vartheta+1}\int_0^1\alpha^\theta(1-\alpha)^\vartheta\left((1-\alpha)\,|\xi(\mu,\cdot)|+\alpha\,|\xi(n,\cdot))\right|d\alpha\quad(a.e.)
\end{aligned}$$

and by the definition of beta function, we get

$$\leq \eta(\nu,\mu)^{\theta+\vartheta+1}\bigg(\beta(\theta+1,\vartheta+2)\left|\xi(\mu,\cdot)\right|+\beta(\theta+2,\vartheta+1)\left|\xi(\nu,\cdot)\right|\bigg) \quad (a.e.)$$

which completes the proof. □

Remark 3.9. *For $\eta(\nu,\mu)=\nu-\mu$ in Theorem 3.8, then we obtain Theorem 3.1 of [10] .*

Theorem 3.10. *Assume that $\xi : I=[\mu,\mu+\eta(\nu,\mu)]\times\Omega\to\mathbb{R}$ be a mean square continuous and mean square integrable stochastic process. If $|\xi|^q$ is a preinvex on $[\mu,\mu+\eta(\nu,\mu)]$, where $\mu<\mu+\eta(\nu,\mu)$ for $q>1$ with $\frac{1}{p}+\frac{1}{q}=1$ and take $\theta,\vartheta>0$, then the inequality holds almost everywhere*

$$\begin{aligned}&\int_{\mu}^{\mu+\eta(\nu,\mu)}(z-\mu)^{\theta}(\mu+\eta(\nu,\mu)-z)^{\vartheta}\xi(z,\cdot)dz\\&\quad\leq \eta(\nu,\mu)^{\theta+\vartheta+1}\bigg(\beta(p\theta+1,p\vartheta+1)\bigg)^{\frac{1}{p}}\bigg(\frac{1}{2}\bigg(|\xi(\mu,\cdot)|^q+|\xi(\nu,\cdot)|^q\bigg)\bigg)^{\frac{1}{q}}\end{aligned} \tag{3.8}$$

for some fixed $\theta,\vartheta>0$.

Proof . *By using Lemma 3.2 and from Hölder integral inequality, we have (a.e.)*

$$\begin{aligned}&\int_{\mu}^{\mu+\eta(\nu,\mu)}(z-\mu)^{\theta}(\mu+\eta(\nu,\mu)-z)^{\vartheta}\xi(z,\cdot)dz\\&\quad\leq \eta(\nu,\mu)^{\theta+\vartheta+1}\int_0^1\alpha^{\theta}(1-\alpha)^{\vartheta}\left|\xi(\mu+\alpha\eta(\nu,\mu),\cdot)\right|d\alpha\\&\quad\leq \eta(\nu,\mu)^{\theta+\vartheta+1}\left(\int_0^1\alpha^{p\theta}(1-\alpha)^{p\vartheta}d\alpha\right)^{\frac{1}{p}}\left(\int_0^1|\xi(m+\alpha\eta(\nu,\mu),\cdot)|^q d\alpha\right)^{\frac{1}{q}}\end{aligned} \tag{3.9}$$

Since $|\xi|^q$ is a preinvex stochastic process, we have (a.e.)

$$\begin{aligned}\int_0^1|\xi(m+\alpha\eta(\nu,\mu),\cdot)|^q d\alpha&\leq\int_0^1\left((1-\alpha)\left|\xi(\mu,\cdot)\right|^q+\alpha\left|\xi(\nu,\cdot)\right|^q\right)d\alpha\\&\leq\frac{1}{2}\bigg(|\xi(\mu,\cdot)|^q+|\xi(\nu,\cdot)|^q\bigg)\end{aligned} \tag{3.10}$$

Now, by the definition of beta function, we can deduce

$$\int_0^1\alpha^{p\theta}(1-\alpha)^{p\vartheta}d\alpha=\beta(p\theta+1,p\vartheta+1). \tag{3.11}$$

So, by inserting (3.10) *and* (3.11) *in* (3.9) *yields the required inequality* (3.8) *.* □

Remark 3.11. *For $\eta(\nu,\mu)=\nu-\mu$ in Theorem 3.10 , then we obtain Theorem 3.2 of [10] .*

Theorem 3.12. *Assume that $\xi : I = [\mu, \mu + \eta(\nu, \mu)] \times \Omega \to \mathbb{R}$ be a mean square continuous and mean square integrable stochastic process. If $|\xi|^q$ is preinvex on $[\mu, \mu + \eta(\nu, \mu)]$ for $q > 1$, where $\mu < \mu + \eta(\nu, \mu)$ and take $\theta, \vartheta > 0$, then the inequality holds almost everywhere*

$$\begin{aligned}
&\int_{\mu}^{\mu+\eta(\nu,\mu)} (z-\mu)^{\theta}(\mu+\eta(\nu,\mu)-z)^{\vartheta}\xi(z,\cdot)dz \\
&\quad \leq \eta(\nu,\mu)^{\theta+\vartheta+1}\left(\beta(\theta+1,\vartheta+1)\right)^{1-\frac{1}{q}} \\
&\qquad \times \left(\beta(\theta+1,\vartheta+2)|\xi(\mu,\cdot)|^q+\beta(\theta+2,\vartheta+1)|\xi(\nu,\cdot)|^q\right)^{\frac{1}{q}}.
\end{aligned} \tag{3.12}$$

for some fixed $\theta, \vartheta > 0$.

Proof . *By using Lemma 3.2 and power-mean integral inequality for $q \geq 1$ yields that*

$$\begin{aligned}
&\int_{\mu}^{\mu+\eta(\nu,\mu)} (z-\mu)^{\theta}(\mu+\eta(\nu,\mu)-z)^{\vartheta}\xi(z,\cdot)dz \\
&\quad \leq \eta(\nu,\mu)^{\theta+\vartheta+1}\int_0^1 \alpha^{\theta}(1-\alpha)^{\vartheta}\left|\xi(\mu+\alpha\eta(\nu,\mu),\cdot)\right| d\alpha \\
&\quad \leq \eta(\nu,\mu)^{\theta+\vartheta+1}\left(\int_0^1 \alpha^{\theta}(1-\alpha)^{\vartheta}d\alpha\right)^{1-\frac{1}{q}} \\
&\qquad \times \left(\int_0^1 \alpha^{\theta}(1-\alpha)^{\vartheta}|\xi(\mu+\alpha\eta(\nu,\mu),\cdot)|^q d\alpha\right)^{\frac{1}{q}} \quad (a.e.).
\end{aligned}$$

Take the definition of preinvex stochastic process of $|\xi|^q$ and the beta function yields that

$$\begin{aligned}
&\int_{\mu}^{\mu+\eta(\nu,\mu)} (z-\mu)^{\theta}(\mu+\eta(\nu,\mu)-z)^{\vartheta}\xi(z,\cdot)dz \\
&\quad \leq \eta(\nu,\mu)^{\theta+\vartheta+1}\left(\int_0^1 \alpha^{\theta}(1-\alpha)^{\vartheta}d\alpha\right)^{1-\frac{1}{q}} \\
&\qquad \times \left(\int_0^1 \alpha^{\theta}(1-\alpha)^{\vartheta}\left((1-\alpha)\left|\xi(\mu,\cdot)\right|^q+\alpha\left|\xi(\nu,\cdot)\right|^q\right)d\alpha\right)^{\frac{1}{q}} \\
&\quad \leq \eta(\nu,\mu)^{\theta+\vartheta+1}\left(\beta(\theta+1,\vartheta+1)\right)^{1-\frac{1}{q}} \\
&\qquad \times \left(\beta(\theta+1,\vartheta+2)|\xi(\mu,\cdot)|^q+\beta(\theta+2,\vartheta+1)|\xi(\nu,\cdot)|^q\right)^{\frac{1}{q}} \quad (a.e.)
\end{aligned}$$

which completes the proof. □

Remark 3.13. *For $\eta(\nu, \mu) = \nu - \mu$ in Theorem 3.12 , then we obtain Theorem 3.3 of [10] .*

Theorem 3.14. *Let $\xi : I = [\mu, \mu + \eta(\nu, \mu)] \times \Omega \to \mathbb{R}$ be a mean square continuous and mean square integrable stochastic process. If $|\xi|^q$ is a preinvex on $[\mu, \mu + \eta(\nu, \mu)]$, where $\mu < \mu + \eta(\nu, \mu)$ for $q > 1$*

with $\frac{1}{p}+\frac{1}{q}=1$ *and take* $\theta,\vartheta>0$*, then the inequality holds almost everywhere*

$$\begin{aligned}&\int_{\mu}^{\mu+\eta(\nu,\mu)}(z-\mu)^{\theta}(\mu+\eta(\nu,\mu)-z)^{\vartheta}\xi(z,\cdot)dz\\&\leq\eta(\nu,\mu)^{\theta+\vartheta+1}\Bigg[\Big(\beta(p\theta+1,p\vartheta+2)\Big)^{\frac{1}{p}}\\&\quad\times\left(\frac{1}{3}\left|\xi(\mu,\cdot)\right|^{q}+\frac{1}{6}\left|\xi(\nu,\cdot)\right|^{q}\right)^{\frac{1}{q}}\\&\quad+\Big(\beta(p\theta+2,p\vartheta+1)\Big)^{\frac{1}{p}}\left(\frac{1}{6}\left|\xi(\mu,\cdot)\right|^{q}+\frac{1}{3}\left|\xi(\nu,\cdot)\right|^{q}\right)^{\frac{1}{q}}\Bigg],\end{aligned}\tag{3.13}$$

for some fixed $\theta,\vartheta>0$.

Proof . *By using Lemma 3.2 and Hölder-İscan integral inequality, we have* (*a.e.*)

$$\begin{aligned}&\int_{\mu}^{\mu+\eta(\nu,\mu)}(z-\mu)^{\theta}(\mu+\eta(\nu,\mu)-z)^{\vartheta}\xi(z,\cdot)dz\\&\leq\eta(\nu,\mu)^{\theta+\vartheta+1}\int_{0}^{1}\alpha^{\theta}(1-\alpha)^{\vartheta}\left|\xi(\mu+\alpha\eta(\nu,\mu),\cdot)\right|d\alpha\\&\leq\eta(\nu,\mu)^{\theta+\vartheta+1}\Bigg[\left(\int_{0}^{1}(1-\alpha)\alpha^{p\theta}(1-\alpha)^{p\vartheta}d\alpha\right)^{\frac{1}{p}}\\&\quad\times\left(\int_{0}^{1}(1-\alpha)|\xi(m+\alpha\eta(\nu,\mu),\cdot)|^{q}d\alpha\right)^{\frac{1}{q}}\\&\quad+\left(\int_{0}^{1}\alpha\alpha^{p\theta}(1-\alpha)^{p\vartheta}d\alpha\right)^{\frac{1}{p}}\left(\int_{0}^{1}\alpha|\xi(m+\alpha\eta(\nu,\mu),\cdot)|^{q}d\alpha\right)^{\frac{1}{q}}\Bigg].\end{aligned}$$

Take the definition of preinvex stochastic process of $|\xi|^{q}$ *and the beta function yields that*

$$\begin{aligned}&\int_{\mu}^{\mu+\eta(\nu,\mu)}(z-\mu)^{\theta}(\mu+\eta(\nu,\mu)-z)^{\vartheta}\xi(z,\cdot)dz\\&\leq\eta(\nu,\mu)^{\theta+\vartheta+1}\int_{0}^{1}\alpha^{\theta}(1-\alpha)^{\vartheta}\left|\xi(\mu+\alpha\eta(\nu,\mu),\cdot)\right|d\alpha\\&\leq\eta(\nu,\mu)^{\theta+\vartheta+1}\Bigg[\left(\int_{0}^{1}\alpha^{p\theta}(1-\alpha)^{p\vartheta+1}d\alpha\right)^{\frac{1}{p}}\\&\quad\times\left(\int_{0}^{1}(1-\alpha)\Big((1-\alpha)\left|\xi(\mu,\cdot)\right|^{q}+\alpha\left|\xi(\nu,\cdot)\right|^{q}\Big)d\alpha\right)^{\frac{1}{q}}\end{aligned}$$

$$
\begin{aligned}
&+\left(\int_0^1 \alpha^{p\theta+1}(1-\alpha)^{p\vartheta}d\alpha\right)^{\frac{1}{p}}\left(\int_0^1 \alpha\Big((1-\alpha)\,|\xi(\mu,\cdot)|^q+\alpha\,|\xi(\nu,\cdot)|^q\Big)d\alpha\right)^{\frac{1}{q}}\Bigg]\\
&\le \eta(\nu,\mu)^{\theta+\vartheta+1}\Bigg[\Big(\beta(p\theta+1,p\vartheta+2)\Big)^{\frac{1}{p}}\\
&\times\left(\frac{1}{3}\,|\xi(\mu,\cdot)|^q+\frac{1}{6}\,|\xi(\nu,\cdot)|^q\right)^{\frac{1}{q}}\\
&+\Big(\beta(p\theta+2,p\vartheta+1)\Big)^{\frac{1}{p}}\left(\frac{1}{6}\,|\xi(\mu,\cdot)|^q+\frac{1}{3}\,|\xi(\nu,\cdot)|^q\right)^{\frac{1}{q}}\Bigg]\quad (a.e.).
\end{aligned}
$$

which completes the proof. □

Theorem 3.15. *Let $\xi : I = [\mu, \mu + \eta(\nu, \mu)] \times \Omega \to \mathbb{R}$ be a mean square continuous and mean square integrable stochastic process. If $|\xi|^q$ is preinvex on $[\mu, \mu + \eta(\nu, \mu)]$ for $q > 1$, where $\mu < \mu + \eta(\nu, \mu)$ and take $\theta, \vartheta > 0$, then the inequality holds almost everywhere*

$$
\begin{aligned}
&\int_\mu^{\mu+\eta(\nu,\mu)}(z-\mu)^\theta(\mu+\eta(\nu,\mu)-z)^\vartheta\xi(z,\cdot)dz\\
&\le \eta(\nu,\mu)^{\theta+\vartheta+1}\Big(\beta(\theta+1,\vartheta+2)\Big)^{1-\frac{1}{q}}\\
&\times\Big(\beta(\theta+1,\vartheta+3)\,|\xi(\mu,\cdot)|^q+\beta(\theta+2,\vartheta+2)\,|\xi(\nu,\cdot)|^q\Big)^{\frac{1}{q}}\\
&+\Big(\beta(\theta+2,\vartheta+1)\Big)^{1-\frac{1}{q}}\\
&\times\Big(\beta(\theta+2,\vartheta+2)\,|\xi(\mu,\cdot)|^q+\beta(\theta+3,\vartheta+1)\,|\xi(\nu,\cdot)|^q\Big)^{\frac{1}{q}}.
\end{aligned}
\tag{3.14}
$$

for some fixed $\theta, \vartheta > 0$.

Proof . *By using Lemma 3.2 and improved power-mean integral inequality for $q \ge 1$ yields that*

$$
\begin{aligned}
&\int_\mu^{\mu+\eta(\nu,\mu)}(z-\mu)^\theta(\mu+\eta(\nu,\mu)-z)^\vartheta\xi(z,\cdot)dz\\
&\le \eta(\nu,\mu)^{\theta+\vartheta+1}\int_0^1\alpha^\theta(1-\alpha)^\vartheta\,|\xi(\mu+\alpha\eta(\nu,\mu),\cdot)|\,d\alpha\\
&\le \eta(\nu,\mu)^{\theta+\vartheta+1}\left(\int_0^1(1-\alpha)\alpha^\theta(1-\alpha)^\vartheta d\alpha\right)^{1-\frac{1}{q}}\\
&\times\left(\int_0^1(1-\alpha)\alpha^\theta(1-\alpha)^\vartheta|\xi(\mu+\alpha\eta(\nu,\mu),\cdot)|^q d\alpha\right)^{\frac{1}{q}}\\
&+\left(\int_0^1\alpha\alpha^\theta(1-\alpha)^\vartheta d\alpha\right)^{1-\frac{1}{q}}\left(\int_0^1\alpha\alpha^\theta(1-\alpha)^\vartheta|\xi(\mu+\alpha\eta(\nu,\mu),\cdot)|^q d\alpha\right)^{\frac{1}{q}}\quad (a.e.).
\end{aligned}
$$

Take the definition of preinvex stochastic process of $|\xi|^q$ and the beta function yields that

$$\begin{aligned}
&\int_{\mu}^{\mu+\eta(\nu,\mu)} (z-\mu)^{\theta}(\mu+\eta(\nu,\mu)-z)^{\vartheta}\xi(z,\cdot)dz \\
&\leq \eta(\nu,\mu)^{\theta+\vartheta+1}\left(\int_0^1 \alpha^{\theta}(1-\alpha)^{\vartheta+1}d\alpha\right)^{1-\frac{1}{q}} \\
&\quad \times\left(\int_0^1 \alpha^{\theta}(1-\alpha)^{\vartheta+1}\Big((1-\alpha)\left|\xi(\mu,\cdot)\right|^q+\alpha\left|\xi(\nu,\cdot)\right|^q\Big)d\alpha\right)^{\frac{1}{q}} \\
&\quad +\left(\int_0^1 \alpha^{\theta+1}(1-\alpha)^{\vartheta}d\alpha\right)^{1-\frac{1}{q}} \\
&\quad \times\left(\int_0^1 \alpha^{\theta+1}(1-\alpha)^{\vartheta}\Big((1-\alpha)\left|\xi(\mu,\cdot)\right|^q+\alpha\left|\xi(\nu,\cdot)\right|^q\Big)d\alpha\right)^{\frac{1}{q}} \quad (a.e.) \\
&\leq \eta(\nu,\mu)^{\theta+\vartheta+1}\Big(\beta(\theta+1,\vartheta+2)\Big)^{1-\frac{1}{q}} \\
&\quad \times\Big(\beta(\theta+1,\vartheta+3)\left|\xi(\mu,\cdot)\right|^q+\beta(\theta+2,\vartheta+2)\left|\xi(\nu,\cdot)\right|^q\Big)^{\frac{1}{q}} \\
&\quad +\Big(\beta(\theta+2,\vartheta+1)\Big)^{1-\frac{1}{q}} \\
&\quad \times\Big(\beta(\theta+2,\vartheta+2)\left|\xi(\mu,\cdot)\right|^q+\beta(\theta+3,\vartheta+1)\left|\xi(\nu,\cdot)\right|^q\Big)^{\frac{1}{q}} \quad (a.e.)
\end{aligned}$$

which completes the proof. □

4. Conclusion

The study of convex stochastic processes has significant applications in optimization theory and linear programming. The new inequalities in stochastic processes is always appreciable as it provides various interesting bounds to study qualitative properties. The mean square integral inequalities for preinvex and η-convex stochastic processes are derived in this research. Further, results are extended to Hölder improved inequalities. The results are new and interesting for this particular area of research.

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