Int. J. Nonlinear Anal. Appl. **2** (2011) No.1, 11–20
ISSN: 2008-6822 (electronic)
http://www.ijnaa.com

A NEW RESTRUCTURED HARDY-LITTLEWOOD'S INEQUALITY

BICHENG YANG[1*], G. M. RASSIAS[2] AND TH. M. RASSIAS[3]

Abstract. In this paper, we reconstruct the Hardy-Littlewood's inequality by using the method of the weight coefficient and the technic of real analysis including a best constant factor. An open problem is raised.

1. Introduction

In 1908, D. Hilbert published the following Hilbert's inequality (cf. [1]): If $0 < \sum_{n=1}^{\infty} a_n^2 < \infty$ and $0 < \sum_{n=1}^{\infty} b_n^2 < \infty$, then

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{m+n} < \pi(\sum_{n=1}^{\infty} a_n^2 \sum_{n=1}^{\infty} b_n^2)^{\frac{1}{2}}, \tag{1.1}$$

where the constant factor π is the best possible. The integral analogue of (1.1) known as Hilbert's integral inequality is stated as follows:

$$\int_0^{\infty}\int_0^{\infty}\frac{f(x)g(y)}{x+y}dxdy < \pi(\int_0^{\infty} f^2(t)dt \int_0^{\infty} g^2(t)dt)^{\frac{1}{2}}, \tag{1.2}$$

where the constant factor π is still the best possible.

In 1925, G. H. Hardy and M. Riesz [2] gave extensions of (1.1) and (1.2) by introducing one pair of conjugate exponents $(p, q)(p > 1, \frac{1}{p} + \frac{1}{q} = 1)$ as:

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{m+n} < \frac{\pi}{\sin(\frac{\pi}{p})}(\sum_{n=1}^{\infty} a_n^p)^{\frac{1}{p}}(\sum_{n=1}^{\infty} b_n^q)^{\frac{1}{q}}, \tag{1.3}$$

$$\int_0^{\infty}\int_0^{\infty}\frac{f(x)g(y)}{x+y}dxdy < \frac{\pi}{\sin(\frac{\pi}{p})}(\int_0^{\infty} f^p(t)dt)^{\frac{1}{p}}(\int_0^{\infty} g^q(t)dt)^{\frac{1}{q}}, \tag{1.4}$$

where the constant factor $\frac{\pi}{\sin(\pi/p)}$ is the best possible. Inequalities (1.3) and (1.4) are respectively called Hardy-Hilbert's inequality and Hardy-Hilbert's integral inequality. Inequalities (1.1) and (1.2) are important in analysis and its applications (cf. [3], [4]).

Date: Received:April 2010; Revised: May 2010.

2000 *Mathematics Subject Classification.* Primary 26D15.

Key words and phrases. Hardy-Littlewood's inequality, weight coefficient, Hölder's inequality, best constant factor.

*: Corresponding author.

In 1998, by introducing an independent parameter $\lambda > 0$ and applying the way of weight functions, Yang gave an extension of (1.2) as (cf. [5], [6]):

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{(x+y)^\lambda} dxdy < B(\frac{\lambda}{2}, \frac{\lambda}{2})(\int_0^\infty t^{1-\lambda} f^2(t)dt \int_0^\infty t^{1-\lambda} g^2(t)dt)^{\frac{1}{2}}, \tag{1.5}$$

where the constant $B(\frac{\lambda}{2}, \frac{\lambda}{2})$ is the best possible, and $B(u, v)$ is the Beta function.

Since then several mathematicians studied this thesis, such as Jichang Kuang, Mingzhe Gao, W. T. Sulaiman and S. R. Salem et al.. In 2003, Yang and Rassias [7] studied the way of weight coefficient and the method of introducing some independent parameters to obtain a number of new improvements and best extensions of (1.1)-(1.5). In 2004, Yang [8] gave an extension of (1.4) by introducing a parameter $\lambda > 0$ and adding another pair of conjugate exponents $(r, s)(r > 1, \frac{1}{r} + \frac{1}{s} = 1)$ as:

$$\begin{aligned}\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x^\lambda + y^\lambda} dxdy &< \frac{1}{\lambda} B(\frac{1}{r}, \frac{1}{s}) \\ &\times (\int_0^\infty t^{p(1-\frac{\lambda}{r})-1} f^p(t)dt)^{\frac{1}{p}} (\int_0^\infty t^{q(1-\frac{\lambda}{s})-1} g^q(t)dt)^{\frac{1}{q}},\end{aligned} \tag{1.6}$$

where the constant factor $\frac{1}{\lambda}B(\frac{1}{r}, \frac{1}{s})$ is the best possible, and for $\lambda = 1, r = q$, inequality (1.6) reduces to (1.4). For those Hilbert-type inequalities, which possess the general form of kernel or the particular homogeneous kernel of $-\lambda$-degree ($\lambda > 0$), Yang et al. [9], [10], [11], [12] used the Operator theory to study them and published many new interested results.

The equivalent form of (1.3) with the best constant $[\frac{\pi}{\sin(\pi/p)}]^p$ is as follows:

$$\sum_{n=1}^\infty (\sum_{m=1}^\infty \frac{a_m}{m+n})^p < [\frac{\pi}{\sin(\pi/p)}]^p \sum_{n=1}^\infty a_n^p. \tag{1.7}$$

Modifying the kernel of (1.7), Hardy's inequality was given as (cf. [13]):

$$\sum_{n=1}^\infty (\frac{1}{n} \sum_{m=1}^n a_m)^p < (\frac{p}{p-1})^p \sum_{n=1}^\infty a_n^p, \tag{1.8}$$

where the constant factor $(\frac{p}{p-1})^p$ is the best possible. The integral analogue of (1.8) is as follows (cf. [13]):

$$\int_0^\infty (\frac{1}{x} \int_0^x f(t)dt)^p dx < (\frac{p}{p-1})^p \int_0^\infty f^p(x)dx. \tag{1.9}$$

In the period 1927-1928, Hardy [14] provided an extension of (1.9) in the following form (cf. [2], Th. 330): If $p > 1, r \neq 1$, $0 < \int_0^\infty x^{-r}(xf(x))^p dx < \infty$, setting $F(x)$ as: $F(x) = \int_0^x f(t)dt (r > 1); F(x) = \int_x^\infty f(t)dt (r < 1)$, then

$$\int_0^\infty x^{-r} F^p(x)dx < (\frac{p}{|r-1|})^p \int_0^\infty x^{-r}(xf(x))^p dx, \tag{1.10}$$

where the constant $(\frac{p}{|r-1|})^p$ is the best possible. Similarly to the type of (1.10), Hardy and Littlewood [15] proved the following inequality (cf. [2], Th. 346): Assuming that $p > 1, r \neq 1, a_n \geq 0, 0 < \sum_{n=1}^{\infty} n^{-r}(na_n)^p < \infty$, if (a) $r > 1, s_n = \sum_{k=1}^{n} a_k$, or (b) $r < 1, s_n = \sum_{k=n}^{\infty} a_k$, then

$$\sum_{n=1}^{\infty} n^{-r} s_n^p \leq K^p \sum_{n=1}^{\infty} n^{-r}(na_n)^p, \tag{1.11}$$

where the constant factor K satisfies the following inequalities

$$\phi_n = \sum_{k=n}^{\infty} \frac{1}{k^r} \leq K n^{1-r}\ (r > 1); \widetilde{\phi}_n = \sum_{k=1}^{n} \frac{1}{k^r} \leq K n^{1-r}\ (r < 1). \tag{1.12}$$

Hardy et al. [2] did not obtained the expression of K^p and proved that the constant factor is the best possible. But Hardy and Littlewood [16] pointed out some applications of (1.11) in the theory of functions, especially for $r = 2$.

The proof of (a) in (1.11) was described in Hardy et. al. [2] as follows:

For $r > 1, s_n = \sum_{k=1}^{n} a_k (s_0 = 0)$, by Abel's transform and (1.12), one finds

$$\begin{aligned} \sum_{n=1}^{m} n^{-r} s_n^p &= \sum_{n=1}^{m} (\phi_n - \phi_{n+1}) s_n^p \\ = & \sum_{n=1}^{m} \phi_n (s_n^p - s_{n-1}^p) - \phi_{m+1} s_m^p \leq \sum_{n=1}^{m} \phi_n (s_n^p - s_{n-1}^p) \\ \leq & K \sum_{n=1}^{m} n^{1-r} s_n^{p-1} a_n = K \sum_{n=1}^{m} n^{-r}(na_n)(s_n^{p-1}). \end{aligned} \tag{1.13}$$

Hence, by $H\ddot{o}lder's$ inequality with weight, it follows

$$\sum_{n=1}^{m} n^{-r} s_n^p \leq K \{\sum_{n=1}^{m} n^{-r}(na_n)^p\}^{\frac{1}{p}} \{\sum_{n=1}^{m} n^{-r} s_n^p\}^{\frac{1}{q}}. \tag{1.14}$$

For more large enough $m \in \mathbf{N}$, we have $\sum_{n=1}^{m} n^{-r} s_n^p > 0$. Dividing by $\{\sum_{n=1}^{m} n^{-r} s_n^p\}^{\frac{1}{q}}$ in both sides of (14), we obtain

$$\{\sum_{n=1}^{m} n^{-r} s_n^p\}^{\frac{1}{p}} \leq K \{\sum_{n=1}^{m} n^{-r}(na_n)^p\}^{\frac{1}{p}}.$$

It follows that (a) in (1.11) is valid.

Remark 1.1. We find that the following inequality

$$s_n^p - s_{n-1}^p \leq s_n^{p-1} a_n \tag{1.15}$$

is wrong. Hence we can't reach the last inequality of (1.13). In fact, we can find that

$$\begin{aligned} s_n^p - s_{n-1}^p &= s_n^{p-1} s_n - s_{n-1}^p = s_n^{p-1}(s_{n-1} + a_n) - s_{n-1}^p \\ = & (s_n^{p-1} - s_{n-1}^{p-1}) s_{n-1} + s_n^{p-1} a_n \\ = & [(s_{n-1} + a_n)^{p-1} - s_{n-1}^{p-1}] s_{n-1} + s_n^{p-1} a_n. \end{aligned} \tag{1.16}$$

Since $p > 1, s_n = \sum_{k=1}^{n} a_k$, in view of $\sum_{n=1}^{\infty} n^{-r}(na_n)^p > 0$, there exists $n \in \mathbf{N}$, such that $s_{n-1} > 0, a_n > 0$ and

$$[(s_{n-1} + a_n)^{p-1} - s_{n-1}^{p-1}] s_{n-1} > (s_{n-1}^{p-1} - s_{n-1}^{p-1}) s_{n-1} = 0.$$

Hence by (1.16), it follows

$$s_n^p - s_{n-1}^p > s_n^{p-1} a_n, \tag{1.17}$$

which contradicts (1.15). Therefore, inequality (1.15) is not valid by using the this way, and we can not prove (a) in (1.11).

If (b) $r < 1, s_n = \sum_{k=n}^{\infty} a_k$, setting $\widetilde{\phi}_0 = 0$, then following the front-way, we can meet the similar result of (1.17). In fact,

$$\begin{aligned}
\sum_{n=1}^{m} n^{-r} s_n^p &= \sum_{n=1}^{m} (\widetilde{\phi}_n - \widetilde{\phi}_{n-1}) s_n^p = \sum_{n=1}^{m} \widetilde{\phi}_n (s_n^p - s_{n+1}^p) - \widetilde{\phi}_m s_{m+1}^p \\
&= \sum_{n=1}^{m} \widetilde{\phi}_n (s_n^{p-1} s_n - s_{n+1}^p) - \widetilde{\phi}_m s_{m+1}^p \\
&= \sum_{n=1}^{m} \widetilde{\phi}_n [s_n^{p-1} (s_{n+1} + a_n) - s_{n+1}^p] - \widetilde{\phi}_m s_{m+1}^p \\
&= \sum_{n=1}^{m} \widetilde{\phi}_n s_n^{p-1} a_n + [\sum_{n=1}^{m} \widetilde{\phi}_n s_{n+1} (s_n^{p-1} - s_{n+1}^{p-1}) - \widetilde{\phi}_m s_{m+1}^p].
\end{aligned} \tag{1.18}$$

Since $s_n^{p-1} - s_{n+1}^{p-1} \geq 0$, we can't prove the following inequality:

$$\sum_{n=1}^{m} \widetilde{\phi}_n s_{n+1} (s_n^{p-1} - s_{n+1}^{p-1}) - \widetilde{\phi}_m s_{m+1}^p \leq 0,$$

and then the inequality $\sum_{n=1}^{m} n^{-r} s_n^p \leq \sum_{n=1}^{m} \widetilde{\phi}_n s_n^{p-1} a_n$ is not valid by (1.18). So we cannot do more work for (b) in (1.11) following this way.

In this paper, by using (1.10), we reformulate (1.11) to obtain a new inequality with a best constant factor, by using the way of weight coefficient and the technic of real analysis. That is the following theorem:

Theorem 1.2. *Assuming that $r \neq 1, p > 1$, $a_n \geq 0, 0 < \sum_{n=1}^{\infty} n^{-r} (na_n)^p < \infty$, if (a) $r > 1, s_n = \sum_{m=1}^{n} a_m$, or (b) $r < 1, s_n = \sum_{m=n}^{\infty} a_m$, then*

$$\sum_{n=1}^{\infty} \frac{n^{-r} s_n^p}{(1 + \frac{|r-1|}{pn})^{p-1}} < k_r^p \sum_{n=1}^{\infty} (1 + \frac{|r-1|}{pn}) n^{-r} (na_n)^p, \tag{1.19}$$

where the constant factor $k_r^p = (\frac{p}{|r-1|})^p$ is the best possible and $.k_r := \frac{p}{|r-1|}$.

Remark 1.3. Inequality (1.19) is a new restructured Hardy-Littlewood's inequality with a best constant factor. For $r = p$, $q = \frac{p}{p-1}$, we have

$$\sum_{n=1}^{\infty} (\frac{qn}{1+qn})^{p-1} (\frac{1}{n} \sum_{m=1}^{n} a_m)^p < q^p \sum_{n=1}^{\infty} \frac{1+qn}{qn} a_n^p, \tag{1.20}$$

which is weaker than (1.8) but with the same best constant factor as (1.8).

2. A lemma and a preliminary theorem

Lemma 2.1. *If* $\alpha > 0, m, n \in \mathbf{N}$, *then*

$$\sum_{n=m}^{\infty} \frac{1}{n^{1+\alpha}} < \frac{1}{\alpha m^{\alpha}}(1+\frac{\alpha}{m}); \tag{2.1}$$

$$\frac{1}{\alpha} n^{\alpha}(1-\frac{1}{n^{\alpha}}) < \sum_{m=1}^{n} \frac{1}{m^{1-\alpha}} < \frac{1}{\alpha} n^{\alpha}(1+\frac{\alpha}{n}). \tag{2.2}$$

Proof. For $\alpha > 0$, we obtain

$$\begin{aligned}\sum_{n=m}^{\infty} \frac{1}{n^{1+\alpha}} &= \frac{1}{m^{1+\alpha}} + \sum_{n=m+1}^{\infty} \frac{1}{n^{1+\alpha}} \\ &< \frac{1}{m^{1+\alpha}} + \int_{m}^{\infty} \frac{1}{x^{1+\alpha}} dx = \frac{1}{\alpha}(1+\frac{\alpha}{m})\frac{1}{m^{\alpha}}.\end{aligned}$$

Then inequality (2.1) is valid.

For $0 < \alpha \leq 1$, it follows

$$\sum_{m=1}^{n} \frac{1}{m^{1-\alpha}} < \int_{0}^{n} \frac{1}{x^{1-\alpha}} dx = \frac{1}{\alpha} n^{\alpha} < \frac{1}{\alpha} n^{\alpha}(1+\frac{\alpha}{n}),$$

$$\sum_{m=1}^{n} \frac{1}{m^{1-\alpha}} > \int_{1}^{n} \frac{1}{x^{1-\alpha}} dx = \frac{1}{\alpha} n^{\alpha}(1-\frac{1}{n^{\alpha}});$$

for $\alpha > 1$, we obtain

$$\begin{aligned}\sum_{m=1}^{n} \frac{1}{m^{1-\alpha}} &= \frac{1}{n^{1-\alpha}} + \sum_{m=1}^{n-1} m^{\alpha-1} \\ &< \frac{1}{n^{1-\alpha}} + \int_{0}^{n} x^{\alpha-1} dx = \frac{1}{\alpha} n^{\alpha}(1+\frac{\alpha}{n}),\end{aligned}$$

$$\sum_{m=1}^{n} \frac{1}{m^{1-\alpha}} = \sum_{m=1}^{n} m^{\alpha-1} > \int_{1}^{n} x^{\alpha-1} dx = \frac{1}{\alpha} n^{\alpha}(1-\frac{1}{n^{\alpha}}).$$

Hence (2.2) is valid. The lemma is proved. □

Theorem 2.2. *If* $p > 1, \frac{1}{p}+\frac{1}{q} = 1, r > 1, a_n, b_n \geq 0, n \in \mathbf{N}$, *such that*

$$0 < \sum_{n=1}^{\infty} n^{-r}(na_n)^p < \infty, 0 < \sum_{n=1}^{\infty} n^{-r}(n^r b_n)^q < \infty, \tag{2.3}$$

then the following inequality holds:

$$\begin{aligned} I \ :&= \sum_{n=1}^{\infty}\sum_{m=1}^{n} a_m b_n = \sum_{m=1}^{\infty}\sum_{n=m}^{\infty} a_m b_n < \frac{p}{r-1} \\ &\times\{\sum_{n=1}^{\infty}(1+\frac{r-1}{pn})n^{-r}(na_n)^p\}^{\frac{1}{p}}\{\sum_{n=1}^{\infty}(1+\frac{r-1}{pn})n^{-r}(n^r b_n)^q\}^{\frac{1}{q}}, \end{aligned} \tag{2.4}$$

where the constant factor $\frac{p}{r-1}$ *is the best possible.*

Since $1 < 1 + \frac{r-1}{pn} \leq 1 + \frac{r-1}{p}$, it is obvious that inequalities (2.3) are equivalent to the following:

$$0 < \sum_{n=1}^{\infty}(1 + \frac{r-1}{pn})n^{-r}(na_n)^p < \infty, 0 < \sum_{n=1}^{\infty}(1 + \frac{r-1}{pn})n^{-r}(n^r b_n)^q < \infty.$$

By *Hölder's* inequality (cf. [17]), we obtain

$$\begin{aligned} I &= \sum_{n=1}^{\infty}\sum_{m=1}^{n}[\frac{m^{(1-\frac{r-1}{p})/q}}{n^{(1+\frac{r-1}{p})/p}}a_m][\frac{n^{(1+\frac{r-1}{p})/p}}{m^{(1-\frac{r-1}{p})/q}}b_n] \\ &\leq \{\sum_{n=1}^{\infty}\sum_{m=1}^{n}\frac{m^{(1-\frac{r-1}{p})(p-1)}}{n^{1+\frac{r-1}{p}}}a_m^p\}^{\frac{1}{p}}\{\sum_{n=1}^{\infty}\sum_{m=1}^{n}\frac{n^{(1+\frac{r-1}{p})(q-1)}}{m^{1-\frac{r-1}{p}}}b_n^q\}^{\frac{1}{q}} \\ &= \{\sum_{m=1}^{\infty}(\sum_{n=m}^{\infty}\frac{1}{n^{1+\frac{r-1}{p}}})m^{(1-\frac{r-1}{p})(p-1)}a_m^p\}^{\frac{1}{p}}\{\sum_{n=1}^{\infty}(\sum_{m=1}^{n}\frac{1}{m^{1-\frac{r-1}{p}}})n^{(1+\frac{r-1}{p})(q-1)}b_n^q\}^{\frac{1}{q}}. \end{aligned}$$

Then by (2.1) and (2.2), setting $\alpha = \frac{r-1}{p}(> 0)$, we have

$$\begin{aligned} I &< \frac{p}{r-1}\{\sum_{m=1}^{\infty}(1 + \frac{r-1}{pm})\frac{m^{(1-\frac{r-1}{p})(p-1)}}{m^{\frac{r-1}{p}}}a_m^p\}^{\frac{1}{p}} \\ &\quad \times\{\sum_{n=1}^{\infty}(1 + \frac{r-1}{pn})n^{(1+\frac{r-1}{p})(q-1)+\frac{r-1}{p}}b_n^q\}^{\frac{1}{q}} \\ &= \frac{p}{r-1}\{\sum_{m=1}^{\infty}(1 + \frac{r-1}{pm})m^{p-r}a_m^p\}^{\frac{1}{p}}\{\sum_{n=1}^{\infty}(1 + \frac{r-1}{pn})n^{qr-r}b_n^q\}^{\frac{1}{q}}. \end{aligned}$$

Hence inequality (2.4) is valid.

For $N \in \mathbf{N}$, setting $\widetilde{a}_n = n^{\frac{r-1}{p}-1}, \widetilde{b}_n = n^{\frac{r-1}{q}-r}, n \leq N; \widetilde{a}_n = \widetilde{b}_n = 0, n > N$, if there exists a positive number $k \leq \frac{p}{r-1}$, such that (2.4) is still valid as we replace $\frac{p}{r-1}$ by k, then in particular, we have

$$\begin{aligned} \widetilde{I} \quad &: \quad = \sum_{n=1}^{\infty}\sum_{m=1}^{n}\widetilde{a}_m\widetilde{b}_n < k\{\sum_{n=1}^{\infty}(1 + \frac{r-1}{pn})n^{-r}(n\widetilde{a}_n)^p\}^{\frac{1}{p}} \\ &\quad \times\{\sum_{n=1}^{\infty}(1 + \frac{r-1}{pn})n^{-r}(n^r\widetilde{b}_n)^q\}^{\frac{1}{q}} \\ &= k\sum_{n=1}^{N}(1 + \frac{r-1}{pn})\frac{1}{n} = k(\sum_{n=1}^{N}\frac{1}{n} + \frac{r-1}{p}\sum_{n=1}^{N}\frac{1}{n^2}) \\ &= k(\sum_{n=1}^{N}\frac{1}{n})[1 + \frac{r-1}{p}(\sum_{n=1}^{N}\frac{1}{n})^{-1}\sum_{n=1}^{N}\frac{1}{n^2}]; \qquad (2.5) \end{aligned}$$

On the other-hand, by (2.2), we obtain

$$\begin{aligned}\widetilde{I} &= \sum_{n=1}^{N}(\sum_{m=1}^{n} m^{\frac{r-1}{p}-1})n^{\frac{r-1}{q}-r} \geq \frac{p}{r-1}\sum_{n=1}^{N} n^{\frac{r-1}{p}}(1-\frac{1}{n^{\frac{r-1}{p}}})n^{\frac{r-1}{q}-r} \\ &= \frac{p}{r-1}(\sum_{n=1}^{N}\frac{1}{n} - \sum_{n=1}^{N}\frac{1}{n^{\frac{r-1}{p}+1}}) \\ &= \frac{p}{r-1}(\sum_{n=1}^{N}\frac{1}{n})[1-(\sum_{n=1}^{N}\frac{1}{n})^{-1}\sum_{n=1}^{N}\frac{1}{n^{\frac{r-1}{p}+1}}]. \end{aligned} \tag{2.6}$$

Combining with (2.5) and (2.6) and dividing by $\sum_{n=1}^{N}\frac{1}{n}$, we have

$$\frac{p}{r-1}[1-(\sum_{n=1}^{N}\frac{1}{n})^{-1}\sum_{n=1}^{N}\frac{1}{n^{\frac{r-1}{p}+1}}] < k[1+\frac{r-1}{p}(\sum_{n=1}^{N}\frac{1}{n})^{-1}\sum_{n=1}^{N}\frac{1}{n^2}],$$

and then $\frac{p}{r-1} \leq k($ for $N \to \infty)$. Hence $k = \frac{p}{r-1}$ is the best value of (2.4) and the theorem is proved.

3. Main results

Theorem 3.1. *If $p > 1, \frac{1}{p}+\frac{1}{q} = 1, r > 1, a_n, b_n \geq 0$, $0 < \sum_{n=1}^{\infty} n^{-r}(na_n)^p < \infty$ and $0 < \sum_{n=1}^{\infty} n^{-r}(n^r b_n)^q < \infty$, then*

$$\begin{aligned} J &:= \sum_{n=1}^{\infty}\frac{n^{-r}}{(1+\frac{r-1}{pn})^{p-1}}(\sum_{m=1}^{n} a_m)^p \\ &< (\frac{p}{r-1})^p\sum_{n=1}^{\infty}(1+\frac{r-1}{pn})n^{-r}(na_n)^p; \end{aligned} \tag{3.1}$$

$$\begin{aligned} L &:= \sum_{m=1}^{\infty}\frac{m^{r(q-1)-q}}{(1+\frac{r-1}{pm})^{q-1}}(\sum_{n=m}^{\infty} b_n)^q \\ &< (\frac{p}{r-1})^q\sum_{n=1}^{\infty}(1+\frac{r-1}{pn})n^{-r}(n^r b_n)^q, \end{aligned} \tag{3.2}$$

where the constant factors $(\frac{p}{r-1})^p$ and $(\frac{p}{r-1})^q$ are the best possible. Inequalities (3.1), (3.2) and (2.4) are equivalent.

Proof. If $J = 0$, then (3.1) is naturally valid; if $J > 0$, then there exists $n_0 \in \mathbf{N}$, such that for $N \geq n_0, \sum_{n=1}^{N} n^{-r}(na_n)^p > 0$ and $J_N := \sum_{n=1}^{N}\frac{n^{-r}}{(1+\frac{r-1}{pn})^{p-1}}(\sum_{m=1}^{n} a_m)^p > 0$.

We set $b_n(N) := \frac{n^{-r}}{(1+\frac{r-1}{pn})^{p-1}}(\sum_{m=1}^{n} a_m)^{p-1}(n \leq N)$, and use (2.4) to obtain

$$\begin{aligned} 0 &< \sum_{n=1}^{N}(1+\frac{r-1}{pn})n^{-r}(n^r b_n(N))^q = J_N \\ &= \sum_{n=1}^{N}\sum_{m=1}^{n} a_m b_n(N) < \frac{p}{r-1}\{\sum_{n=1}^{N}(1+\frac{r-1}{pn})n^{-r}(na_n)^p\}^{\frac{1}{p}} \\ &\quad \times\{\sum_{n=1}^{N}(1+\frac{r-1}{pn})n^{-r}(n^r b_n(N))^q\}^{\frac{1}{q}}. \end{aligned} \tag{3.3}$$

Dividing $\{\sum_{n=1}^{N}(1+\frac{r-1}{pn})n^{-r}(n^r b_n(N))^q\}^{\frac{1}{q}}$ in both sides of (3.3), it follows

$$\begin{aligned} 0 &< \{\sum_{n=1}^{N}(1+\frac{r-1}{pn})n^{-r}(n^r b_n(N))^q\}^{\frac{1}{p}} = J_N^{\frac{1}{p}} \\ &< \frac{p}{r-1}\{\sum_{n=1}^{N}(1+\frac{r-1}{pn})n^{-r}(na_n)^p\}^{\frac{1}{p}} \\ &< \frac{p}{r-1}\{\sum_{n=1}^{\infty}(1+\frac{r-1}{pn})n^{-r}(na_n)^p\}^{\frac{1}{p}} < \infty. \end{aligned} \tag{3.4}$$

We conform that $0 < \sum_{n=1}^{\infty} n^{-r}(n^r b_n(\infty))^q < \infty$ and for $N \to \infty$, both (3.3) and (3.4) still preserve the strict sign-inequalities. Hence (3.1) follows.

By the same way, if $L = 0$, then (3.2) is naturally valid; if $L > 0$, then there exists n_0, such that for $N \geq n_0$, $\sum_{n=1}^{N} n^{-r}(n^r b_n)^q > 0$ and $L_N := \sum_{m=1}^{N} \frac{m^{r(q-1)-q}}{(1+\frac{r-1}{pm})^{q-1}}(\sum_{n=m}^{N} b_n)^q$ > 0. We set $a_m(N) := \frac{m^{r(q-1)-q}}{(1+\frac{r-1}{pm})^{q-1}}(\sum_{n=m}^{N} b_n)^{q-1}$ and use (2.4) to obtain

$$\begin{aligned} 0 &< \sum_{m=1}^{N}(1+\frac{r-1}{pm})m^{-r}(ma_m(N))^p = L_N = \sum_{m=1}^{N}\sum_{n=m}^{\infty} a_m(N) b_n \\ &< \frac{p}{r-1}\{\sum_{m=1}^{N}(1+\frac{r-1}{pm})m^{-r}(ma_m(N))^p\}^{\frac{1}{p}} \\ &\quad \times\{\sum_{n=1}^{N}(1+\frac{r-1}{pn})n^{-r}(n^r b_n)^q\}^{\frac{1}{q}}; \end{aligned} \tag{3.5}$$

$$\begin{aligned} 0 &< \sum_{m=1}^{N}(1+\frac{r-1}{pm})m^{-r}(ma_m(N))^p \\ &< (\frac{p}{r-1})^q \sum_{n=1}^{\infty}(1+\frac{r-1}{pn})n^{-r}(n^r b_n)^q < \infty. \end{aligned} \tag{3.6}$$

We conform that $0 < \sum_{m=1}^{\infty} m^{-r}(ma_m(\infty))^p < \infty$, and for $N \to \infty$, both (3.5) and (3.6) still preserve the strict sign-inequalities. Hence we have (3.2).

By *Hölder's* inequality (cf. [17]), we have

$$\begin{aligned} I &= \sum_{n=1}^{\infty}[\frac{n^{\frac{-r}{p}}}{(1+\frac{r-1}{pn})^{\frac{1}{q}}}\sum_{m=1}^{n}a_m][(1+\frac{r-1}{pn})^{\frac{1}{q}}n^{\frac{r}{p}}b_n] \\ &\leq J^{\frac{1}{p}}\{\sum_{n=1}^{\infty}(1+\frac{r-1}{pn})n^{-r}(n^r b_n)^q\}^{\frac{1}{q}}; \qquad (3.7) \end{aligned}$$

$$\begin{aligned} I &= \sum_{m=1}^{\infty}[(1+\frac{r-1}{pm})^{\frac{1}{p}}m^{1-\frac{r}{p}}a_m][\frac{m^{\frac{r}{p}-1}}{(1+\frac{r-1}{pm})^{\frac{1}{p}}}\sum_{n=m}^{\infty}b_n] \\ &\leq \{\sum_{n=1}^{\infty}(1+\frac{r-1}{pm})m^{-r}(ma_n)^p\}^{\frac{1}{p}}L^{\frac{1}{q}}. \qquad (3.8) \end{aligned}$$

On the other hand, assuming that (3.1)(or (3.2)) is valid, by (3.7)(or (3.8)), we obtain (2.4). Hence (3.1) , (3.2) and (2.4) are equivalent. We conform that both constants $(\frac{p}{r-1})^p$ in (3.1) and $(\frac{p}{r-1})^q$ in (3.2) are the best possible, otherwise, we can obtain a contradiction by (3.7) or (3.8) that the constant factor in (2.4) is not the best possible. The theorem is proved. □

Proof of Theorem 1. Exchange with m and n, a_m and b_n, p and q in (3.2), and putting $R = r(> 1)$, we have

$$\sum_{n=1}^{\infty}\frac{n^{R(p-1)-p}}{(1+\frac{R-1}{qn})^{p-1}}(\sum_{m=n}^{\infty}a_m)^p < (\frac{q}{R-1})^p\sum_{m=1}^{\infty}(1+\frac{R-1}{qm})m^{-R}(m^R a_m)^p. \qquad (3.9)$$

Setting $r = p - R(p-1)$ in (3.9), we obtain $R(p-1) = p - r, r < 1$ and

$$\sum_{n=1}^{\infty}\frac{n^{-r}}{(1+\frac{1-r}{pn})^{p-1}}(\sum_{m=n}^{\infty}a_m)^p < (\frac{p}{1-r})^p\sum_{m=1}^{\infty}(1+\frac{1-r}{pm})m^{-r}(ma_m)^p. \qquad (3.10)$$

Combining with (3.1) and (3.10), we have (1.19), and the constant factor is obviously the best possible. This proves the theorem.

Open problem. Since $1+\frac{|r-1|}{pn} \leq 1+\frac{|r-1|}{p}$, if we set $K_r = 1+\frac{p}{|r-1|}$, then inequality (1.19) can be deduced to

$$\sum_{n=1}^{\infty}n^{-r}s_n^p < K_r^p\sum_{n=1}^{\infty}n^{-r}(na_n)^p, \qquad (3.11)$$

which is the same as (1.11), but obviously the constant factor K_r^p is not the best possible in (3.11) unless $K_r = k_r = \frac{p}{|r-1|}$. If we replace K_r by $\widetilde{k_r}$, that makes (3.11) still valid, then by simple proof, we find $\frac{p}{|r-1|} \leq \widetilde{k_r} \leq 1 + \frac{p}{|r-1|}$ and in view of (1.8), it follows for $r = p$, $\inf \widetilde{k_p} = k_p$. We conjecture that

$$\inf \widetilde{k_r} = k_r = \frac{p}{|r-1|}. \qquad (3.12)$$

We leave behind it as an open problem.

References

1. H. Weyl, *Singulare integral gleichungen mit besonderer berucksichtigung des fourierschen integral theorems,* Inaugeral-Dissertation, Gottingen.
2. G. H. Hardy, J. E. Littlewood and G. Polya, *Inequalities*, Cambridge Univ. Press, Cambridge, 1952.
3. G. H. Hardy, *Note on a theorem of Hilbert concerning series of positive terms*, Proc. London Math. Soc., 23,2(1925), Records of Proc. xlv–xlvi.
4. D. S. Mintrinovic, J. E. Pecaric and A. M. Fink, *Inequalities involving functions and their integrals and derivertives*, Kluwer Academic Publishers, Boston, 1991.
5. Bicheng Yang, *On Hilbert's integral inequality,* J. Math. Anal. Appl.,220(1998): 778–785.
6. Bicheng Yang, *A note on Hilbert's integral inequality,* Chinese Quarterly Journal of Mathematics, 13, 4(1998): 83–86.
7. Bicheng Yang and Th. M. Rassias, *On the way of weight coefficient and research for the Hilbert-type inequalities*, Math. Ineq. Appl., 6, 4(2003): 625–658.
8. Bicheng Yang, *On an extension of Hilbert's integral inequality with some parameters*, The Australian Journal of Mathematical Analysis and Applications, 1,1(2004): Art.1,1–8.
9. Bicheng Yang, *On the norm of a self-adjoint operator and a new bilinear integral inequality*, Acta Math. Sinica, English Series,23,7 (2007): 1311–1316.
10. Bicheng Yang, *On a Hilbert-type operator with a symmetric homogeneous kernel of -1-order and applications,* Journal of Inequalities and Applications, Volume 2007, Article ID 47812, 9 pages, doi:10.1155/2007/47812.
11. M Krnetic, Mingzhe Gao, J. Pecaric and Xuemei Gao, *On the best constant in Hilbert's inequality,* Math. Ineq. Appl., 8, 2(2005): 317–329.
12. I Brnet, M. Krnic and J. Pecaric, *Multiple Hilbert and Hardy-Hilbert inequalities with non-conjugate parameters,* Bull. Austral. Math. Soc., 71(2005): 447–457.
13. G. H. Hardy, *Note on a theorem of Hilbert*, Math. Zeitschr, 6(1920): 314–317.
14. G. H. Hardy, *Note on some points in the integral calculus(LXIV)*, Messenger of Math., 57(1928): 12–16.
15. G. H. Hardy and J. E. Littlewood, *Elementary theorems concerning power series with positive coefficients and moment constants of positive function,* J. Math., 157(1927): 141–158.
16. G. H. Hardy and J. E. Littlewood, *Some new properties of Fourier constants,* Math. Annalen, 97(1927): 159–209.
17. Jichang Kuang, *Applied inequalities*, Shangdong Science Press, Jinan, 2004.

[1] Department of Mathematics, Guangdong Education Institute, and Guangzhou, Guangdong 510303, P. R. China

E-mail address: bcyang@pub.guangzhou.gd.cn

[2] Zagoras St. Paradissos, Amaroussion 15125 Athens, Greece

E-mail address: trassias@math.ntua.gr

[3] Department of Mathematics, National Technical University of Athens, Zografou, Campus 15780 Athens, Greece.

E-mail address: trassias@math.ntua.gr