Abstract

It is known that there are disk-cyclic operators that are not satisfy the disk-cyclic criterion. This work introduce weakly disk-cyclic operators, which are kind of disk-cyclic operators, and study its relation with disk-cyclic criterion with several results and properties.

Keywords: disk-cyclic, codisk-cyclic, mixing operators, weakly mixing operators.

1. Introduction

Let H be a separable infinite dimensional Hilbert space, and $T \in B(H)$ is said to be a hypercyclic operator if the orbit of $T$, $\text{orbt}(T, x) = \{T^n x : n \geq 0\}$, is dense in $H$. The first sufficient condition for hypercyclic (the Hypercyclic Criterion) discovered independently by Kitai and Godefroy and Shapiro. In 1974 Hilden and Wallen, generalized the definition of hypercyclic operators to supercyclic by cone orbit, $\text{Corbt}(T, x) = \{\alpha T^n x : \alpha \in C, n \geq 0\}$, is dense in $H$. Jamil in her Ph. D. thesis, partition the cone orbit into three parts, according to unit circle as: disk-cyclic operators when $|\alpha| \leq 1$, circle cyclic operators when $|\alpha| = 1$ and codisk-cyclic operators when $|\alpha| \geq 1$. But Saavedra and Müller proved that every circle cyclic operators are hypercyclic.

There are many authors studied disk-cyclic operators from multiple aspects like: Bamerni defined subspace disk-cyclic and multidisk-cyclic operators on Banach Spaces, and Yu-Xia Iiang and Ze-Hua Zhou introduced Disk-cyclic and Codisk-cyclic tuples of the adjoint weighted composition operators on Hilbert spaces.

A sufficient conditions for disk-cyclic operators were found by Jamil in 2002, which is called disk-cyclic criterion. In 2016 Bamerni, provided another version to the disk-cyclic criterion, which is simpler than the main disk-cyclic criterion. Moreover, proved that $T$ satisfies disk-cyclic criterion
if and only if $\bigoplus_{i=1}^{k} T$ is disk-cyclic for all $k \geq 2$). Now the national quotation.

Is $\bigoplus_{i=1}^{n} T \in DC(H)$ for all $n \geq 2$, disk-cyclic operator whenever $T$ is?

This quotation motivates as to introduce a new concept of disk-cyclic phenomena which is called weakly $D$-mixing operator $T \in B(H)$ as:

$$T \oplus T : H \oplus H \to H \oplus H \text{ is disk-cyclic.}$$

Weakly $D$-mixing operators deals with $D$- return sets and proper $D$- return sets which defined as respectively:

$$N^D(U, V) = N^D_{\oplus}(U, V) = \{n \in N : T^n(\alpha U) \cap V \neq \varnothing ; \alpha \in D\}$$

$$N^D(U_1, V_1) \cap N^D(U_2, V_2) \neq \varnothing.$$

Although the weakly $D$-mixing concept and Disk-cyclic Criterion are equivalent, there is still a need for studying its properties by using $D$- return sets.

This work discuss the necessary and sufficient conditions for an operator to be weakly $D$-mixing. We first tried to reduce the four open sets to three or two open sets only.

After that we studied the relation between weakly $D$-mixing and thick $D$- return sets.

Finally, we gave the sufficient conditions for an operator to be weakly $D$-mixing.

This work consists of two sections. In section two, we introduced four concepts ($D$-return set, proper $D$- return set and thick set) and its properties. In section three, we introduced the concept of weakly $D$-mixing and reduced the four open sets to three or two open sets only.

We remark $D := \alpha \in C, \ 0 < |\alpha| \leq 1$, unless otherwise stated.

2. $D$-Return Set With Disk-cyclic Operators

The following theorem gives specific certain of disk-cyclic criterion use in this paper.

Definition 2.1 (Disk-cyclic Criterion). Suppose that $T \in B(H)$ is called satisfies disk-cyclic criterion. If there exists an increasing sequence of positive integers $\{n_k\} \in N$ and $\{\alpha_{n_k}\} \in D$; For which there are a dense subsets $Y, X$ in $H$ and a sequence of mappings, $S_{n_k} : Y \to H$, as $k \to \infty$ such that:

(i) $\alpha_{n_k} T^{n_k}x \to 0$ for all $x \in X$

(ii) (a) $\frac{1}{\alpha_{n_k}} S_{n_k}y \to 0$ for all $y \in Y$

(b) $T^{n_k} S_{n_k}y \to y$ for all $y \in Y$.

Theorem 2.2. Every operator satisfies disk-cyclic criterion is disk-cyclic operator.

Proposition 2.3. Let $\{H_i\}_{i=1}^{n}$ be a family of separable, infinite dimensional Hilbert spaces, let $T_i \in B(H_i)$. If $\bigoplus_{i=1}^{n} T_i \in DC\left(\bigoplus_{i=1}^{n} H_i\right)$, then $T_i \in DC(H_i)$, for all $i = 1, \ldots, n$.

Proposition 2.4. $T \in B(H)$ satisfies the disk-cyclic criterion if and only if $\bigoplus_{i=1}^{r} T$ is disk-cyclic operator for all $r \geq 1$.

In flowing example we show that the disk-cyclic operators not necessary satisfy Disk-cyclic Criterion.

Example 2.5. Let $T \in B(C)$, such that $T(x) = 2x$. Then $T$ is a disk-cyclic operator but does not satisfy Disk-cyclic Criterion.
Proposition 2.4 and example 2.5 lead to the following main problems

Problem 2.6.

(i) Does every disk-cyclic operator satisfy the Disk-cyclic Criterion?

(ii) Is $\bigoplus_{n=1}^{\infty} T \in DC(H)$ for all $n \geq 2$, disk-cyclic operator whenever $T$ is?

Nareen in (2016) [1], proved the two above problems are equivalent.

The part two of problem 2.4 motivates as to introduce a new concept of disk-cyclic phenomena.

Our starting point is the following definitions.

Definition 2.7. [2] A subsets $S$ of $N$, is said to be a thick set if for every $M \in N$ there exists $t \in N$ such that $t, t+1, \ldots, t+M \in S$.

Definition 2.8. Let $T \in B(H)$ and $U, V$ be any non-empty open subsets of $H$, then the set

- $N^D(U, V) = N^D_U(V) = \{ n \in N : T^n(\alpha U) \cap V \neq \emptyset \}$ is called a $D$-return set. While,
- $C^D(U, V) = C^D_U(V) = \{ n \in N : T^n(\alpha U) \subset V \}$ is called a proper $D$-return set.

The following propositions gives some properties on $D$-return and proper $D$-return.

First we recall that if $A$ and $B$ are two subsets of $N$ then the sum (difference) set $A \pm B$ is defined by $A \pm B = \{ n \in N : (n, m) \in A \times B, n \geq m \}$.

Proposition 2.9. Let $T \in B(H)$, and $U, V, W$ be non-empty open subsets of $H$, then

(i) $N^D(U, V) + C^D(V, W) \subset N^D(U, W)$

(ii) $N^D(U, W) - C^D(U, V) \subset N^D(V, W)$

Proof.

(i) Let $k \in N^D(U, V) + C^D(V, W)$.

Thus $k = n + m$; $n \in N^D(U, V)$, $m \in C^D(V, W)$.

Then there exist $\alpha, \beta \in D$ such that $T^n(\alpha U) \cap V \neq \emptyset$ and $T^m(\beta V) \subset W$.

Hence there is an open set $X$ in $U$, such that $T^n(\alpha X) \subset V$ and $T^m(\beta X) \subset V$,

so $T^k(\beta \alpha X) = T^n(T^m(\beta \alpha X)) \subset T^n\beta V \subset W$.

Hence $T^k(\beta \alpha X) \cap W \neq \emptyset$, so $T^k(\sigma U) \cap W \neq \emptyset$ where $\delta = D$. Then $k \in N^D(U, W)$.

(ii) Let $k \in N^D(U, W) - C^D(U, V)$

i.e. $k = n - m$; $n \in N^D(U, V)$ and $m \in C^D(U, V)$.

Then there exist $\alpha, \beta \in D$ such that $T^n(\alpha U) \cap W \neq \emptyset$ and $T^m(\beta U) \subset V$, so $\beta U \subset T^{-m}V$.

Then $\alpha U \subset T^{-m}(\frac{\beta}{3}V)$. Hence $T^n(\alpha U) \subset T^n(T^{-m}\frac{\beta}{3}V) = T^k(\frac{\beta}{3}V)$. Then $T^k(\frac{\beta}{3}V) \cap W \neq \emptyset$.

So $T^k(\alpha V) \cap \beta W \neq \emptyset$. Since $|\beta| \leq 1$, so $\beta W \subset W$, thus $T^k(\alpha V) \cap W \neq \emptyset$.

Example 2.10. Let $T \in B(H)$, $U, V$ be any non-empty open subsets of $H$, then

$N^D(U, V) \neq N^D(V, U)$

In fact, $T(x) = 2x$ is a disk-cyclic operator by Example 2.3. So if we take $U = (1, 3), V = (10, 25)$, then for all $\alpha \in D, 1 \notin N^D(U, V)$. While when $\alpha = \frac{1}{10}, 1 \in N^D(V, U)$. 
3. Weakly Disk-cyclic Mixing Operators

This section, study some properties of a new concept on disk-cycliclty phenomena (weakly disk-mixing).

Definition 3.1. \( T \in B(H) \), is called weakly disk-mixing if \( T \oplus T : H \oplus H \to H \oplus H \) is disk-cyclic. We refer to it by weakly \( D \)-mixing.

Remark 3.2. The technique of the concept of weakly \( D \)-mixing is:
For any four non-empty open subsets \( U_1, U_2, V_1, V_2 \) of \( H \), there exists some \( n \in N \) and \( \alpha, \beta \in D \) such that \( T^n(\alpha U_1) \cap V_1 \neq \emptyset \) and \( T^n(\beta U_2) \cap V_2 \neq \emptyset \). i.e.,
\[
N^D(U_1,V_1) \cap N^D(U_2,V_2) \neq \emptyset.
\]

Although weakly \( D \)-mixing concept is another form of Disk-cyclic Criterion concept but we use the technique of the weakly \( D \)-mixing to prove a generalization of Example 2.5. The technique of weakly \( D \)-mixing is an approximation property involving 4-tuples of open sets whereas the technique of disk-cyclic concept is an approximation property involving pairs of open sets. The following theorem shows that in the definition of weakly \( D \)-mixing one may reduce the four open sets to 3-tuples and even to 2-tuples.

Theorem 3.3. Let \( T \in B(H) \). \( U, V, U_1, U_2 \) be non-empty open subsets of \( H \). Then the following are equivalent:

(i) For any \( U, V \) in \( H \) we have \( N^D(U,V) \cap N^D(V,V) \neq \emptyset \).

(ii) For any \( U_1, U_2, V \) in \( H \) we have \( N^D(U_1,V) \cap N^D(U_2,V) \neq \emptyset \).

(iii) \( T \) is weakly \( D \)-mixing.

Proof.

(i) \( \implies \) (ii) Let \( U_1, U_2, V \) be non-empty open subsets of \( H \). By (i) we can get \( N^D(U,V) \neq \emptyset \), so \( T \) is a disk-cyclic operator. Hence there is some \( n \in N, \alpha \in D \) such that \( V_1 = U_2 \cap T^{-n}(\frac{1}{n}V) \) is non-empty and open. By the hypothesis there is some \( m \in N^D(U_1, V_1) \cap N^D(V_1, V_1) \). Thus there exist \( \beta_1, \beta_2 \in D \) such that
\[
T^m(\beta_1 U_1) \cap V_1 \neq \emptyset \text{ and } T^m(\beta_2 V_1) \cap V_1 \neq \emptyset.
\]
Therefore there is \( x \in V_1 \subset T^{-n}(\frac{1}{n}V) \), such that \( x \in T^m(\beta_1 U_1) \), then \( T^{n+m}(\alpha \beta_1 U_1) \cap V \neq \emptyset \).

Also, there is \( y \in V_1 \) such that \( T^m(\beta_2 y) \in V_1 \), thus \( y \in U_2 \) and \( T^n(\alpha y) \in V \), which implies that \( T^{n+m}(\alpha \beta_2 y) \in V \), then
\[
T^{n+m}(\alpha \beta_2 U_2) \cap V \neq \emptyset.
\]
We have that
\[
n + m \in N^D(U_1,V) \cap N^D(U_2,V).
\]

(ii) \( \implies \) (iii) Let \( U_1, U_2, V_1, V_2 \) be non-empty open subsets of \( H \). By (ii) we can get \( N^D(U_1,V) \neq \emptyset \), so \( T \) is a disk-cyclic operator. Hence there is some \( n \in N, \alpha \in D \) such that \( V = V_1 \cap T^{-n}(\frac{1}{n}V_2) \) is a non-empty open set. Moreover, since disk-cyclic operator have dense range, also \( T^{-n}(U_2) \) is non-empty and open. By the hypothesis we find
\[
m \in N^D(U_1, V) \cap N^D(T^{-n}(U_2), V).
\]
Then there exist $\beta_1, \beta_2 \in D$ such that
\[ T^m (\beta_1 U_1) \cap V \neq \emptyset \quad \text{and} \quad T^{m-n} (\beta_2 U_2) \cap V \neq \emptyset. \]

But $V \subset V_1$ then $m \in N^D(U_1, V_1)$.

On the other hand $V \subset T^{-n} \left( \frac{1}{\alpha} V_2 \right)$, thus $T^{m-n} (\beta_2 U_2) \cap T^{-n} \left( \frac{1}{\alpha} V_2 \right) \neq \emptyset$.

That is $T^m (\beta_2 \alpha U) \cap V_2 \neq \emptyset$, which yields that
\[ m \in N^D(U_1, V_1) \cap N^D(U_2, V_2). \]

Hence $T$ is weakly $D$-mixing.

$(iii) \implies (i)$ Trivial \hfill \square

**Theorem 3.4.** Let $T \in B(H)$, $U, V, V_1, V_2$ be non-empty open subsets of $H$. Then the following are equivalent:

(i) For any $U, V$ in $H$ we have $N^D(U, U) \cap N^D(U, V) \neq \emptyset$.

(ii) For any $U, V, V_1, V_2$ in $H$ we have $N^D(U, V_1) \cap N^D(U, V_2) \neq \emptyset$.

(iii) $T$ is weakly $D$-mixing.

**Proof.**

$(i) \implies (ii)$ Let $U, V_1, V_2$ be non-empty open subsets of $H$. By the hypothesis we can get $N^D(U, V) \neq \emptyset$, so $T$ is a disk-cyclic operator. Hence there is some $n \in N, \alpha \in D$ such that $U_1 = U \cap T^{-n} \left( \frac{1}{\alpha} V_1 \right)$ is a non-empty open set. Since disk-cyclic have dense range, implies that $T^{-n}(V_2)$ is non-empty and open, so that there exists some $m \in N^D(U_1, V_1) \cap N^D(U_2, V_2)$. Then there exist $\beta_1, \beta_2 \in D$ such that
\[ T^m (\beta_1 U_1) \cap U_1 \neq \emptyset \quad \text{and} \quad T^m (\beta_2 U_1) \cap T^{-n} \left( \frac{1}{\alpha} V_2 \right) \neq \emptyset. \]

Thus $T^{m+n} (\alpha \beta_2 U_1) \cap V_2 \neq \emptyset$. So $n + m \in N^D(U, V_2)$.

On the other hand, there is $x \in U_1 \subset U$ such that $T^m (\beta_1 x) \in U_1 \subset T^{-n} \left( \frac{1}{\alpha} V_1 \right)$, then $T^{m+n} (\alpha \beta_1 x) \in V_1$. Hence $T^{m+n} (\alpha \beta_1 U) \cap V_1 \neq \emptyset$. Therefore
\[ n + m \in N^D(U, V_1). \]

Which implies that $n + m \in N^D(U, V_1) \cap N^D(U, V_2)$.

$(ii) \implies (iii)$ Let $U_1, U_2, V_1, V_2$ be non-empty open subsets of $H$. By the hypothesis we can get $N^D(U, V_1)$, so $T$ is a disk-cyclic operator. Hence there is some $n \in N, \alpha \in D$ such that $U = U_1 \cap T^{-n} \left( \frac{1}{\alpha} U_2 \right)$ is a non-empty open set. Moreover, since disk-cyclic operator have dense range, also $T^{-n}(V_2)$ is non-empty and open. By the hypothesis we find $m \in N^D(U, V_1) \cap N^D(U, T^{-n} V_2)$.

Then there exist $\beta_1, \beta_2 \in D$ such that $T^m (\beta_1 U_1) \cap V_1 \neq \emptyset$ and $T^m (\beta_2 U_2) \cap T^{-n} V_2 \neq \emptyset$.

In particular, there exists $x \in U \subset U_1$ with $T^m (\beta_1 x) \in V_1$. Thus
\[ m \in N^D(U_1, V_1). \]

Also, there exists $y \in U \subset T^{-n} \left( \frac{1}{\alpha} U_2 \right)$ such that $T^m (\beta_2 y) \in T^{-n} V_2$. We then conclude that $T^n (\alpha y) \in U_2$ and since $\alpha \in D$,
\[ T^m \alpha (T^n (\beta_2 y)) = T^m \beta_2 (T^n (\alpha y)) \in \alpha V_2 \subset V_2. \]
While $T^m \beta_2(T^n(\alpha y)) \in T^m \beta_2 U_2$, therefore $T^m(\beta_2 U_2) \cap V_2 \neq \emptyset$. Thus

$$m \in N^D(U_2, V_2).$$

Which yields that

$$m \in N^D(U_1, V_1) \cap N^D(U_2, V_2).$$

Hence $T$ is weakly $D$-mixing.

(iii) $\implies$ (i) Trivial

\[\square\]

**Theorem 3.5.** Let $T \in B(H)$. The following are equivalent:

(i) $T$ is weakly $D$-mixing.

(ii) Each $N^D(U, V) \neq \emptyset$ and for each $U_1, V_1, U_2, V_2$ non-empty open subsets of $H$, there are non-empty open sets $U_3, V_3$ such that $N^D(U_3, V_3) \subset N^D(U_1, V_1) \cap N^D(U_2, V_2)$.

(iii) All sets $N^D(U, V)$ are thick.


**Proof.**

(i)$\implies$ (ii) Let $U_1, V_1, U_2, V_2$ be non-empty open subsets of $H$. Since $T$ is weakly $D$-mixing, there is $m \in N^D(U_1, U_2) \cap N^D(V_1, V_2)$. Since $U_1, V_1$ are non-empty and open we can get $U_3, V_3$, non-empty open sets such that $U_3 \subset U_1$ and $V_3 \subset V_1$. So there exist $\alpha_1, \alpha_2 \in D$ such that $T^m(\alpha_1 U_3) \subset U_2$ and $T^m(\alpha_2 V_3) \subset V_2$. Then $N^D(U_3, V_3) \subset N^D(U_1, V_1)$.

Moreover, if $n \in N^D(U_3, V_3)$, then by proposition 2.9

$$n + m \in N^D(U_3, V_3) + C^D(V_3, V_2) \subset N^D(U_3, V_2),$$

so that

$$n = (n + m) - m \in N^D(U_3, V_2) - C^D(U_3, U_2) \subset N^D(U_2, V_2)$$

Hence $N^D(U_3, V_3) \subset N^D(U_2, V_3)$.

So $N^D(U_3, V_3) \subset N^D(U_1, V_1) \cap N^D(U_2, V_2)$.

(ii) $\implies$ (i) This is trivial.

(i)$\implies$ (iii) Let $U, V$ non-empty open subsets of $H$. Let $L$ be a positive integer.

Since $T$ is weakly $D$-mixing, then by proposition 2.9 $\bigoplus_{i=1}^{L} T \in DC(H)$. Thus, by proposition 2.9, $T \in DC(H)$. Hence one can find $n \in N$ and $\alpha \in D$ such that $T^n(\alpha U) \cap T^{-i}(V) \neq \emptyset$ for all $i = 0, \ldots, L$. So $n, \ldots, n + L \in N^D(U, V)$.

Therefore $N^D(U, V)$ is a thick set.

(iii) $\implies$ (iv) Let $M \in N$. By (iii) $N^D(U, V)$ is thick, then there exist $k \in N$ such that $k, \ldots, k + M \in N^D(U, V)$. So $M = k - k + M \in N^D(U, V) - N^D(U, V)$.

(iv) $\implies$ (i) Let $U, V, V_2$ be non-empty open subsets of $H$. by (iv) $N^D(U, V) \neq \emptyset$ otherwise $N = \emptyset$, thus $T \in DC(H)$. So one can find an $m \in N$, $\alpha \in D$ and a non-empty open set $V_1 \subset V$ such that $T^m(\alpha V_1) \subset V_2$. By (iv), we can take $k \in N$ such that $k \in N^D(U, V_1)$ and $k + m \in N^D(U, V_1)$. Then by proposition 2.9, part (ii)

$$k + m \in N^D(U, V_1) \cap [N^D(U, V_1) + C^D(V_1, V_2)] \subset N^D(U, V_1) \cap N^D(U, V_2).$$

Hence by (2.9, part (ii)) the result done.

\[\square\]
4. Conclusion

We gave the define of weakly D-mixing and reduced the four open sets to three or two open sets only. Every weakly D-mixing operators is another form of Disk-cyclic Criterion and every weakly D-mixing operators gave the $N^D(U, V)$ are thick sets.

References