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Weakly disk-cyclic mixing operators

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Abstract

It is known that there are disk-cyclic operators that are not satisfy the disk-cyclic criterion. This work introduce weakly disk-cyclic operators, which are kind of disk-cyclic operators, and study its relation with disk-cyclic criterion with several results and properties.

Keywords: disk-cyclic, codisk-cyclic, mixing operators, weakly mixing operators.

1. Introduction

Let H be a separable infinite dimensional Hilbert space, and $T \in B(H)$ is said to be a hypercyclic operator if the orbit of T, orbt $(T, x) = \{T^n x \ : n \geq 0\}$, is dense in H [3]. The first sufficient condition for hypercyclic (the Hypercyclic Criterion) discovered independently by kitai [7] and Godefroy and Shapioro [4]. In 1974 Hilden and Wallen [5], generalized the definition of hypercyclic operators to supercyclic by cone orbit, Corbt $(T, x) = \{\alpha T^n x \ : \ \alpha \in C \ , \ n \geq 0 \ \}$, is dense in H. Jamil in her Ph. D. thesis [6], partition the cone orbit into three parts, according to unit circle as: disk-cyclic operators when $|\alpha| \leq 1$, circle cyclic operators when $|\alpha| = 1$ and codisk-cyclic operators when $|\alpha| \geq 1$. But Saavedra and Müller[9], proved that every circle cyclic operators are hypercyclic.

There are many authors studied disk-cyclic operators from multiple aspects like: Bamerni defined subspace disk-cyclic and multidisk-cyclic operators on Banach Spaces [2], and Yu-Xia Iiang and Ze-Hua Zhou introduced Disk-cyclic and Codisk-cyclic tuples of the adjoint weighted composition operators on Hilbert spaces [8].

A sufficient conditions for disk-cyclic operators were found by Jamil in 2002 [6], which is called disk-cyclic criterion. In 2016 Bamerni [1], provided another version to the disk-cyclic criterion, which is simpler than the main disk-cyclic criterion. Moreover, proved that (T satisfies disk-cyclic criterion

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if and only if $\bigoplus_{i=1}^{k} T$ is disk-cyclic for all $k \geq 2$) [1]. Now the national quotation.
Is $\bigoplus_{i=1}^{n}$ T $\in DC(H)$ for all $n \geq 2$, disk-cyclic operator whenever T is?
This quotation motivates as to introduce a new concept of disk-cyclic phenomena which is called weakly D-mixing operator $T \in B(H)$ as:

$$T \oplus T : H \oplus H \to H \oplus H \quad \text{is disk-cyclic.}$$

Weakly D-mixing operators deals with D- return sets and proper D- return sets which defined as respectively:

$$N^D(U,V) = N_T^D(U,V) = \{n \in N : \ T^n(\alpha U) \cap V \neq \varnothing \ \ ; \alpha \in D\}$$
$$N^D(U_1, V_1) \cap N^D(U_2, V_2) \neq \varnothing.$$

Although the weakly D-mixing concept and Disk-cyclic Criterion are equivalent, there is still a need for studying it properties by using D- return sets.
This work discuss the necessary and sufficient conditions for an operator to be weakly D-mixing. We first tried to reduce the four open sets to three or two open sets only.
After that we studied the relation between weakly D-mixing and thick D- return sets.
Finally, we gave the sufficient conditions for an operator to be weakly D-mixing.
This work consists of two sections. In section two, we introduced four concepts (D-return set, proper$D$- return set and thick set) and its properties. In section three, we introduced the concept of weakly$D$-mixing and reduced the four open sets to three or two open sets only.
We remark $D := \alpha \in C, \ \ 0 < |\alpha| \leq 1$, unless otherwise stated.

2. *D*-Return Set With Disk-cyclic Operators

The following theorem gives specific certain of disk-cyclic criterion use in this paper.

Definition 2.1 (Disk-cyclic Criterion). *Suppose that $T \in B(H)$ is called satisfies disk-cyclic criterion. If there exists an increasing sequence of positive integers $\{n_k\}$ in N and $\{\alpha_{n_k}\}$ in D; For which there are a dense subsets* Y, X *in* H *and a sequence of mappings, $S_{n_k} : Y \to H$, as $k \to \infty$ such that:*

(i) $\alpha_{n_k} T^{n_k} x \to 0$ *for all* $x \in X$

(ii) (a) $\frac{1}{\alpha_{n_k}} S_{n_k} y \to 0$ *for all* $y \in Y$
(b) $T^{n_k} S_{n_k} y \to y$ *for all* $y \in Y$.

Theorem 2.2. *[1] Every operator satisfies disk-cyclic criterion is disk-cyclic operator.*

Proposition 2.3. *[6] Let $\{H_i\}_{i=1}^{n}$ be a family of separable, infinite dimensional Hilbert spaces, let $T_i \in B(H_i)$. If $\bigoplus_{i=1}^{n} T_i \in DC\left(\bigoplus_{i=1}^{n} H_i\right)$, then $T_i \in DC(H_i)$, for all $i = 1, \ldots, n$.*

Proposition 2.4. *[1] $T \in B(H)$ satisfies the disk-cyclic criterion if and only if $\bigoplus_{i=1}^{r} T$ is disk-cyclic operator for all $r \geq 1$.*

In flowing example we show that the disk-cyclic operators not necessary satisfy Disk-cyclic Criterion.

Example 2.5. *[1] Let $T \in B(C)$, such that $T(x) = 2x$. Then T is a disk-cyclic operator but does not satisfy Disk-cyclic Criterion.*

Proposition 2.4 and example 2.5 lead to the following main problems

Problem 2.6.

(i) Dose every disk-cyclic operator satisfy the Disk-cyclic Criterion?

(ii) Is $\bigoplus_{i=1}^{n} T \in DC(H)$ *for all* $n \geq 2$, *disk-cyclic operator whenever* T *is?*

Nareen in (2016) [1], proved the two above problem are equivalent.
The part two of problem 2.6 motivates as to introduce a new concept of disk-cyclic phenomena.
Our starting point is the following definitions.

Definition 2.7. *[2] A subsets* S *of* N, *is said to be a thick set if for every* $M \in N$ *there exists* $t \in N$ *such that* $t, t+1, \ldots, t+M \in S$.

Definition 2.8. *Let* $T \in B(H)$ *and* U, V *be any non-empty open subsets of* H, *then the set*

- $N^D(U,V) = N_T^D(U,V) = \{n \in N : T^n(\alpha U) \cap V \neq \varnothing \; ; \alpha \in D\}$ *is called a D-return set. While,*
- $C^D(U,V) = C_T^D(U,V) = \{n \in N : T^n(\alpha U) \subset V \; ; \alpha \in D\}$ *is called a proper D- return set.*

The following propositions gives some properties on D- return and proper D- return.
First we recall that if A and B are two subsets of N then the sum (difference) set $A \pm B$ is defined by $A \pm B = \{n - m : (n,m) \in A \times B, \; n \geq m\}$.

Proposition 2.9. *Let* $T \in B(H)$, *and* U, V, W *be non-empty open subsets of* H, *then*

(i) $N^D(U,V) + C^D(V,W) \subset N^D(U,W)$

(ii) $N^D(U,W) - C^D(U,V) \subset N^D(V,W)$

Proof .

(i) Let $k \in N^D(U,V) + C^D(V,W)$.
Thus $k = n + m$; $n \in N^D(U,V)$, $m \in C^D(V,W)$.
Then there exist $\alpha, \beta \in D$ *such that* $T^n(\alpha U) \cap V \neq \varnothing$ *and* $T^m(\beta V) \subset W$.
Hence there is an open set X *in* U, *such that* $T^n(\alpha X) \subset V$ *and* $T^n(\beta\alpha X) \subset \beta V$,
so $T^k(\beta\alpha X) = T^m(T^n(\beta\alpha X)) \subset T^m \beta V \subset W$.
Hence $T^k(\beta\alpha X) \cap W \neq \varnothing$, *so* $T^k(\sigma U) \cap W \neq \varnothing$ *where* $\delta \in D$. *Then* $k \in N^D(U,W)$.

(ii) Let $k \in N^D(U,W) - C^D(U,V)$
i.e. $k = n - m$; $n \in N^D(U,V)$ *and* $m \in C^D(U,V)$.
Then there exist $\alpha, \beta \in D$ *such that* $T^n(\alpha U) \cap W \neq \varnothing$ *and* $T^m(\beta U) \subset V$, *so* $\beta U \subset T^{-m} V$. *Then* $\alpha U \subset T^{-m}\left(\frac{\alpha}{\beta} V\right)$. *Hence* $T^n(\alpha U) \subset T^n\left(T^{-m} \frac{\alpha}{\beta} V\right) = T^k\left(\frac{\alpha}{\beta} V\right)$. *Then* $T^k\left(\frac{\alpha}{\beta} V\right) \cap W \neq \varnothing$. *So* $T^k(\alpha V) \cap \beta W \neq \varnothing$. *Since* $|\beta| \leq 1$, *so* $\beta W \subset W$, *thus* $T^k(\alpha V) \cap W \neq \varnothing$.

□

Example 2.10. *Let* $T \in B(H)$, U, V *be any non-empty open subsets of* H, *then*

$$N^D(U,V) \neq N^D(V,U)$$

In fact, $T(x) = 2x$ *is a disk-cyclic operator by Example 2.5. So if we take* $U = (1,3)$, $V = (10,25)$. *then for all* $\alpha \in D$, $1 \notin N^D(U,V)$. *While when* $\alpha = \frac{1}{10}, 1 \in N^D(V,U)$.

3. Weakly Disk-cyclic Mixing Operators

This section, study some properties of a new concept on disk-cycliclty phenomena (weakly disk-mixing).

Definition 3.1. *$T \in B(H)$, is called weakly disk-mixing if $T \oplus T : H \oplus H \to H \oplus H$ is disk-cyclic. We refer to it by weakly D-mixing.*

Remark 3.2. *The technique of the concept of weakly D-mixing is:*
For any four non-empty open subsets U_1, U_2, V_1, V_2 of H, there exists some $n \in N$ and $\alpha_1, \alpha_2 \in D$ Such that $T^n(\alpha_1 U_1) \cap V_1 \neq \varnothing$ and $T^n(\alpha_2 U_2) \cap V_2 \neq \varnothing$. i.e,

$$N^D(U_1, V_1) \cap N^D(U_2, V_2) \neq \varnothing.$$

Although weakly D-mixing concept is another form of Disk-cyclic Criterion concept but we use the technique of the weakly D-mixing to prove a generalization of Example 2.5.
The technique of weakly D-mixing is an approximation property involving 4-tuples of open sets whereas the technique of disk-cyclic concept is an approximation property involving pairs of open sets. The following theorem shows that in the definition of weakly D-mixing one may reduce the four open sets to 3-tuples and even to 2-tuples.

Theorem 3.3. *Let $T \in B(H)$. U, V, U_1, U_2 be non-empty open subsets of H. Then the following are equivalent:*

(i) For any U, V in H we have $N^D(U, V) \cap N^D(V, V) \neq \varnothing$.

(ii) For any U_1, U_2, V in H we have $N^D(U_1, V) \cap N^D(U_2, V) \neq \varnothing$.

(iii) T is weakly D-mixing.

Proof .
(i) $\Longrightarrow$ (ii) Let U_1, U_2, V be non-empty open subsets of H. By (i) we can get $N^D(U, V) \neq \varnothing$, so T is a disk-cyclic operator. Hence there is some $n \in N$, $\alpha \in D$ such that $V_1 = U_2 \cap T^{-n}\left(\frac{1}{\alpha}V\right)$ is non-empty and open. By the hypothesis there is some $m \in N^D(U_1, V_1) \cap N^D(V_1, V_1)$. Thus there exist $\beta_1, \beta_2 \in D$ such that
$T^m(\beta_1 U_1) \cap V_1 \neq \varnothing$ and $T^m(\beta_2 V_1) \cap V_1 \neq \varnothing$.
Therefore there is $x \in V_1 \subset T^{-n}\left(\frac{1}{\alpha}V\right)$, such that $x \in T^m(\beta_1 U_1)$, then $T^{n+m}(\alpha\beta_1 U_1) \cap V \neq \varnothing$, Also, there is $y \in V_1$ such that $T^m(\beta_2 y) \in V_1$, thus $y \in U_2$ and $T^n(\alpha y) \in V$, which implies that $T^{m+n}(\alpha\beta_2 y) \in V$, then

$$T^{n+m}(\alpha\beta_2 U_2) \cap V \neq \varnothing.$$

We have that

$$n + m \in N^D(U_1, V) \cap N^D(,\ U_2, V).$$

(ii) $\Longrightarrow$ (iii) Let U_1, U_2, V_1, V_2 be non-empty open subsets of H. By (ii) we can get $N^D(U_1, V) \neq \varnothing$, so T is a disk-cyclic operator. Hence there is some $n \in N, \alpha \in D$ such that $V = V_1 \cap T^{-n}(\frac{1}{\alpha}V_2)$ is a non-empty open set. Moreover, since disk-cyclic operator have dense range, also $T^{-n}(U_2)$ is non-empty and open. By the hypothesis we find

$$m \in N^D(U_1,\ V) \cap N^D\left(T^{-n}(U_2),\ V\right).$$

Then there exist $\beta_1, \beta_2 \in D$ *such that*

$$T^m(\beta_1 U_1) \cap V \neq \varnothing \text{ and } T^{m-n}(\beta_2 U_2) \cap V \neq \varnothing.$$

But $V \subset V_1$ *then* $m \in N^D(U_1,\ V_1)$.
On the other hand $V \subset T^{-n}\left(\frac{1}{\alpha}V_2\right)$, *thus* $T^{m-n}(\beta_2 U_2) \cap T^{-n}\left(\frac{1}{\alpha}V_2\right) \neq \varnothing$.
That is $T^m(\beta_2 \alpha U) \cap\ V_2 \neq \varnothing$, *which yields that*

$$m \in N^D(U_1,\ V_1) \cap N^D(U_2,\ V_2).$$

Hence T *is weakly D-mixing.*
(iii)$\Longrightarrow$ *(i) Trivial* ☐

Theorem 3.4. *Let* $T \in B(H)$. $U,\ V, V_1,\ V_2$ *be non-empty open subsets of* H. *then the following are equivalent:*

(i) For any $U,\ V$ *in* H *we have* $N^D(U,U) \cap N^D(U,V) \neq \varnothing$.

(ii) For any $U,\ V_1,\ V_2$ *in* H *we have* $N^D(U,V_1) \cap N^D(\ U,V_2) \neq \varnothing$.

(iii) T *is weakly D-mixing.*

Proof .
(i)$\Longrightarrow$ *(ii) Let* $U,\ V_1,\ V_2$ *be non-empty open subsets of* H. *By the hypothesis we can get* $N^D(U,V) \neq \varnothing$, *so* T *is a disk-cyclic operator. Hence there is some* $n \in N$, $\alpha \in D$ *such that* $U_1 = U \cap T^{-n}\left(\frac{1}{\alpha}V_1\right)$ *is a non-empty open set. Since disk-cyclic have dense range, implies that* $T^{-n}(V_2)$ *is non-empty and open, so that there exists some* $m \in N^D(U_1,\ U_1) \cap N^D\left(U_1,\ T^{-n}\left(\frac{1}{\alpha}V_2\right)\right)$. *Then there exist* $\beta_1, \beta_2 \in D$ *such that*

$$T^m(\beta_1 U_1) \cap U_1 \neq \varnothing \quad \text{and} \quad T^m(\beta_2 U_1) \cap T^{-n}\left(\frac{1}{\alpha}V_2\right) \neq \varnothing$$

Thus $T^{m+n}(\alpha\beta_2 U_1) \cap V_2 \neq \varnothing$.*So* $n+m \in N^D(U,\ V_2)$.
On the other hand, there is $x \in U_1 \subset U$ *such that* $T^m(\beta_1 x) \in U_1 \subset T^{-n}\left(\frac{1}{\alpha}V_1\right)$, *then* $T^{m+n}(\alpha\beta_1 x) \in V_1$. *Hence* $T^{m+n}(\alpha\beta_1 U) \cap V_1 \neq \varnothing$. *Therefore*

$$n+m \in N^D(U,\ V_1)$$

Which implies that $n+m \in N^D(U,\ V_1) \cap N^D(U,\ V_2)$.
(ii) $\Longrightarrow$ *(iii) Let* $U_1,\ U_2,\ V_1,\ V_2$ *be non-empty open subsets of* H. *By the hypothesis we can get* $N^D(U,V_1)$, *so* T *is a disk-cyclic operator. Hence there is some* $n \in N$,$\alpha \in D$ *such that* $U = U_1 \cap T^{-n}(\frac{1}{\alpha}U_2)$ *is a non-empty open set. Moreover, since disk-cyclic operator have dense range, also* $T^{-n}(V_2)$ *is non-empty and open. By the hypothesis we find* $m \in N^D(U,\ V_1) \cap N^D(U,\ T^{-n}V_2)$.
Then there exist $\beta_1, \beta_2 \in D$ *such that* $T^m(\beta_1 U) \cap V_1 \neq \varnothing$ *and* $T^m(\beta_2 U)) \cap T^{-n}V_2 \neq \varnothing$.
In particular, there exists $x \in U \subset U_1$ *with* $T^m(\beta_1 x) \in\ V_1$. *Thus*

$$m \in N^D(U_1,\ V_1).$$

Also, there exists $y \in U \subset T^{-n}(\frac{1}{\alpha}U_2)$ *such that* $T^m(\beta_2 y) \in T^{-n}V_2$. *We then conclude that* $T^n(\alpha y) \in U_2$ *and since* $\alpha \in D$,

$$T^m\alpha(T^n(\beta_2 y)) = T^m\beta_2(T^n(\alpha y)) \in \alpha V_2 \subset V_2.$$

While $T^m\beta_2\left(T^n(\alpha y)\right) \in T^m\beta_2 U_2$, *therefore* $T^m(\beta_2 U_2) \cap V_2 \neq \varnothing$. *Thus*

$$m \in N^D\left(U_2,\ V_2\right).$$

Which yields that

$$m \in N^D\left(U_1,\ V_1\right) \cap N^D\left(U_2,\ V_2\right).$$

Hence T *is weakly D-mixing.*
(iii) $\Longrightarrow$ *(i) Trivial* □

Theorem 3.5. *Let* $T \in B(H)$. *The following are equivalent:*

(i) T *is weakly D-mixing.*

(ii) Each $N^D(U,V) \neq \varnothing$ *and for each* $U_1,\ V_1,\ U_2,\ V_2$ *non-empty open subsets of* H, *there are non-empty open sets* $U_3,\ V_3$ *such that* $N^D\left(U_3,\ V_3\ \right) \subset N^D\left(U_1,\ V_1\ \right) \cap N^D\left(U_2,\ V_2\ \right)$.

(iii) All sets $N^D(U,V)$ *are thick.*

(iv) $N^D(U,V) - N^D(U,V) = N$, *for any* U,V.

Proof .
(i)$\Longrightarrow$ *(ii) Let* $U_1,\ V_1,\ U_2,\ V_2$ *be non-empty open subsets of H. Since* T *is weakly D-mixing, thus there is* $m \in N^D\left(U_1,\ U_2\ \right) \cap N^D\left(V_1,\ V_2\ \right)$. *Since* $U_1,\ V_1$ *are non-empty and open we can get* $U_3,\ V_3$, *non-empty open sets such that* $U_3 \subset U_1$ *and* $V_3 \subset V_1$, *So there exist* $\alpha_1, \alpha_2 \in D$ *such that* $T^m\left(\alpha_1 U_3\right) \subset U_2$ *and* $T^m\left(\alpha_2 V_1\right) \subset V_2$. *Then* $N^D\left(U_3,\ V_3\ \right) \subset N^D\left(U_1,\ V_1\ \right)$.
Moreover, if $n \in N^D\left(U_3,\ V_3\ \right)$, *then by proposition 2.9*

$$n+m \in N^D\left(U_3,\ V_3\ \right) + C^D\left(V_3, V_2\right) \subset N^D\left(U_3,\ V_2\ \right),$$

so that

$$n = (n+m) - m \in N^D\left(U_3,\ V_2\ \right) - C^D\left(U_3, U_2\right) \subset N^D\left(U_2,\ V_2\ \right)$$

Hence $N^D\left(U_3,\ V_3\ \right) \subset N^D\left(U_2,\ V_2\ \right)$.
So $N^D\left(U_3,\ V_3\ \right) \subset N^D\left(U_1,\ V_1\ \right) \cap N^D\left(U_2,\ V_2\ \right)$.
(ii)$\Longrightarrow$ *(i) This is travel.*
(i)$\Longrightarrow$ *(iii) Let* U,V *non-empty open subsets of* H, L *be a positive integer .*
Since T *is weakly D-mixing, then by proposition 2.9* $\bigoplus_{i=1}^{L} T \in DC(H)$. *Thus, by proposition 2.9,* $T \in DC(H)$. *Hence one can find* $n \in N$ *and* $\alpha \in D$ *such that* $T^n(\alpha U) \cap T^{-i}(V) \neq \varnothing$ *for all* $i = 0, .., L$. *So* $n, \ldots, n+L\ \in N^D(U,V)$.
Therefore $N^D(U,V)$ *is a thick set.*
(iii)$\Longrightarrow$ *(iv) Let* $M \in N$. *By (iii)* $N^D(U,V)$ *is thick, then there exist* $k \in N$ *such that* $k, \ldots, k+M\ \in N^D(U,V)$. *So* $M = k - k + M \in N^D(U,V) - N^D(U,V)$.
(iv)$\Longrightarrow$ *(i) Let* U, V, V_2 *be non-empty open subsets of H. by (iv)* $N^D(U,V) \neq \varnothing$ *otherwise* $N = \varnothing$, *thus* $T \in DC(H)$. *So one can find an* $m \in N$, $\alpha \in D$ *and a non-empty open set* $V_1 \subset V$ *such that* $T^m\left(\alpha V_1\right) \subset V_2$. *By (iv), we can take* $k \in N$ *such that* $k \in N^D\left(U,\ V_1\ \right)$ *and* $k+m \in N^D\left(U,\ V_1\ \right)$. *Then by proposition (2.9 part (i))*

$$k+m \in N^D\left(U, V_1\ \right) \cap \left[N^D\left(U,\ V_1\ \right) + C^D\left(V_1, V_2\right)\right] \subset N^D\left(U,\ V_1\ \right) \cap N^D\left(U,\ V_2\ \right).$$

Hence by (2.9 part (ii)) the result done. □

4. Conclusion

We gave the define of weakly D-mixing and reduced the four open sets to three or two open sets only. Every weakly D-mixing operators is another form of Disk-cyclic Criterion and every weakly D-mixing operators gave the $N^D(U, V)$ are thick sets.

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