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Some Properties of δ-integral with respect $\delta-$ fuzzy Measure

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Abstract

The δ-integral with respect - δ - fuzzy measure has significant practical applications such as economic, data mining, etc. In this paper, we define δ-fuzzy measure and studied the measurable function with respect to $\delta-$ fuzzy measure, also we defined an integration on a type of measurement, which is called the $\delta-$ integral because of its importance in many other applications, we studied the properties of this integration and proved some important theories, Besides that, the convergence theorems are studied and the obtained results are proved.

Keywords: $\sigma-$ algebra, measurable function, $\boldsymbol{\delta}-$ fuzzy measure, $\boldsymbol{\delta}$-integral.
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1. Introduction

The concept of fuzzy measurement was firstly introduced by Zadeh in 1965[10]. Then it was sophisticatedly studied more by others [3-5] and [8]. Lotfi Zadeh proposed the concept of fuzziness in contrast to fuzzy sets in 1965 . Since then, the concept of fuzziness has been introduced into many disciplines of mathematics. Fuzzy topology, fuzzy logic, fuzzy algorithmic, fuzzy geometry, and so on are all examples of fuzzy topology. Because of their powerful depiction of imprecision, fuzzy mathematics techniques are frequently used in many engineering disciplines and developed to become in the current wonderful form. Thereby, they have entered in many fields of science produced results of great importance especially in the field of technical sciences. Moreover, it has found that the integration and measure theory plays a crucial role in applications of economics, game theory or decision theory. It naturally plays a key role in a variety of fields, including probability theory

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and other. Integration was also introduced in some kind of measurement in [1],[9]. In this research, we studied new properties of fuzzy integration, but on a new type of measurement. This paper is organized as follows; section 2 defines the essential of δ-Fuzzy measure. Section 3 studies the δ -integral. Finally in section 4 , the Monotone Convergence in this measure is discussed.

2. Preliminaries

We begin with some basic definitions and results [2],[6] [7].

Definition 2.1. *Let X be a set, a family $\mathcal{H}$ of subset of a set X is $\sigma-$ algebra on X if satisfy*

(1) $X \in \mathcal{H}$

(2) $\forall A \in \mathcal{H}$ *then* $A^c \in \mathcal{H}$

(3) If $A_n \in \mathcal{H} \quad \forall n = 1, 2, \ldots$ *then* $\bigcup_{n=1}^{\infty} A_n \in \mathcal{H}$

$(\mathrm{X}, \mathcal{H})$ *is a measurable space, a subset A of* X *is called (measurable with respect to $\sigma-$ algebra $\mathcal{H}$.*

Definition 2.2. *Let $\mathcal{H}$ be $\sigma-$ algebra of subset on* $\mathrm{X}, \mathcal{M} : \mathcal{H} \to [0, 1]$ *is called $\boldsymbol{\delta}-$ fuzzy measure if satisfy the axioms*

(i) $\mathcal{M}(X) = 1$

(ii) $\mathcal{M}\left(\cup_{i=1}^{\infty} A_i\right) = \begin{cases} \frac{1}{\delta}\left[\prod_{i=1}^{\infty}\left(1 + \delta\mathcal{M}\left(A_i\right)\right) - 1\right] & \delta \neq 0 \\ \sum_{i=1}^{\infty} \mathcal{M}\left(A_i\right) & \delta = 0 \end{cases}$

Where $\delta\epsilon(-1, \infty)$ and a tripe $(\mathrm{X}, \mathcal{H}, \mathcal{M})$ is $\boldsymbol{\delta}-$ fuzzy measure space.

Example of $\boldsymbol{\delta}-$ Fuzzy measure

Let $\quad X = \{a, b, c\}, \quad \mathcal{M}(\{a\}) = 0.4, \quad \mathcal{M}(\{b\}) = 0.3, \quad \mathcal{M}(\{c\}) = 0.2$
Take $\delta = 0.3719$
Then

$$\begin{aligned} \mathcal{M}(X) = \mathcal{M}(\{a, b, c\}) &= \mathcal{M}(\{a\} \cup \{b\} \cup \{c\}) \\ &= \frac{1}{\delta}\mathcal{M}[(1 + \mathcal{M}(\{a\}))(1 + \mathcal{M}(\{b\}))(1 + \mathcal{M}(\{c\})) - 1] \\ &= \frac{1}{0.3719}\mathcal{M}[(1 + 0.4)(1 + 0.3)(1 + 0.2) - 1 \end{aligned}$$

then $\mathcal{M}(X) = 1$

$$\begin{aligned} \mathcal{M}(\{a, b\}) &= \mathcal{M}(\{a\}) + \mathcal{M}(\{b\}) + \delta\mathcal{M}(\{a\})\mathcal{M}(\{b\}) = 0.7446 \\ \mathcal{M}(\{a, c\}) &= \mathcal{M}(\{a\}) + \mathcal{M}(\{c\}) + \delta\mathcal{M}(\{a\})\mathcal{M}(\{c\}) = 0.6289 \\ \mathcal{M}(\{b, c\}) &= \mathcal{M}(\{b\}) + \mathcal{M}(\{c\}) + \delta\mathcal{M}(\{b\})\mathcal{M}(\{c\}) = 0.5223 \end{aligned}$$

Theorem 2.3. *Let $(X, \mathcal{H}, \mathcal{M})$ is $\delta-$ fuzzy measure space,*

(1) $\mathcal{M}(\emptyset) = 0$

(2) If $A, B \in \mathcal{H}$ *then* $\mathcal{M}(A.B) = \mathcal{M}(A) + \mathcal{M}(B) + \delta.\mathcal{M}(A)\mathcal{M}(B)$

(3) If $A_1, A_2, \ldots, A_n \in \mathcal{H}$ *then* $\mathcal{M}\left(\bigcup_{i=1}^{n} A_i\right) = \begin{cases} \frac{1}{\delta}\left[\prod_{i=1}^{n}\left(1+\delta\mathcal{M}\left(A_i\right)\right)-1\right\} & \delta \neq 0 \\ \sum_{i=1}^{n} \mathcal{M}\left(A_i\right) & \delta = 0 \end{cases}$

(4) If $A \in \mathcal{H}$ *then* $\mathcal{M}(A) + \mathcal{M}\left(A^c\right) = 1 - \delta\mathcal{M}(A)\mathcal{M}\left(A^c\right)$

(5) $\mathcal{M}\left(A_1 \cap A_2\right) = \frac{1-\mathcal{M}(A_1 \cup A_2)}{1+\delta\mathcal{M}(A_1 \cup A_2)}$

3. δ - Integral

Let $(X, \mathcal{H}, \mathcal{M})$ be a positive $\boldsymbol{\delta}-$ fuzzy measure space(stydy in case $\delta = 0$), $g : X \to R$ be a simple measurable function, if $a_1, a_2, \ldots, a_n$ are distinct values of the simple function, g, and if $C_i = g^{-1}\left(\{a_i\}\right)$, i $= 1, \ldots,$ n, then

$$g = \sum_{i=1}^{n} a_i \chi_{C_i}$$

Where χ_{C_i} indicator function
If $A \in \mathcal{H}$, we have $\theta(A) = \int_A \quad g d\mathcal{M} = \sum_{i=1}^{n} a_i \mathcal{M}\left(C_i \cap A\right), \theta$ is positive measure since each term in the right side is a positive measure as function of A.
Note that

- $\int_A a g d\mathcal{M} = a \int_A g d\mathcal{M} \quad$ if $0 \leq a < \infty$
- $\int_A g d\mathcal{M} = a\mathcal{M}(A)$ if $a \in [0, \infty]$ and g a simple measurable function such that $g = a \quad$ on A
- If g^* is another simple measurable function and $g < g^*$ then $\int_A \quad g d\mathcal{M} \leq \int_A g^* d\mathcal{M}$
- Let $\eta_1, \eta_2, \ldots, \eta_k$ be distinct values of g and $E_j = g^{-1}\left(\eta_j\right), j = 1, \ldots, k$, Put $B_{ij} = C_i \cap E_j$, $\int_A g d\mathcal{M} = \theta\left(\mathrm{U}_{ij}\left(A \cap B_{ij}\right)\right) = \sum_{ij} \theta\left(A \cap B_{ij}\right) = \sum_{ij} \int_{A \cap B_{ij}} f d\mathcal{M}$
  $\sum_{ij} \int_{A \cap B_{ij}} a_i d\mathcal{M} \leq \sum_{ij} \int_{A \cap B_{ij}} \eta_i d\mathcal{M} = \int_A \quad g^* d\mathcal{M}$

And by similar way, it can be proved that

$$\int_A \left(g + g^*\right) d\mathcal{M} = \int_A g d\mathcal{M} + \int_A g^* d\mathcal{M}$$

From above, it follows that

$$\int_A g \chi_A d\mathcal{M} = \int_A \sum_{ij} a_i \chi_{A \cap c_i} d\mathcal{M} = \int_A g d\mathcal{M}$$

if A is a measurable set, $A \in \mathcal{H}$ and $f : X \to R$ is non-negative measurable function and $\delta-$ integral is define as

$$\int_A f d\mathcal{M} = \operatorname{Sup}\left\{\int_A g d\mathcal{M} \mid \ 0 < g \leq f \quad \text{and } g \text{ is simple measurable function and } g = 0 \ \text{ on } A^c\right\}$$

The left side in this equation is called the Lebesque integral of f over A with respect to $\mathcal{M}$. In this work, we consider δ-integral of f over A. The two definition of integral of a simple function g over A are agree.

Properties of δ-integral

Theorem 3.1. *(1) If $f_1 \leq f_2$ then $\int_A f_1 d\mathcal{M} \leq \int_A f_2 d\mathcal{M}$*

(2) $\int_A (f_1 + f_2)\, d\mathcal{M} = \int_A f_1 d\mathcal{M} + \int_A f_2 d\mathcal{M}$

(3) r $\in R^+$ then $\int_A r f d\mathcal{M} = r \int_A f d\mathcal{M}$

(4) Let $f \geq 0$ is measurable and non-negative function on $A \in F$ and $\int_A f d\mathcal{M} < \infty$ set $B = \{x \in A : f(x) = +\infty\}$ then $B \in \mathcal{H}$ and $\mathcal{M}(B) = 0$

Proof .

(1) since $\int_A gd\mathcal{M} \mid g \leq f_1, g$ is simple function $\subset \left\{\int_A gd\mathcal{M} \mid g \leq f_2, g \text{ is simple function}\right\}$
Sup $\int_A gd\mathcal{M} \mid g \leq f_1, g$ is simple function $\leq \sup\left\{\int_A \mathrm{gd}\,\mathcal{M} \mid g \leq f_2, g \text{ is simple function }\right\}$
then $\int_A f_1 d\mathcal{M} \leq \int_A f_2 d\mathcal{M}$

(2) Let $\{g_n\}_{n\in N}$ and $\{h_n\}_{n\in N}$ be non - increasing sequence of simple non- negative measurable function and

$$\lim_{n\to\infty} g_n = f_1, \lim_{n\to\infty} h_n = f_2$$

$$\begin{aligned}\int_A f_1 d\mathcal{M} + \int_A f_2 d\mathcal{M} &= \mathrm{Sup}\left\{\int_A g_n d\mathcal{M} \mid \quad n \in N\right\} + \sup\left\{\int_A h_n d\mathcal{M} \mid \quad n \in N\right\}\\ &= \sup\left\{\int_A (g_n + h_n)\, d\mathcal{M} \mid \quad n \in N\right\}\\ &= \int_A (f_1 + f_2)\, d\mathcal{M}\end{aligned}$$

(3)

$$\begin{aligned}\int_A r f d\mathcal{M} &= \sup\left\{\int_A gd\mathcal{M} \mid g \leq rf, g \text{ is simple function}\right\}\\ &= r\int_A \frac{g}{r} \mid \frac{g}{r} \leq f, \quad g\text{is simple function}\\ &= r\int_A f dM\end{aligned}$$

(4) since f is measurable function, $f^{-1}(\{\infty\}) \in \mathcal{H}$ and so $B = A \cap f^{-1}(\{\infty\}) \in F$
Define

$$\omega_n = \begin{cases} n & , n \in B \\ 0 & n \notin B \end{cases}$$

Since $B \in \mathcal{H}$ and ω_n is measurable and non-negative function, $\omega_n \leq f$
Then $n\mathcal{M} = \int_A \omega_n d\mathcal{M} \leq \int_A f d\mathcal{M} < \infty \forall n \geq 1$ then $\mathcal{M}(B) = 0$

□

Theorem 3.2. *If f is measurable and non-negative function on $A \in F$ and $\mathcal{M}(A) = 0$ then $\int_A f d\mathcal{M} = 0$*

Proof . Let $0 \leq g \leq f$ be sequence and $g = \sum_{n=1}^{\infty} a_n \chi_{A_n}, \quad a_n \geq 0, A_n \in \mathcal{H}$

$$\int_A g d\mathcal{M} = \sum_{n=1}^{\infty} a_n \mathcal{M}\left(A_n \cap A\right)$$

Since $\mathcal{M}$ is monotone
$\mathcal{M}\left(A_n \cap A\right) \leq \mathcal{M}(A) = 0$
Then $\int_A g d\mathcal{M} = 0$ for every g is non-negative measurable function
Then $\int_A f d\mathcal{M} = 0$ □

4. Monotone Convergence Theorem

Theorem 4.1. *Let $f_n : X \to [0, \infty], n = 1, 2, 3, \ldots,$ be a sequence of measurable function and suppose that $f_n \uparrow f$*

That is $0 \leq f_1 \leq f_2 \leq \cdots$ and $f_n(x) \to f(x)$ as $n \to \infty$ for every $x \in X$ then f is measurable and $\int_X f_n d\mathcal{M} \to \int_X f d\mathcal{M}$ as $n \to \infty$.

Proof . The function f is measurable
since $f = \sup_{n \geq 1} f_n$
$f_n \leq f_{n+1} \leq f$ satisfy $\int_X f_n d\mathcal{M} \leq \int_X f_{n+1} d\mathcal{M} \leq \int_X f d\mathcal{M}$
And conclude exists $a \in [0, \infty]$
$\int_X f_n d\mathcal{M} \to a$ as $n \to \infty$
And $a \leq \int_X f d\mathcal{M}$
Let h any simple measurable function such that $0 \leq h \leq f$ and $0 < k < 1$ is constant, for fixed $n \in N_+$

$$A_n = \{x \in X; f_n(x) \geq kh(x)\}$$

If $a_1, \ldots, a_p$ are the distinct values of $h, A_n = \bigcup_{i=1}^{p} \left(\{x \in X; f_n(x) \geq ka_i\} \cap \{h = a_i\}\right)$
And A_n is measurable.
Clear $A_1 \subseteq A_2 \subseteq \cdots$ and if $f(x) = 0$
Then $x \in A_1$, if $f(x) > 0$ then $kh(x) < f(x), x \in A_n$
For sufficiently large n thus $\bigcup_{n=1}^{\infty} A_n = X$
Now $a \geq \int_{A_n} f_n d\mathcal{M} \geq k \int_{A_n} h d\mathcal{M}$ we get $a \geq k \int_X h d\mathcal{M}$
And since $A \to \int_A h d\mathcal{M}$ is poisitive measure on $\mathcal{M}$, let $k \uparrow 1$ $a \geq k \int_X h d\mathcal{M}$
Then $a \geq k \int_X f d\mathcal{M}$
That is

$$\int_X f_n d\mathcal{M} \to \int_X f d\mathcal{M}$$

□

Theorem 4.2. *If $f_n : X \to R, n = 1, 2, \ldots$ are measurable then $\int_X \lim_{n\to\infty} \inf f_n d\mathcal{M} \leq \lim_{n\to\infty} \int_X f_n d\mathcal{M}$*

Proof . Let $g_k = \inf_{n \geq k} f_n$ this is $g_k \uparrow \lim_{n\to\infty} \inf f_n$ and $\int_X g_k d\mathcal{M} \leq \int_X f_n d\mathcal{M}$ $, n \geq k$ And $\int_X g_k d\mathcal{M} \leq \inf_{n \geq k} \int_X f_n d\mathcal{M}$ by theorem (4.1) we have

$$\int_X \lim_{n\to\infty} \inf f_n d\mathcal{M} \leq \lim_{n\to\infty} \int_X f_n d\mathcal{M}$$

□

Open Problem

We generalize this Compute fuzzy integrals with another measure (as capacity measure) using the Fundamental Theorem of Calculus, Monotone and Dominated Convergence Theorems, and the Tonelli and Fubini Theorems.

Conclusion

At the current study, we have introduced the connotation of Fuzzy integral functions. Topic presented in this research belongs to a contemporary field of mathematics that has been successfully applied in economic analysis. Further, the goal of this study is to give a simple way to grasp fuzzy integrals by reviewing the basic notions of fuzzy measure, which are the foundation of fuzzy integrals.

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