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An inverse problem for homogeneous time-fractional diffusion problem on the sphere

Danh Hua Quoc Nam

Division of Applied Mathematics, Thu Dau Mot University, Binh Duong Province, Vietnam

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Abstract

In this paper, we consider an inverse problem for the time-fractional diffusion equation on the sphere where the final data on the sphere are given. The problem is ill-posed in the sense of Hadamard. Hence, the regularization method has to be used for the stable approximate solution. Then the well-posedness of the proposed regularizing problem and convergence property of the regularizing solution to the exact one is proved. Error estimates for this method are provided together with a selection rule for the regularization parameter.

Keywords: Time fractional diffusion; inverse problem, Ill-posed problem; Convergence estimates.
2010 MSC: 35K05, 35K99, 47J06, 47H10x.

1. Introduction

Partial differential equations on spheres have many applications in physical geodesy, potential theory, oceanography, and meteorology. Evolution equations on spherical geometry such as shallow water equations have been studied in weather forecasting services. Some numerical methods such as meshfree methods, radial basic function method, kernel-based methods, in particular, are applied to solving partial differential equations in the simple domain. The parabolic partial differential equation defined on the unit sphere $S^2 \subset R^3$ is given by

$$\frac{\partial}{\partial t}u(x,t) - \Delta^* u(x,t) = F(u(x,t)), \quad (x,t) \in S^2 \times [0,T]. \tag{1.1}$$

Given the initial data $u(x,0) = f(x)$, the direct parabolic problem is to determine the heat distribution $u(x,t)$ at later time, while the inverse problem is to recover the $u(x,t)$ at any earlier

*Corresponding author
Email address: namdhq@tdmu.edu.vn (Danh Hua Quoc Nam)

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time from the measurement of the final value data $u(x,T) = f(x)$. For example, in practice, one may have to investigate the temperature distribution and the heat flux history from the known data at a particular time. In other words, it may be possible to specify the temperature distribution at a particular time, say $t = T > 0$, and from this data the question arises as to whether the temperature distribution at any earlier time $t < T$ can be retrieved. This is usually referred to as the backward heat conduction problem (BHCP), or the final boundary value problem. As we known, the inverse problem is not well-posed in the sense of Hadamard. By the definition of Hadamard then a problem is called well-posed if it satisfies

1. There exists a solution to the problem (existence),
2. There is at most one solution to the problem (uniqueness),
3. The solution depends continuously on the data (stability). The solution's behavior hardly changes when there's a slight change in the initial condition. A problem which is not well-posed is called ill-posed.

The direct problem for parabolic equation on the sphere and numerical approximation of it has been considered by many authors, such as Le Gia Quoc Thong [14, 15]. Recently, the time-fractional diffusion equation is a mathematical model of a wide class of important physical phenomena. Such equations describe anomalous diffusion and subdiffusion processes, relaxation phenomena in complex viscoelastic materials, and so on. In this paper, we are looking for solution u of the following initial inverse problem on the sphere

$$u_t - \frac{\partial}{\partial t}\left(\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\Delta^* u(s)ds\right) = 0, \quad u(x,T) = f(x).$$

Here the convolution is given by for any $\alpha > 0$,

$$\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} w(s)ds$$

defines the Riemann-Liouville fractional integral of w of order α. There are many works which studied in the time-fractional diffusion equations area, see [1, 2, 3, 4, 6, 7, 10, 13, 19, 20]. To the best of our knowledge, there aren't any results on inverse problems for the time-fractional diffusion equations on the sphere. Our main goal in this paper is to establish a quasi-boundary value method for finding an approximate solution. The quasi-boundary value method also called the non-local boundary value method in [9], is a regularization technique by replacing the final condition or boundary condition by a new approximate condition. This method has been used to solve some inverse problems, for example, in [17, 18].

The paper is organized as follows. In section 3, we investigate the ill-posedness of the backward problem for time-fractional diffusion equation on the spheres. In section 4, we present a Quasi-boundary regularization problem and establish the convergence estimates between the regularized solution and the exact solution.

2. Preliminaries

We introduced a two-parameter function of the Mittag-Leffler type, which plays an important role in the time-fractional PDEs equations, and it is defined by

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}, \quad z \in C \tag{2.1}$$

where $\alpha > 0$ and $\beta \in R$ are arbitrary constant. General properties of the Mittag-Leffler function are discussed in [9].

Lemma 2.1. *Let $0 < \alpha_0 < \alpha_1 < 1$. Then there exits constants $C_1^-, C_1^+, C_2^-, C_2^+ > 0$ depending only on γ_0, γ_1 such that*

$$\frac{C_1^-}{\alpha} e^{x^{\frac{1}{\gamma}}} \leq E_{\alpha,1}(x) \leq \frac{C_1^+}{\alpha} e^{x^{\frac{1}{\alpha}}}, \quad \forall x \geq 0$$

$$\frac{C_2^-}{\Gamma(1-\alpha)} \frac{1}{1-x} \leq E_{\alpha,1}(x) \leq \frac{C_2^+}{\Gamma(1-\alpha)} \frac{1}{1-x}, \quad \forall x \leq 0.$$

These estimates are uniform for all $\alpha \in [\alpha_0, \alpha_1]$.

Lemma 2.2. *Assume that $\alpha \in (0,1)$. Then the Mittag - Leffler functions satisfy that*

$$E_{\alpha,1}(x) = \frac{1}{\alpha} e^{x^{1/\alpha}} - \frac{1}{x\Gamma(1-\alpha)} + O(\frac{1}{x^2}), 0 < x \to +\infty$$

$$E_{\alpha,1}(x) = -\frac{1}{x\Gamma(1-\alpha)} + O(\frac{1}{x^2}), -\infty \leftarrow x < 0$$

$$E_{\alpha,0}(x) = \frac{1}{\alpha} x^{1/\alpha} e^{x^{1/\alpha}} - \frac{1}{x\Gamma(-\alpha)} + O(\frac{1}{x^2}), 0 < x \to +\infty$$

$$E_{\alpha,0}(x) = -\frac{1}{x\Gamma(-\alpha)} + O(\frac{1}{x^2}), -\infty \leftarrow x < 0.$$

Spherical harmonics are polynomials which satisfy $\Delta_x Y(x) = 0$ (where Δ_x is the Laplacian operator in R^{n+1}) and are restricted to the surface of the Euclidean sphere S^n.

The eigenvalues for $-\Delta$ are

$$\lambda_l = l(l+n-1), \quad l = 0, 1, 2, \cdots,$$

and the respective eigenfunctions are the spherical harmonics $Y_l(x)$ of order l i.e.,

$$\Delta Y_l(x) = -\lambda_l Y_l(x).$$

The space of all spherical harmonics of degree l on S^n, denoted by V_l, has an orthonormal basis

$$\{Y_{lk}(x) : k = 1, 2, 3, \cdots, N(n,l)\},$$

where

$$N(n,0) = 1, \quad N(n,l) = \frac{(2l+n-1)\Gamma(l+n-1)}{\Gamma(l+1)\Gamma(n)}, \quad l \geq 1.$$

Every function $f \in L^2(S^n)$ can be expanded in terms of spherical harmonics

$$f = \sum_{l=0}^{\infty} \sum_{k=1}^{N(n,l)} \widehat{f}_{lk} Y_{lk}, \quad \widehat{f}_{lk} = \int_{S^n} f \overline{Y}_{lk} dS,$$

where dS is the surface measure of the unit sphere. The Sobolev space $H^\sigma(S^n)$ with real parameter σ consists of all distributions f such that

$$\|f\|^2_{H^\sigma(S^n)} = \sum_{l=0}^{\infty} \sum_{k=1}^{N(n,l)} (1+\lambda_l)^\sigma |\widehat{f}_{lk}|^2 < \infty.$$

Obviously, the norm stems from an inner product

$$< f, g >_{H^\sigma} = \sum_{l=0}^{\infty} \sum_{k=1}^{N(n,l)} (1+\lambda_l)^\sigma \widehat{f}_{lk} \widehat{g}_{lk}.$$

3. The initial value problem fractional diffusion

Consider the following problem

$$\begin{cases} u_t - \dfrac{\partial}{\partial t}\left(\displaystyle\int_0^t \dfrac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\Delta^* u(s)ds\right) = 0, \quad x \in S^n, 0 < t < T \\ u(x,0) = h(x), \quad x \in S^n. \end{cases} \tag{3.1}$$

Here Δ^* is the Laplace-Beltrami on the sphere S^n. The direct problem is to find the final value data $u(x,T) = g(x)$ from the known initial value data $u(x,0) = h(x)$.

Every $u \in L^2(S^n)$ can be expanded in terms of spherical harmonics

$$u(x,t) = \sum_{l=0}^{\infty} \sum_{k=1}^{N(n,l)} \widehat{u}_{lk}(t) Y_{lk}(x), \quad \widehat{u}_{lk}(t) = \int_{S^n} u(x,t)\overline{Y}_{lk}(x)dS,$$

where dS is the surface measure of the unit sphere. We like put $\widehat{h}_{lk} = \int_{S^n} h(x)\overline{Y}_{lk}(x)dS$. By taking the inner product of Y_{lk} with

$$\frac{d\widehat{u}_{lk}}{dt} + \lambda_l \frac{\partial}{\partial t}\left(\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\widehat{u}_{lk}(s)ds\right) = 0, \quad \widehat{u}_{lk}(0) = \widehat{h}_{lk}.$$

From the results of W. Clean [8], we derive that

$$u(x,t) = \sum_{l=0}^{\infty} \sum_{k=1}^{N(n,l)} E_{\alpha,1}(-\lambda_l t^\alpha)\widehat{h}_{lk} Y_{lk}(x).$$

By a similar method as in the work of M. Yamamoto et al [11], we can show that the mild solution of initial problem (3.1) that $u \in L^\infty(0,T;L^2(S^n))$ if $h \in L^2(S^n)$.

Theorem 3.1. *Given $h \in H^\sigma(S^n)$. Then g satisfies that the following operator*

$$Lg = h, \tag{3.2}$$

where

$$Lv = \sum_{l=0}^{\infty} \sum_{k=1}^{N(n,l)} \widehat{L}(l)\widehat{v}_{lk} Y_{lk}(x), \tag{3.3}$$

and

$$\widehat{L}(l) = \frac{1}{E_{\alpha,1}(-\lambda_l T^\alpha)}. \tag{3.4}$$

Lemma 3.2. *There exists two positive constants C_1, C_2 such that*

$$C_1(l+1)^{2\beta} \le \widehat{L}(l) \le C_2(l+1)^{2\beta}.$$

Proof .

Since

$$\frac{1}{2}(l+1)^2 \leq \lambda_l = l(l+n-1) \leq \frac{n-1}{2}(l+1)^2,$$

and from (4.4), we get

$$E_{\alpha,1}(-\lambda_l T^\alpha) \leq \frac{C_2^+}{\Gamma(1-\alpha)} \frac{1}{1+\lambda_l T^\alpha} \leq \frac{C_2^+}{\Gamma(1-\alpha)} \frac{1}{\lambda_l T^\alpha} \leq \frac{2C_2^+}{\Gamma(1-\alpha) T^\alpha}(l+1)^{-2},$$

and the following inequality holds

$$\begin{aligned} E_{\alpha,1}(-\lambda_l T^\alpha) &\geq \frac{C_2^-}{\Gamma(1-\alpha)} \frac{1}{1+\lambda_l T^\alpha} \\ &\geq \frac{C_2^-}{\Gamma(1-\alpha)(1+T^\alpha)} \frac{1}{\lambda_l} \geq \frac{2C_2^-}{\Gamma(1-\alpha)(1+T^\alpha)(n-1)}(l+1)^{-2}. \end{aligned} \tag{3.5}$$

Then $\widehat{L}(l)$ defined in (3.4) satisfying

$$\frac{\Gamma(1-\alpha)T^\alpha}{2C_2^+}(l+1)^2 \leq \widehat{L}(l) \leq \frac{\Gamma(1-\alpha)(1+T^\alpha)(n-1)}{2C_2^-}(l+1)^2.$$

□

4. The inverse initial time-fractional diffusion equation

Consider the following inverse problem

$$\begin{cases} u_t - \dfrac{\partial}{\partial t}\left(\displaystyle\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \Delta^* u(s) ds\right) = 0, \quad x \in S^n, 0 < t < T \\ u(x,T) = g(x), \quad x \in S^n. \end{cases} \tag{4.1}$$

Here Δ^* is the Laplace-Beltrami on the sphere S^n. The inverse problem is to reconstruction the initial value data $u(x,0) = f(x)$ from the known final data value $u(x,T) = g(x)$. Now, we find an explicit formula of the mild solution of Problem (4.1). Let us assume that $u \in L^2(S^n)$ can be expanded in terms of spherical harmonics

$$u(x,t) = \sum_{l=0}^{\infty} \sum_{k=1}^{N(n,l)} \widehat{u}_{lk}(t) Y_{lk}(x), \quad \widehat{u}_{lk}(t) = \int_{S^n} u(x,t) \overline{Y}_{lk}(x) dS,$$

where dS is the surface measure of the unit sphere. We like put $\widehat{f}_{lk} = \int_{S^n} f(x) \overline{Y}_{lk}(x) dS$. By taking the inner product of Y_{lk} with

$$\frac{d\widehat{u}_{lk}}{dt} + \lambda_l \frac{\partial}{\partial t}\left(\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \widehat{u}_{lk}(s) ds\right) = 0, \quad \widehat{u}_{lk}(T) = \widehat{g}_{lk}.$$

Let us assume that $u(x,0) = u_0(x)$ where $u_0 \in L^2(S^n)$. Then, from section 3, we get that

$$u(x,t) = \sum_{l=0}^{\infty} \sum_{k=1}^{N(n,l)} E_{\alpha,1}(-\lambda_l t^\alpha) \widehat{u}_{0lk} Y_{lk}(x).$$

Letting $t = T$ in above expression which yields that

$$g(x) = u(x,T) = \sum_{l=0}^{\infty} \sum_{k=1}^{N(n,l)} E_{\alpha,1}(-\lambda_l T^{\alpha}) \widehat{u}_{0lk} Y_{lk}(x) = \sum_{l=0}^{\infty} \sum_{k=1}^{N(n,l)} \widehat{g}_{lk} Y_{lk}(x). \tag{4.2}$$

Due to the uniqueness of Fourier expansion of the function on $L^2(S^n)$, we deduce that $E_{\alpha,1}(-\lambda_l T^{\alpha})\widehat{u}_{0lk} = \widehat{g}_{lk}$. From some above observations, we find that

$$u(x,t) = \sum_{l=0}^{\infty} \sum_{k=1}^{N(n,l)} \frac{E_{\alpha,1}(-\lambda_l t^{\alpha})}{E_{\alpha,1}(-\lambda_l T^{\alpha})} \widehat{g}_{lk} Y_{lk}(x). \tag{4.3}$$

4.1. Ill-posedness and a conditional stability for the inverse problem

Theorem 4.1. *The problem* (4.1) *has a unique solution* $u \in C([0,T; L^2(S^n))) \cap C([0,T; H^2(S^n)))$ *if* $g \in H^2(S^n)$

In the following Theorems, we prove that the backward problem is stable for $t \in (0,T)$. But the state at $t = 0$ is an exception.

Theorem 4.2. *Let any* $g \in L^2(S^n)$. *Then Problem* (4.1) *has a mild solution* u *which depends continously on the final data* g *for* $t > 0$.

Proof . We have for $t > 0$

$$\frac{C_2^-}{\Gamma(1-\alpha)} \frac{1}{1+\lambda_l t^{\alpha}} \le E_{\alpha,1}(-\lambda_l t^{\alpha}) \le \frac{C_2^+}{\Gamma(1-\alpha)} \frac{1}{1+\lambda_l t^{\alpha}},$$

and

$$\frac{C_2^-}{\Gamma(1-\alpha)} \frac{1}{1+\lambda_l T^{\alpha}} \le E_{\alpha,1}(-\lambda_l T^{\alpha}) \le \frac{C_2^+}{\Gamma(1-\alpha)} \frac{1}{1+\lambda_l T^{\alpha}}. \tag{4.4}$$

From two preceding estimates, we obtain

$$\frac{E_{\alpha,1}(-\lambda_l t^{\alpha})}{E_{\alpha,1}(-\lambda_l T^{\alpha})} \le \frac{C_2^+}{C_2^-} \frac{1+\lambda_l T^{\alpha}}{1+\lambda_l t^{\alpha}} \le \frac{T^{\alpha} C_2^+}{t^{\alpha} C_2^-}.$$

Then we find that

$$\begin{aligned} \|u(x,t)\|^2_{L^2(S^n)} &= \sum_{l=0}^{\infty} \sum_{k=1}^{N(n,l)} \left| \frac{E_{\alpha,1}(-\lambda_l t^{\alpha})}{E_{\alpha,1}(-\lambda_l T^{\alpha})} \widehat{g}_{lk} \right|^2 \\ &\le \left(\frac{T^{\alpha} C_2^+}{t^{\alpha} C_2^-} \right)^2 \sum_{l=0}^{\infty} \sum_{k=1}^{N(n,l)} \left| \widehat{g}_{lk} \right|^2 = \left(\frac{T^{\alpha} C_2^+}{t^{\alpha} C_2^-} \right)^2 \|f\|^2_{L^2(S^n)}. \end{aligned}$$

□

Remark 4.3. *If $t=0$ then $u(x,0)$ does not depend on continuity on the given data g on L^2 norm. Since (4.3), we have*

$$f(x)=u(x,0)=\sum_{l=0}^{\infty}\sum_{k=1}^{N(n,l)}\frac{1}{E_{\alpha,1}(-\lambda_l T^\alpha)}\widehat{g}_{lk}Y_{lk}(x). \tag{4.5}$$

Then $g(x)=\sum_{l=0}^{\infty}\sum_{k=1}^{N(n,l)}E_{\alpha,1}(-\lambda_l T^\alpha)\widehat{f}_{lk}Y_{lk}(x)$. To find $f(x)$, we just need to solve the following integral equation

$$(Kf)(x)=\int_{S^n}k(x,\xi)f(\xi)dS=g(x),\quad x\in S^n \tag{4.6}$$

where the kernel is

$$k(x,\xi)=\sum_{l=0}^{\infty}\sum_{k=1}^{N(n,l)}E_{\alpha,1}(-\lambda_l T^\alpha)Y_{lk}(x)Y_{lk}(\xi).$$

From $k(x,\xi)=k(\xi,x)$, we know K is self-adjoint. If $f\in H^\sigma,\ \ 0\le\sigma<2$ then $g\in H^2$. Because $H^\sigma, 0\le\sigma<2$ is compactly embedded into H^2, so $K:H^\sigma\to H^\sigma$ is compact and the problem (4.6) is ill-posed.

Theorem 4.4. *Let $f(x)=u(x,0)\in H^p(S^n)$ satisfy an a priori bound condition*

$$\|f\|_{H^p(S^n)}\le E,\quad p>0 \tag{4.7}$$

then we have the following estimate

$$\|f\|_{L^2(S^n)}\le\left[\frac{\Gamma(1-\alpha)(1+T^\alpha)}{C_2^-}\right]^{\frac{p}{p+1}}\|f\|_{H^p(S^n)}^{\frac{1}{p+1}}\|g\|_{L^2(S^n)}^{\frac{p}{4p+4}}. \tag{4.8}$$

Proof . Using the Hölder inequality, we get

$$\begin{aligned}\|f\|^2_{L^2(S^n)}&=\sum_{l=0}^{\infty}\sum_{k=1}^{N(n,l)}|\widehat{f}_{lk}|^2=\sum_{l=0}^{\infty}\sum_{k=1}^{N(n,l)}\frac{|\widehat{g}_{lk}|^2}{|E_{\alpha,1}(-\lambda_l T^\alpha)|^2}\\&=\sum_{l=0}^{\infty}\sum_{k=1}^{N(n,l)}\frac{|\widehat{g}_{lk}|^{\frac{4}{2p+2}}}{|E_{\alpha,1}(-\lambda_l T^\alpha)|^2}|\widehat{g}_{lk}|^{\frac{2p}{2p+2}}\\&\le\left(\sum_{l=0}^{\infty}\sum_{k=1}^{N(n,l)}\frac{|\widehat{g}_{lk}|^2}{|E_{\alpha,1}(-\lambda_l T^\alpha)|^{2p+2}}\right)^{\frac{1}{p+1}}\left(\sum_{l=0}^{\infty}\sum_{k=1}^{N(n,l)}|\widehat{g}_{lk}|^2\right)^{\frac{p}{2p+2}}.\end{aligned} \tag{4.9}$$

Applying (4.4), we have

$$\begin{aligned}\sum_{l=0}^{\infty}\sum_{k=1}^{N(n,l)}\frac{|\widehat{g}_{lk}|^2}{|E_{\alpha,1}(-\lambda_l T^\alpha)|^{2p+2}}&\le\sum_{l=0}^{\infty}\sum_{k=1}^{N(n,l)}\frac{|\widehat{g}_{lk}|^2}{|E_{\alpha,1}(-\lambda_l T^\alpha)|^2}\left[\frac{\Gamma(1-\alpha)(1+\lambda_l T^\alpha)}{C_2^-}\right]^{2p}\\&\le\left[\frac{\Gamma(1-\alpha)(1+T^\alpha)}{C_2^-}\right]^{2p}\sum_{l=0}^{\infty}\sum_{k=1}^{N(n,l)}\lambda_l^{2p}|\widehat{f}_{lk}|^2\\&=\left[\frac{\Gamma(1-\alpha)(1+T^\alpha)}{C_2^-}\right]^{2p}\|f\|^2_{H^p(S^n)}.\end{aligned} \tag{4.10}$$

Combining (4.9) and (4.10), we obtain

$$\|f\|^2_{L^2(S^n)} \le \left[\frac{\Gamma(1-\alpha)(1+T^\alpha)}{C_2^-}\right]^{\frac{2p}{p+1}} \|f\|^{\frac{2}{p+1}}_{H^p(S^n)} \|g\|^{\frac{p}{2p+2}}_{L^2(S^n)}. \tag{4.11}$$

□

4.2. Regularization by quasi-boundary value method

In this subsection, we propose a quasi-boundary value method to solve Problem (11) and give two convergence estimates under a priori regularization parameter choice rule and an a posteriori regularization parameter choice rule, respectively.

Let $u^\epsilon_\beta(x,t)$ be the solution of the following regularized problem

$$\begin{cases} v_t - \dfrac{\partial}{\partial t}\left(\displaystyle\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\Delta^* v(s)ds\right) = 0, \quad x \in S^n, 0 < t < T \\ v(x,T) + \beta v(x,0) = g_\epsilon(x), \quad x \in S^n. \end{cases} \tag{4.12}$$

Theorem 4.5. *Let $g_\epsilon \in L^2(S^n)$ such that $\|g_\epsilon - g\|_{L^2(S^n)} \le \epsilon$. Let assume that*

$$\|f\|_{H^p(S^n)} \le E.$$

Then Problem (4.12) has a mild solution $u^\epsilon_\beta \in L^\infty(0,T;L^2(S^n))$. Moreover, we get the following estimate

$$\|u^\epsilon_\beta(.,0) - u(.,0)\|_{L^2(S^n)} \le \epsilon^{1-\nu} + \max(C_1(p), C_2(p))\left(\epsilon^{\frac{\nu p}{4}}, \epsilon\right), \tag{4.13}$$

where $\beta = \epsilon^\nu$, $0 < \nu < 1$.

Proof . By the separation of variables, we know $u^\epsilon_\beta(x,t)$ has the following form

$$u^\epsilon_\beta(x,t) = \sum_{l=0}^{\infty}\sum_{k=1}^{N(n,l)} E_{\alpha,1}(-\lambda_l t^\alpha)\widehat{c}_{lk} Y_{lk}(x). \tag{4.14}$$

From (4.12), we get the following equality

$$\widehat{c^\epsilon}_{lk} E_{\alpha,1}(-\lambda_l T^\alpha) + \beta\widehat{c^\epsilon}_{lk} = (\widehat{g}_\epsilon)_{lk}. \tag{4.15}$$

Thus, we obtain that $\widehat{c^\epsilon}_{lk} = \frac{(\widehat{g}_\epsilon)_{lk}}{\beta + E_{\alpha,1}(-\lambda_l T^\alpha)}$. Substituting $\widehat{c^\epsilon}_{lk}$ into (4.14), we get the following equality

$$u^\epsilon_\beta(x,t) = \sum_{l=0}^{\infty}\sum_{k=1}^{N(n,l)} \frac{E_{\alpha,1}(-\lambda_l t^\alpha)}{\beta + E_{\alpha,1}(-\lambda_l T^\alpha)}(\widehat{g}_\epsilon)_{lk} Y_{lk}(x). \tag{4.16}$$

The function $f^\epsilon_\beta(x) = u^\epsilon_\beta(x,0)$ be as approximation of $f(x)$ defined by

$$f^\epsilon_\beta(x) = u^\epsilon_\beta(x,0) = \sum_{l=0}^{\infty}\sum_{k=1}^{N(n,l)} \frac{1}{\beta + E_{\alpha,1}(-\lambda_l T^\alpha)}(\widehat{g}_\epsilon)_{lk} Y_{lk}(x). \tag{4.17}$$

Denoted by

$$f_\beta(x) = \sum_{l=0}^{\infty}\sum_{k=1}^{N(n,l)} \frac{1}{\beta + E_{\alpha,1}(-\lambda_l T^\alpha)}\widehat{g}_{lk} Y_{lk}(x). \tag{4.18}$$

First, we look at the estimate $\|f_\beta^\epsilon - f_\beta\|_{L^2}$. Using Parseval's inequality, we get that

$$\begin{aligned}\|f_\beta^\epsilon - f_\beta\|^2_{L^2(S^n)} &= \sum_{l=0}^{\infty}\sum_{k=1}^{N(n,l)} \left(\frac{1}{\beta + E_{\alpha,1}(-\lambda_l T^\alpha)}\right)^2 |(\widehat{g}_\epsilon)_{lk} - \widehat{g}_{lk}|^2 \\ &\le \frac{1}{\beta^2}\sum_{l=0}^{\infty}\sum_{k=1}^{N(n,l)} |(\widehat{g}_\epsilon)_{lk} - \widehat{g}_{lk}|^2 = \frac{\epsilon^2}{\beta^2}. \end{aligned} \tag{4.19}$$

In the following, we give two convergence estimates for $\|f_\beta^\epsilon(.) - f(.)\|_{L^2(S^n)}$. Indeed, using Parseval's equality and using some simple calculation, we find that

$$\begin{aligned}&\|f_\beta(x) - f(x)\|^2_{L^2(S^n)} \\ = &\sum_{l=0}^{\infty}\sum_{k=1}^{N(n,l)} \left(\frac{1}{E_{\alpha,1}(-\lambda_l T^\alpha)} - \frac{1}{\beta + E_{\alpha,1}(-\lambda_l T^\alpha)}\right)^2 |\widehat{g}_{lk}|^2 \\ = &\sum_{l=0}^{\infty}\sum_{k=1}^{N(n,l)} \frac{\beta^2}{|E_{\alpha,1}(-\lambda_l T^\alpha)|^2\Big(\beta + E_{\alpha,1}(-\lambda_l T^\alpha)\Big)^2}|\widehat{g}_{lk}|^2 \\ = &\sum_{l=0}^{\infty}\sum_{k=1}^{N(n,l)} \frac{|\widehat{g}_{lk}|^2}{|E_{\alpha,1}(-\lambda_l T^\alpha)|^2}\lambda_l^p \frac{\beta^2\lambda_l^2}{\Big(\beta\lambda_l + \lambda_l E_{\alpha,1}(-\lambda_l T^\alpha)\Big)^2}\frac{1}{\lambda_l^p} \\ \le &\sum_{l=0}^{\infty}\sum_{k=1}^{N(n,l)} \frac{|\widehat{g}_{lk}|^2}{|E_{\alpha,1}(-\lambda_l T^\alpha)|^2}\lambda_l^p \frac{\beta^2\lambda_l^2}{\Big(\beta\lambda_l + E_{\alpha,1}(-\lambda_l T^\alpha)\Big)^2}\frac{1}{\lambda_l^p} \\ \le &E^2 \sup_{l\in N}\Big|A(l)\Big|^2, \end{aligned} \tag{4.20}$$

where we set

$$A(l) = \frac{\beta\lambda_l^{1-\frac{p}{2}}}{\beta\lambda_l + E_{\alpha,1}(-\lambda_l T^\alpha)} \le \frac{\beta\lambda_l^{1-\frac{p}{2}}}{\beta\lambda_l + \frac{M}{\lambda_l}} = \frac{\beta\lambda_l^{2-\frac{p}{2}}}{\beta\lambda_l^2 + M}. \tag{4.21}$$

It is easy to see that the following estimate

$$A(l) \le C_1(p)\beta^{\frac{p}{4}}, \quad \text{for all} \quad 0 \le p < 4 \tag{4.22}$$

$$A(l) \le C_2(p)\beta, \quad \text{for all} \quad p \ge 4. \tag{4.23}$$

Indeed, if $p > 4$ then $\lambda_l^{2-\frac{p}{2}} \le \lambda_1^{2-\frac{p}{2}}$ and this implies that we choose $C_2(p) = \frac{\lambda_1^{2-\frac{p}{2}}}{\beta\lambda_1^2 + M}$ in order to deduce that $A(l) \le C_2(p)\beta$. With the case $0 < p < 4$, we let $h(y) = \frac{\beta y^{2-\frac{p}{2}}}{\beta y^2 + M}$. The derivative of it is

$$h'(y) = \frac{\beta y^{1-\frac{p}{2}}(4M - pM - p\beta y^2)}{2(M + \beta y^2)^2}.$$

The positive solution of the equation $h'(y) = 0$ is y_0 such that $4M - pM - p\beta y_0^2 = 0$, or $y_0 = \sqrt{\frac{M(4-p)}{p\beta}}$. Then if $0 < p < 4$ and for $y > 0$,

$$h(y) \le h(y_0) = \frac{\beta y_0^{2-\frac{p}{2}}}{\beta y_0^2 + M} = \frac{\left[\frac{M(4-p)}{p}\right]^{\frac{4-p}{p}} \beta^{\frac{p}{4}}}{M + \frac{M(4-p)}{p}} = C_1(p, M)\beta^{\frac{p}{4}}. \tag{4.24}$$

Combining (4.19) and (4.24), we get that

$$\begin{aligned} \|f_\beta^\epsilon - f\|_{L^2(S^n)} &\le \|f_\beta - f\|_{L^2(S^n)} + \|f_\beta^\epsilon - f_\beta\|_{L^2(S^n)} \\ &\le \frac{\epsilon}{\beta} + \max(C_1(p), C_2(p))\Big(\beta^{\frac{p}{4}}, \beta\Big). \end{aligned} \tag{4.25}$$

□

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