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Conservation laws and exact solutions of a generalized (2+1)-dimensional Bogoyavlensky-Konopelchenko equation

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Abstract

This paper aims to study a generalized (2+1)-dimensional Bogoyavlensky-Konopelchenko equation. We perform symmetry reduction and derive exact solutions of a generalized (2+1)-dimensional Bogoyavlensky-Konopelchenko equation. In addition, conservation laws for the underlying equation are constructed.

Keywords: Symmetry reduction, Exact solutions, Conservation laws
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1. Introduction

The generalized (2+1)-dimensional Bogoyavlensky-Konopelchenko equation is given by [1]

$$p_t + \alpha p_{xxx} + 6\beta p_{xxy} + 6\alpha p_x p + 4\beta p_y p + 4\beta p_x \partial_x^{-1} p_y = 0, \tag{1.1}$$

where α and β are non-zero arbitrary constants while $p = p(t, x, y)$ denotes the wave profile and the variables t, x and y represent time and space respectively. In [2], equation (1.1) with $\alpha = 0$ is also referred to as the Calogero-Bogoyavlensky-Schiff equation. Several methods for example, the Dardoux

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transformation and the inverse scattering method have been employed to solve equation (1.1). See for example [3, 4] and references therein. The term $\partial_x^{-1}p$ is a spatial antiderivative of p which is defined through the Fourier transform by the multiplier $\frac{i}{\xi}$ and $\partial_x^{-1} = \int dx$ is the inverse scattered transformation. When substituting $\partial_x^{-1}p = u$ into equation (1.1) one can obtain the equivalent form of (1.1), namely

$$u_{tx} + \alpha u_{xxxx} + 6\beta u_{xxxy} + 6\alpha u_{xx}u_x + 4\beta u_{xy}u_x + 4\beta u_{xx}u_y = 0. \tag{1.2}$$

Motivated by recent work in [1, 5], we revisit the (2+1)-dimensional Bogoyavlensky-Konopelchenko equation (1.1).

The objective of this work is twofold. Firstly, we seek to establish new exact solutions [6, 7, 8] of a generalized (2+1)-dimensional Bogoyavlensky-Konopelchenko equation (1.2) using the Lie symmetry method [9, 10, 11, 12, 13, 14, 15, 16]. Thereafter, we aim to derive *low-order* local conservation laws of equation (1.2) using the invariance and multiplier approach based on the well known results that the Euler-Lagrange operator annihilates the total divergence.

2. Symmetry analysis of equation (2)

The vector field operator

$$\mathbf{X} = \xi^1(t,x,y,u)\frac{\partial}{\partial t} + \xi^2(t,x,y,u)\frac{\partial}{\partial x} + \xi^3(t,x,y,u)\frac{\partial}{\partial y} + \eta(t,x,y,u)\frac{\partial}{\partial u} \tag{2.1}$$

is a Lie point symmetry of (1.2) if

$$\mathbf{X}^{[4]}\left\{u_{tx} + \alpha u_{xxxx} + 6\beta u_{xxxy} + 6\alpha u_{xx}u_x + 4\beta u_{xy}u_x + 4\beta u_{xx}u_y = 0\right\}\Big|_{(1.2)} = 0,$$

where $\mathbf{X}^{[4]}$ is the fourth extension of (2.1). Expanding the above equation and splitting the monomials leads to linear overdetermined system of partial differential equations. These are

$$\begin{aligned}
&\xi^3_x = 0,\ \xi^1_x = 0,\ \xi^1_y = 0,\ \xi^2_u = 0,\ \xi^3_u = 0,\ \xi^1_u = 0,\ \xi^2_{xx} = 0,\ \eta_{tx} = 0,\ \eta_{xx} = 0,\\
&\eta_{xu} = 0,\ \eta_{uu} = 0,\ \eta_u + \xi^2_x = 0,\ -4\beta\eta_x + \xi^3_t = 0,\ 3\xi^2_{xy} - \eta_{yu} = 0,\\
&\xi^2_{xy} - \eta_{yu} = 0,\ 6\alpha\eta_x + 4\beta\eta_y - \xi^2_t = 0,\ 4\beta\eta_{xy} + \eta_{tu} - \xi^2_{tx} = 0,\\
&4\beta\xi^2_y - 3\alpha\xi^3_y + 3\alpha\xi^2_x = 0,\ \xi^1_t - \xi^3_y + \eta_u - \xi^2_x = 0,\ \beta\xi^2_y - \alpha\xi^3_y + \alpha\eta_u + 2\alpha\xi^2_x = 0.
\end{aligned}$$

Solving the above systems of partial differential equations prompt the following two cases.

Case 1. $\alpha \neq -\beta$

In this case equation (1.2) admits six Lie point symmetries, namely

$$\begin{aligned}
&\mathbf{X}_1 = \frac{\partial}{\partial t},\ \mathbf{X}_2 = (-\alpha y + 2\beta x)\frac{\partial}{\partial u} + 8\alpha\beta t\frac{\partial}{\partial x} + 8\beta^2 t\frac{\partial}{\partial y},\\
&\mathbf{X}_3 = 3t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} - u\frac{\partial}{\partial u},\ \mathbf{X}_4 = \frac{\alpha}{(\alpha+\beta)}\frac{\partial}{\partial x} + \frac{\beta}{(\alpha+\beta)}\frac{\partial}{\partial y},\\
&\mathbf{X}_5 = 4p(t)\beta\frac{\partial}{\partial x} + yp'(t)\frac{\partial}{\partial u},\ \mathbf{X}_6 = q(t)\frac{\partial}{\partial u}.
\end{aligned}$$

Case 2. $\alpha = -\beta$

Again equation (1.2) has six symmetries. These are

$$\begin{aligned}
&\mathbf{X}_1 = \frac{\partial}{\partial t},\ \mathbf{X}_2 = \frac{\partial}{\partial y},\ \mathbf{X}_3 = 3t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} - u\frac{\partial}{\partial u},\\
&\mathbf{X}_4 = (2x+3y)\frac{\partial}{\partial u} + 8t\beta\frac{\partial}{\partial y},\ \mathbf{X}_5 = 4p(t)\beta\frac{\partial}{\partial x} + yp'(t)\frac{\partial}{\partial u},\ \mathbf{X}_6 = q(t)\frac{\partial}{\partial u}.
\end{aligned}$$

2.1. Symmetry reductions of (1.2)

In this section we construct symmetry reductions and exact solutions of equation (1.2). Firstly, we consider Case 1 when $\alpha \neq -\beta$. Here we get the following subcases.

Case 1.1.

We begin with $\mathbf{X_4}$ which transform (1.2) into a partial differential equation in two independent variables. The symmetry $\mathbf{X_4}$ yields the following three invariants:

$$f = t, \quad g = -\frac{\alpha y - \beta x}{\beta}, \quad \phi = u.$$

Using the above invariants, we then transform equation (1.2) into

$$\phi_{fg} - 2\alpha\phi_{gg}\phi_g = 0. \tag{2.2}$$

The Lie point symmetries of equation (2.2) are

$$\Upsilon_1 = 2f\alpha\frac{\partial}{\partial g} - g\frac{\partial}{\partial \phi}, \quad \Upsilon_2 = 4f^2\alpha\frac{\partial}{\partial f} + 4fg\alpha\frac{\partial}{\partial g} - g^2\frac{\partial}{\partial \phi}, \quad \Upsilon_3 = \frac{\partial}{\partial g},$$
$$\Upsilon_4 = g\frac{\partial}{\partial g} + 2\phi\frac{\partial}{\partial \phi} \quad \Upsilon_5 = \frac{\partial}{\partial f}, \quad \Upsilon_6 = f\frac{\partial}{\partial f} - \phi\frac{\partial}{\partial \phi}, \quad \Upsilon_7 = G(f)\frac{\partial}{\partial \phi}.$$

Considering a linear combination $\mu\Upsilon_3 + \Upsilon_5$, one obtains the invariants

$$z = \frac{\mu f - g}{\mu}, \quad \Psi = \phi$$

and this leads to following nonlinear ordinary differential equation

$$2\alpha\Psi''(z)\Psi'(z) - \mu^2\Psi''(z) = 0, \tag{2.3}$$

whose solution is

$$\Psi(z) = C_1 z + C_2, \tag{2.4}$$

where C_1 and C_2 are constants of integration. Using equation (2.4) and reverting back into the original variables, the group-invariant solution of equation (1.2) is

$$u(t, x, y) = \frac{\beta\mu C_1 t + \alpha C_1 y - \beta C_2 x + \beta\mu C_2}{\beta\mu}. \tag{2.5}$$

Case 1.2.

We now consider Υ_6 and one obtains the following invariants

$$z = g, \quad \psi = f\phi.$$

Employing these invariants, equation (2.2) reduces to the following nonlinear ordinary differential equation

$$2\alpha\psi''(z)\psi'(z) + \psi'(z) = 0. \tag{2.6}$$

The solution of equation (2.6) is

$$\psi(z) = -\frac{1}{4}\frac{z^2}{\alpha} + C_1 z + C_2, \tag{2.7}$$

where C_1 and C_2 are constants. Invoking equation (2.7) and reverting back into the original variables, the group-invariant solution of equation (1.2) is given by

$$u(t,x,y)=\frac{1}{4}\frac{-\frac{4\alpha C_1(\alpha y-\beta x)}{\beta}+4\alpha C_2-\frac{(\alpha y-\beta x)^2}{\beta^2}}{\alpha t}. \tag{2.8}$$

Case 1.3.

We now choose $\mathbf{\Upsilon_4}$ and one gets the following invariants

$$z=f,\quad \Psi=\frac{\phi}{g^2}$$

and this leads to the nonlinear ordinary differential equation

$$-4\alpha\Psi^2(z)+\Psi'(z)=0, \tag{2.9}$$

whose solution is

$$\Psi(z)=\frac{1}{(-4\alpha z+C_1)}, \tag{2.10}$$

where C_1 an integration constant. Employing (2.10) and relapsing back into the original variables, we get

$$u(t,x,y)=\frac{(\alpha y-\beta x)^2}{\beta^2(-4\alpha t+C_1)}, \tag{2.11}$$

as the solution of equation (1.2).

Case 1.4.

Choosing $\mathbf{\Upsilon_2}$, one obtains two invariants, namely

$$z=\frac{f}{g},\quad \psi=\frac{1}{4}\frac{4\alpha f\phi+g^2}{\alpha f}, \tag{2.12}$$

which gives the following nonlinear ordinary differential equation

$$z\psi''(z)\psi'(z)+2(\psi'(z))^2=0, \tag{2.13}$$

whose solutions is

$$\psi(z)=C_1+\frac{C_2}{z}. \tag{2.14}$$

Consequently, we conclude that the solution of equation (1.2) is

$$u(t,x,y)=\frac{1}{4}\frac{4\alpha C_1 t-\frac{4\alpha C_2(\alpha y-\beta x)}{\beta}-\frac{(\alpha y-\beta x)^2}{\beta^2}}{\alpha t}, \tag{2.15}$$

where C_1 and C_2 are constants.

Case 1.5.

Taking $\mathbf{X_2}$, equation (1.2) transforms to a partial differential equation in two independent variables. The symmetry $\mathbf{X_2}$ yields the following three invariants, viz.,

$$f=t,\quad g=-\frac{\alpha y-\beta x}{\beta},\quad \phi=\frac{1}{16}\frac{16\beta^2tu+3\alpha y^2-4\beta xy}{\beta^2 t}.$$

By employing the above invariants, we transform equation (1.2) into

$$2\alpha f\phi_{gg}\phi_g - f\phi_{fg} - g\phi_{gg} - \phi_g = 0.$$

The Lie point symmetries of the above equation are

$$\mathbf{\Upsilon_1} = g\frac{\partial}{\partial g} + 2\phi\frac{\partial}{\partial \phi}, \quad \mathbf{\Upsilon_2} = 4\alpha\frac{\partial}{\partial f} - \frac{g^2}{f^2}\frac{\partial}{\partial \phi}, \quad \mathbf{\Upsilon_3} = 2f\frac{\partial}{\partial f} + g\frac{\partial}{\partial g},$$
$$\mathbf{\Upsilon_4} = f^2\frac{\partial}{\partial f} + fg\frac{\partial}{\partial g}, \quad \mathbf{\Upsilon_5} = 2\alpha\frac{\partial}{\partial g} + \frac{g}{f}\frac{\partial}{\partial \phi}, \quad \mathbf{\Upsilon_6} = f\frac{\partial}{\partial g}, \quad \mathbf{\Upsilon_7} = \frac{H(f)}{f^2}\frac{\partial}{\partial \phi}.$$

Now considering symmetry $\mathbf{\Upsilon_1}$, one gets two invariants, namely

$$z = f, \quad \Psi = \frac{\phi}{g^2}$$

and this leads to the following nonlinear ordinary differential equation

$$-4\alpha z(\Psi)^2 + z\Psi'(z) + 2\Psi = 0, \tag{2.16}$$

whose solution is

$$\Psi(z) = \frac{1}{z(zC_1 + 4\alpha)}, \tag{2.17}$$

where C_1 is a constant of integration. As a results, we conclude that the group-invariant solution of equation (1.2) is

$$u(t,x,y) = -\frac{1}{16}\frac{3\alpha ty^2C_1 - 4\beta txyC_1 - 4\alpha^2y^2 + 16\alpha\beta xy - 16\beta^2x^2}{\beta^2t(tC_1 + 4\alpha)}. \tag{2.18}$$

Case 1.6.

We now work with $\mathbf{\Upsilon_3}$ and we obtain two invariants, namely

$$z = \frac{f}{g^2}, \quad \psi = \phi$$

and this yileds the following nonlinear ordinary differential equation

$$8\alpha z^3\psi''(z)\psi'(z) + 12\alpha z^2(\psi'(z))^2 + z\psi''(z) + \psi'(z) = 0, \tag{2.19}$$

whose solution is

$$\begin{aligned}\psi(z) \; &= \; -\frac{C_1}{-1+\sqrt{16\alpha C_1 z+1}} - \frac{C_1}{1+\sqrt{16\alpha C_1 z+1}} - C_1\ln\Big(1+\sqrt{16\alpha C_1 z+1}\Big) \\ &\quad +C_1\ln\Big(-1+\sqrt{16\alpha C_1 z+1}\Big) + \frac{1}{8\alpha z} + C_2,\end{aligned} \tag{2.20}$$

where C_1 and C_2 are arbitrary constants of integration. Thus, the group-invariant solution of equation (1.2) is given by

$$u(t,x,y) \; = \; \left[-\frac{1}{(\alpha y-\beta x)^2\Big[-1+\sqrt{\frac{16\alpha\beta^2C_1t}{(\alpha y-\beta x)^2}+1}\Big]\Big(1+\sqrt{\frac{16\alpha\beta^2C_1t}{(\alpha y-\beta x)^2}+1}\Big)}\right] \times \Bigg\{$$

$$C_1\Bigg(16C_1\ln\left[1+\sqrt{\frac{16\alpha\beta^2C_1t}{(\alpha y-\beta x)^2}+1}\right]\alpha\beta^2t-$$
$$16C_1\ln\left[-1+\sqrt{\frac{16\alpha\beta^2C_1t}{(\alpha y-\beta x)^2}+1}\right]\alpha\beta^2t+2\alpha^2y^2\sqrt{\frac{16\alpha\beta^2C_1t}{(\alpha y-\beta x)^2}+1}$$
$$-4\alpha\beta xy\sqrt{\frac{16\alpha\beta^2C_1t}{(\alpha y-\beta x)^2}+1}+2\beta^2x^2\sqrt{\frac{16\alpha\beta^2C_1t}{(\alpha y-\beta x)^2}+1}$$
$$-16\alpha\beta^2C_2t+\alpha^2y^2-2\beta^2x^2\Bigg)\Bigg\}. \tag{2.21}$$

Case 1.7.

Considering the scalings symmetry $\mathbf{X_3}$, we convert equation (1.2) into a partial differential equation in two independent variables. This symmetry $\mathbf{X_3}$ yields the following three invariants, namely

$$f=\frac{y}{x},\quad g=\frac{t}{x^3},\quad \phi=ux.$$

Employing the above invariants, equation (1.2) reduces to the following nonlinear partial differential equation

$$\begin{aligned}
&-3g\phi_{gg}-f\phi_{fg}-\beta f^3\phi_{ffff}-36\beta f\phi_{ff}+81\alpha g^4\phi_{gggg}+\alpha f^4\phi_{ffff}+1692\alpha g^2\phi_{gg}\\
&-27\beta g^3\phi_{fggg}+96\alpha f\phi_f-324\alpha g^2(\phi_g)^2-24\beta f^2(\phi_f)^2+24\beta f(\phi_f)^2+816\alpha g\phi_g\\
&-12\beta f^2\phi_{fff}-162\beta g^2\phi_{fgg}+72\alpha f^2\phi_{ff}-186\beta g\phi_{fg}+756\alpha g^3\phi_{ggg}+16\beta\phi\phi_f+\\
&16\alpha f^3\phi_{fff}+24\alpha\phi-24\beta\phi_f-12\alpha\phi^2-4\phi_g+36\beta g^2\phi_{fg}\phi_g+108\alpha g^3\phi_{fggg}-\\
&162\alpha g^3\phi_g\phi_{gg}-36\alpha f\pi\phi_f-144\alpha g\phi\phi_g+96\beta g\phi_f\phi_g-6\alpha f^3\phi_f\phi_{ff}-6\alpha f^2\phi\phi_{ff}+\\
&8\beta f^2\phi f\phi_{ff}+4\beta f\phi\phi_{ff}+12\alpha f^3g\phi_{fffg}+180\alpha f^2g\phi_{ffg}-9\beta f^2g\phi_{fffg}+744\alpha fg\phi_{fg}\\
&-90\beta fg\phi_{ffg}+12\beta g\phi\phi_{fg}+54\alpha f^2g^2\phi_{ffgg}+648\alpha fg^2\phi_{fgg}-27\beta fg^2\phi_{ffgg}-\\
&54\alpha g^2\phi\phi_{gg}+36\beta g^2\phi_f\phi_{gg}-54\alpha fg^2\phi_f\phi_{gg}-108\alpha fg^2\phi_g\phi_{fg}-36\alpha f^2g\phi_f\phi_{fg}-\\
&18\alpha f^2g\phi_{ff}\phi_g-36\alpha fg\phi\phi_{fg}+36\beta fg\phi_f\phi_{fg}+12\beta fg\phi_{ff}\phi_g-180\alpha fg\phi_f\phi_g=0.
\end{aligned} \tag{2.22}$$

Consequently, we conclude that the group-invariant solution of equation (1.2) is

$$u(t,x,y)=\frac{1}{x}\phi\left(\frac{y}{x},\frac{t}{x^3}\right), \tag{2.23}$$

where ϕ is any solution of equation (2.22).

Lastly, we consider Case 2 when $\alpha=-\beta$. Here we obtain the following subcases.

Case 2.1.

Taking the linear combination of the translation symmetries $\mathbf{\Gamma}=\mathbf{X_1}+\mathbf{X_2}$ and thereafter, solving the characteristics equations yields the following three invariants:

$$f=x,\quad g=t-y,\quad \phi=u.$$

Employing the above invariants, we transformed equation (1.2) into a partial differential equation with two independent variables, namely

$$\phi_{fg}-\beta\phi_{ffff}-\beta\phi_{fffg}-6\beta\phi_{ff}\phi_f-4\beta\phi_{fg}\phi_f-4\beta\phi_{ff}\phi_g=0.$$

The above equation admits the following Lie point symmetries

$$\mathbf{\Upsilon_1} = 4f\beta\frac{\partial}{\partial f} + 4g\beta\frac{\partial}{\partial g} + (-4\beta\phi + 2f - 3g)\frac{\partial}{\partial \phi}, \quad \mathbf{\Upsilon_2} = \frac{\partial}{\partial f}, \, \mathbf{\Upsilon_3} = \frac{\partial}{\partial f} + \frac{\partial}{\partial g}, \quad \mathbf{\Upsilon_4} = \frac{\partial}{\partial \phi}.$$

Considering a linear combination of $\mathbf{\Upsilon_3} + \mathbf{\Upsilon_4}$, one obtains the following the invariants

$$z = f - g, \quad \Psi = -g + \phi$$

and this leads to the following nonlinear ordinary differential equation
$2\beta\Psi''(z)\Psi'(z) - 4\beta\Psi''(z) - \Psi''(z) = 0$, whose solution is $\Psi(z) = 2z + \frac{1}{2}\frac{z}{\beta} + C_1$.

Consequently, we conclude that the group-invariant solution of equation (1.2) is

$$u(t,x,y) = \frac{1}{2}\frac{2\beta C_1 - 2\beta t + 4\beta x + 2\beta y - t + x + y}{\beta}, \tag{2.24}$$

where C_1 is a constant of integration.

Case 2.2.

We now choose the combination of symmetries $\mathbf{\Gamma} = \mathbf{X_1} + \mathbf{X_2}$. Solving the Lagrange system, we get the following three invariants:

$$f = x, \quad g = t - y, \quad \phi = u.$$

Invoking the above invariants, equation (1.2) transforms into a partial differential equation, namely
$\phi_{fg} - \beta\phi_{ffff} - \beta\phi_{fffg} - 6\beta\phi_{ff}\phi_f - 4\beta\phi_{fg}\phi_f - 4\beta\phi_{ff}\phi_g = 0$,
which possess the follow Lie point symmetries

$$\mathbf{\Upsilon_1} = 4f\beta\alpha\frac{\partial}{\partial f} + 4g\beta\alpha\frac{\partial}{\partial g} + (-\beta\phi + 2f - 3g)\frac{\partial}{\partial \phi}, \quad \mathbf{\Upsilon_2} = \alpha\frac{\partial}{\partial f}, \, \mathbf{\Upsilon_3} = \frac{\partial}{\partial f} + \frac{\partial}{\partial g}, \quad \mathbf{\Upsilon_4} = \frac{\partial}{\partial \phi}.$$

Considering $\mathbf{\Upsilon_3}$, one obtains the invariants

$$z = f - g, \quad \psi = \phi$$

and this leads to following nonlinear ordinary differential equation
$2\beta\psi''(z)\psi'(z) - \psi''(z) = 0$, whose solution is $\psi(z) = \frac{1}{2}\frac{z}{\beta} + C_1$.

Therefore the group-invariant solution of equation (1.2) is

$$u(t,x,y) = \frac{1}{2}\frac{2\beta C_1 - t + x + y}{\beta}, \tag{2.25}$$

where C_1 is an integration constant.

Case 2.3.

Taking symmetries, $\mathbf{\Gamma} = \mathbf{X_4}$, we get the following three invariants:

$$f = t, \quad g = x, \quad \phi = \frac{1}{16}\frac{16\beta t u - 4xy - 3y^2}{\beta t}.$$

Using the above invariants, equation (1.2) transformed into
$6\beta f\phi_{gg}\phi_g + \beta f\phi_{gggg} - f\phi_{fg} - g\phi_{gg} - \phi_g = 0$.

The Lie point symmetries of the above equation are

$$\mathbf{\Upsilon_1} = 12\beta\frac{\partial}{\partial f} - \frac{g^2}{f^2}\frac{\partial}{\partial \phi}, \mathbf{\Upsilon_2} = 3f\frac{\partial}{\partial f} + g\frac{\partial}{\partial g} - \phi\frac{\partial}{\partial \phi}, \, \mathbf{\Upsilon_3} = 6\beta\frac{\partial}{\partial g} + \frac{g}{f}\frac{\partial}{\partial \phi}, \mathbf{\Upsilon_4} = f\frac{\partial}{\partial g}, \mathbf{\Upsilon_5} = \frac{R(f)}{f^2}\frac{\partial}{\partial \phi}.$$

Using $\boldsymbol{\Upsilon_2}$, one obtains the invariants

$$z = \frac{f}{g^3}, \quad \Psi = g\phi$$

and this leads to following nonlinear ordinary differential equation

$$\begin{aligned}
&81\beta z^5\Psi''''(z) - 162\beta z^4\Psi'(z)\Psi''(z) + 756\beta z^4\Psi'''(z) - 54\beta z^3\Psi(z)\Psi''(z)\\
&-324\beta z^3(\Psi'(z))^2 + 1692\beta z^3\Psi''(z) - 144\beta z^2\Psi(z)\Psi'(z) + 816\beta z^2\Psi'(z)\\
&-12\beta z(\Psi(z))^2 - 6z^2\Psi''(z) + 24\beta z\Psi(z) - 11z\Psi'(z) - \Psi(z) = 0. \qquad (2.26)
\end{aligned}$$

Consequently, the group-invariant solution of equation (1.2) is

$$u(t,x,y) = \frac{xy}{4\beta t} + \frac{3y^2}{16\beta t} + \frac{\Psi(z)}{x}, \;\; z = \frac{t}{x^3}, \qquad (2.27)$$

where $\Psi(z)$ is any solution of equation (2.26).

Case 2.4.

Considering the linear combination of the translation symmetries, $\boldsymbol{\Gamma} = \mathbf{X_1} + \mathbf{X_2}$ and solving the characteristics equations, yields the following three invariants:

$$f = x, \quad g = t - y, \quad \phi = u.$$

Employing the above invariants, equation (1.2) becomes

$\phi_{fg} - \beta\phi_{ffff} - \beta\phi_{fffg} - 6\beta\phi_{ff}\phi_f - 4\beta\phi_{fg}\phi_f - 4\beta\phi_{ff}\phi_g = 0.$

The Lie point symmetries of the above equation are given by

$$\boldsymbol{\Upsilon_1} = 4f\beta\frac{\partial}{\partial f} + 4g\beta\frac{\partial}{\partial g} + (-4\beta\phi + 2f - 3g)\frac{\partial}{\partial \phi}, \quad \boldsymbol{\Upsilon_2} = \frac{\partial}{\partial f}, \boldsymbol{\Upsilon_3} = \frac{\partial}{\partial f} + \frac{\partial}{\partial g}, \quad \boldsymbol{\Upsilon_4} = \frac{\partial}{\partial \phi}.$$

Considering $\boldsymbol{\Upsilon_1}$, one obtains the invariants

$$z = \frac{f}{g}, \quad \psi = \frac{1}{8}\frac{g(8\beta\phi - 2f + 3g)}{\beta}$$

and this leads to a nonlinear ordinary differential equation, namely

$$8z\psi''(z)\psi'(z) + z\psi''''(z) + 4\psi(z)\psi''(z) - 6\psi''(z)\psi'(z) + 8(\psi'(z))^2 - \psi''''(z) + 4\psi'''(z) = 0. \quad (2.28)$$

Therefore we conclude that the group-invariant solution of equation (1.2) is

$$u(t,x,y) = \frac{x}{4\beta} - \frac{3(t-y)}{8\beta} + \frac{\psi(z)}{t-y}, \;\; z = \frac{x}{t-y} \qquad (2.29)$$

with $\psi(z)$ being any solution of equation (2.28).

3. Conservation laws

In this section we derive the *low-order* conservation laws of the equation (1.2) using the multiplier approach. Here we will consider the multiplier of the second order, namely
$\Lambda = \Lambda(t, x, y, u, u_t, u_x, u_y, u_{tx}, u_{ty}, u_{xy}, u_{tt}, u_{xx}, u_{yy})$. The determining equation for the multiplier Λ is

$$\frac{\delta}{\delta u}\left\{(\Lambda)(u_{tx} + \alpha u_{xxxx} + 6\beta u_{xxxy} + 6\alpha u_{xx}u_x + 4\beta u_{xy}u_x + 4\beta u_{xx}u_y)\right\} = 0.$$

Expanding the above equation with the aid of Maple computer algebra package prompts the following second order multiplier Λ, namely

$$\Lambda = 4\beta k_1(t)u_x + k_1'(t)y + C_1u_x + k_2(t), \tag{3.1}$$

where $k_1(t), k_2(t)$ are arbitrary functions of t and β, C_1 are arbitrary constants. Corresponding to the above second order multiplier Λ, we obtain the following conservation laws

$$\begin{aligned}
T_1^t &= \frac{1}{2}{u_x}^2,\\
T_2^x &= -\frac{4}{3}\beta u u_x u_{xy} + 2\alpha u_x^3 + \frac{4}{3}\beta u_x^2 u_y + 2\beta u_x u_y + \alpha u_x u_{xxx} + 3\beta u_y u_{xxx} - \frac{1}{2}\alpha u_{xx}^2\\
&\quad -3\beta u_{xx}u_{xy} + 3\beta u_x u_{xxy} + 3\beta u u_{xxxy} + 2\beta u u_{xy},\\
T_3^y &= \frac{4}{3}\beta u u_x u_{xx} - 2\beta u u_{xx} - 3\beta u u_{xxxx};\\
T_1^t &= u_x\left(2\beta F(t)u_x + F'(t)y\right),\\
T_2^x &= -\frac{16}{3}\beta^2 u u_x u_{xy} + 8\alpha\beta F(t)u_x^3 + \frac{16}{3}\beta^2 F(t)u_x^2 u_y + 12\beta^2 u u_{xxxy} + 8\beta^2 F(t)u u_{xy}\\
&\quad +4\alpha\beta F(t)u_x u_{xxx} + 12\beta^2 F(t)u_x u_{xxy} + 8\beta^2 F(t)u_x u_y - 2\alpha\beta F(t){u_{xx}}^2 - 12\beta^2 F(t)u_{xx}u_{xy}\\
&\quad +12\beta^2 F(t)u_y u_{xxx} - 2\beta F'(t)y u u_{xy} + 3\alpha {u_x}^2 F'(t)y + 2\beta F'(t)u_x u_y y - 2\beta F'(t)u u_x +\\
&\quad \alpha F'(t)y u_{xxx} + 6\beta F'(t)y u_{xxy} + 4\beta F'(t)y u_y - F''(t)yu,\\
T_3^y &= \frac{2}{3}\beta u\left(8\beta F(t)u_x u_{xx} - 12\beta F(t)u_{xx} - 18\beta F(t)u_{xxxx} + 3F'(t)y u_{xx}\right);\\
T_1^t &= F(t)u_x,\\
T_2^x &= -2\beta F(t)u u_{xy} + 3\alpha F(t)u_x^2 + 2\beta F(t)u_x u_y + \alpha F(t)u_{xxx} + 6\beta F(t)u_{xxy}\\
&\quad +4\beta F(t)u_y - F'(t)u,\\
T_3^y &= 2\beta F(t)u_{xx} - 2\beta u u_{xx}.
\end{aligned}$$

associated with $C_1, k_1(t)$ and $k_2(t)$ respectively. Here we observe that due to the presence of the arbitrary functions in the conservation laws, one can generate an infinite number of conservation laws for equation (1.2).

4. Concluding remarks

In this paper new exact solutions and conservation laws were computed for a generalized (2+1)-dimensional Bogoyavlensky-Konopelchenko equation (1.2). The Lie symmetry method was used to derive exact solutions and the multiplier method was employed to compute conservation laws. The generalized (2+1)-dimensional Bogoyavlensky-Konopelchenko equation (1.2) consists of an infinite number of local conservation laws due to the arbitrary elements embedded in the conserved quantities. Furthermore, higher order conservation laws for a generalized (2+1)-dimensional Bogoyavlensky-Konopelchenko equation can be derived by increasing the order of the multiplier. However, this remains to be studied elsewhere.

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