Int. J. Nonlinear Anal. Appl. 13 (2022) No. 1, 1-28
ISSN: 2008-6822 (electronic)
http://dx.doi.org/10.22075/ijnaa.2021.22858.2422

Solvability and numerical method for non-linear Volterra integral equations by using Petryshyn's fixed point theorem

Amar Deep[a], Ashish Kumar[b], Syed Abbas[c], Mohsen Rabbani[d,*]

[a]*Mathematics Discipline, PDPM-IIITDM, Jabalpur 482005 (M.P), India*
[b]*Department of Mathematics, Govt. PG college Bina 470113 (M.P), India*
[c]*School of Basic Sciences, IIT, Mandi 175005 (H.P), India*
[d]*Department of Mathematics, Sari Branch, Islamic Azad University, Sari, Iran*

(Communicated by Madjid Eshaghi Gordji)

Abstract

In this paper, utilizing the technique of Petryshyn's fixed point theorem in Banach algebra, we analyze the existence of solution for functional integral equations, which includes as special cases many functional integral equations that arise in various branches of non-linear analysis and its application. Finally, we introduce the numerical method formed by modified homotopy perturbation approach to resolving the problem with acceptable accuracy.

Keywords: Fixed point theorem, Measure of non-compactness(MNC), Banach algebra, Functional integral equation(FIE), Modified Homotopy perturbation(MHP)
2010 MSC: 47H10; 90C39

FIEs are regarded as a part of the applications of non-linear analysis. It create a very important and significant part of the theories of radiative transfer, mathematical physics, population dynamics, kinetic theory of gases and neutron transport [1, 4, 10, 11, 12, 21]. Recently, the concept of MNC is one of the most efficient tools in non-linear analysis to study the solvability of FIEs and differential equations [14, 16, 17, 24, 25, 27, 33]. We study a different existence result for the solution of FIEs

*Corresponding author
Email addresses: amar54072@gmail.com (Amar Deep), ashishgulia1992@gmail.com (Ashish Kumar), abbas@iitmandi.ac.in (Syed Abbas), mrabbani@iausari.ac.ir (Mohsen Rabbani)

Received: March 2021 *Accepted:* August 2021

as,

$$u(s) = q\left(s, u(\theta(s)), \nu(s, u(\zeta(s)), \int_0^{\alpha(s)} h(s,r,u(\phi(r)))dr, \int_0^{\beta(s)} f(s,r,u(\varphi(r)))dr\right) \tag{0.1}$$

$$u(s) = q\left(s, \nu(s, u(\zeta(s)), \int_0^{\alpha(s)} h(s,r,u(\phi(r)))dr, u(\theta(s)) \int_0^{\beta(s)} f(s,r,u(\varphi(r)))dr\right) \tag{0.2}$$

$$u(s) = q\left(s, u(\zeta(s)), u(\theta(s)) \int_0^{c} h(s,r,u(\phi(r)))dr, \int_0^{\beta(s)} f(s,r,u(\varphi(r)))dr\right) \tag{0.3}$$

where, $s \in I_c = [0, c]$. Eq.(1) is the generalization of the some equations which introduced by [3, 7, 8, 15, 19, 22],etc. We apply the concept of MNC and Petryshyn's fixed point theorem[29], which is generalization of Darbo's fixed point theorem[5]. The main aim is to find the existence result of Eq.(1) and also, work to gain the analytic solution of it by applying the semi-analytic method. Many useful papers have been studied to the existence result for various FIE by Darbo's fixed point theorem (see[6, 7, 8, 18, 22, 23]). Now, we discuss the principal reason why we examine Eqs.(1,0.2,0.3) and what we perform. The first importance is that the conditions given in various papers will be investigated and the second reason is that it joins similar work in this area. The third condition is the bounded condition explains that the "sublinear condition" that has been recognized in literature will not play a meaningful role here.
This article is divided into 4 sections including the introduction. Section 2, we show some preliminaries and define the concept of MNC. Section 3 is applied to state and prove an existence result for Eq.(1) using Petryshyn's theorem. In last, we provide some examples that test the utilization of FIE.

1. Preliminaries

In this paper, assume Δ be a real Banach space. Let $B(u, \sigma)$ be a closed ball centre at u with radius $\sigma > 0$. MNC are very powerful tools in non-linear analysis, for example in the operator equations theory and fixed point theory in Banach space.

Definition 1.1. *[5] The Kuratowski MNC*

$$\gamma(\Omega) = inf\{\rho > 0 : \Omega = \bigcup_{j=1}^{m} \Omega_j \ with\ diam\Omega_j \leq \rho,\ j = 1, 2, ..., m\}.$$

Definition 1.2. *[5] The Hausdroff MNC*

$$\chi(\Omega) = \inf\{\rho > 0 : there\ exists\ a\ finite\ \rho\ net\ for\ \Omega\ in\ \Delta\ \}, \tag{1.1}$$

here, from a finite ρ net for Ω in Δ it implies, as a set $\{u_1, u_2, ..., u_m\} \subset \Delta$ such that $B_\rho(\Delta, u_1), B_\rho(\Delta, u_2), ..., B$ over Ω. These MNC are similar in the sense that

$$\chi(\Omega) \leq \gamma(\Omega) \leq 2\chi(\Omega),\ for\ any\ bounded\ set\ \Omega \subset \Delta.$$

Theorem 1.3. *Let $\Omega, Z \subset \Delta$ bounded and $\lambda \in \mathbb{R}$. Then*

(i) $\chi(\Omega) = 0$ *iff Ω is precompact;*

(ii) $\Omega \subseteq Z \implies \chi(\Omega) \leq \chi(Z)$*;*

(iii) $\chi(\bar{co}\Omega) = \chi(\Omega)$*;*

(iv) $\chi(\Omega \cup Z) = \max\{\chi(\Omega), \chi(Z)\}$;

(v) $\chi(\lambda\Omega) = |\lambda|\chi(\Omega)$;

(vi) $\chi(\Omega + Z) \leq \chi(\Omega) + \chi(Z)$.

Let $C[0, c]$be the space of all real valued and continuous function defined on the $[0, c]$ with the standard norm, $||u|| = \max\{|u(s)| : s \in [0, c]\}$. $C[0, c]$ is also the structure of Banach algebra. The modulus of continuity of u defined as

$$\omega(u, \rho) = \sup\{|u(s) - u(\eta)| : \eta, s \in [0, c], |s - \eta| \leq \rho\}.$$

Theorem 1.4. *[22] The Hausdorff MNC is similar to*

$$\mu(\Omega) = \lim_{\rho \to 0} sup\ \Omega(u, \rho), \ \ for\ all\ \Omega \subset C[0, c]. \tag{1.2}$$

Theorem 1.5. *[26] Assume that $F : \Delta \to \Delta$ is a continuous mapping of Δ which holds the condition if for all $\Omega \subset \Delta$ with Ω bounded, $F(\Omega)$ is bounded and $\gamma(F\Omega) \leq k\gamma(\Omega), k \in (0, 1)$. If $\gamma(F\Omega) < \gamma(\Omega), \quad for\ all\ \gamma(\Omega) > 0$, then F is called condensing or densifying map.*

Theorem 1.6. *Petryshyn's[29] Suppose that $F : B_\sigma \to \Delta$ be a densifying mapping, which fulfill the boundary condition.*

$$If\ F(u) = ku, \quad for\ some\ u \in \partial B_\sigma \ \ then\ k \leq 1,$$

then the set of fixed points of F in B_σ is nonempty.

2. Main Results

Here, we shall treat Eq.(1) for u belongs to $C[0, c]$ under the following assumptions;

(T_1) $q \in C(I_c \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$, $\nu \in C(I_c \times \mathbb{R}, \mathbb{R})$, $h \in C(I_c \times [0, A_1] \times \mathbb{R}, \mathbb{R})$, $f \in C(I_c \times [0, A_2] \times \mathbb{R}, \mathbb{R})$, and $\alpha, \beta : I_c \to \mathbb{R}^+$, $\phi : [0, A_1] \to I_d$, $\varphi : [0, A_2] \to I_c$, $\theta, \zeta : I_c \to I_c$, are continuous so $\alpha(s) \leq A_1, \beta(s) \leq A_2 \ \ \forall\ s \in I_c$;

(T_2) $\exists$ non-negative constants $d_1 + d_2 d_5 < 1$, so

$$|q(s, v_1, v_2, v_3, v_4) - q(s, x_1, x_2, x_3, x_4| \leq d_1|v_1 - x_1| + d_2|v_2 - x_2| + d_3|v_3 - x_3| + d_4|v_4 - x_4|;$$

$$|\nu(s, v_1) - \nu(s, v_2)| \leq d_5|v_1 - x_1|.$$

(T_3) $\exists$ a $\sigma > 0$ of the inequality

$$sup\{|q(s, v_1, v_2, v_3, v_4)| : s \in I_c, v_1, v_2 \in [-\sigma, \sigma], v_3 \in [-A_1 N_1, A_1 N_1]$$
$$, v_4 \in [-A_2 N_2, A_2 N_2]\} \leq \sigma,$$

where,

$$N_1 = sup\{|h(s, r_1, u)| : \forall s \in I_c, r_1 \in [0, A_1] \ \ and \ \ u \in [-\sigma, \sigma]\},$$
$$N_2 = sup\{|f(s, r_2, u)| : \forall s \in I_c, r_2 \in [0, A_2] \ \ and \ \ u \in [-\sigma, \sigma]\}.$$

Theorem 2.1. *Under the above hypotheses Eq.(1) has at least one solution in $\Delta = C(I_c)$.*

Proof . Define $F : B_\sigma \to \Delta$ as

$$(Fu)(s) = q\left(s, u(\theta(s)), \nu(s, u(\zeta(s)), \int_0^{\alpha(s)} h(s,r,u(\phi(r)))dr, \int_0^{\beta(s)} f(s,r,u(\varphi(r)))dr\right).$$

To prove that F is continuous on B_σ. Takeing $\rho > 0$ and for any $u, v \in B_\sigma$ such that $||u - v|| < \rho$. We have

$$\begin{aligned}
&|(Fu)(s) - (Fv)(s)| = \\
&\left| q\left(s, u(\theta(s)), \nu(s, u(\zeta(s)), \int_0^{\alpha(s)} h(s,r,u(\phi(r)))dr, \int_0^{\beta(s)} f(s,r,u(\varphi(r)))dr\right)\right. \\
&\left. -q\left(s, v(\theta(s)), \nu(s, v(\zeta(s)), \int_0^{\alpha(s)} h(s,r,v(\phi(r)))dr, \int_0^{\beta(s)} f(s,r,v(\varphi(r)))dr\right)\right| \\
\leq\ &\left| q\left(s, u(\theta(s)), \nu(s, u(\zeta(s)), \int_0^{\alpha(s)} h(s,r,u(\phi(r)))dr, \int_0^{\beta(s)} f(s,r,u(\varphi(r)))dr\right)\right. \\
&\left. -q\left(s, v(\theta(s)), \nu(s, u(\zeta(s)), \int_0^{\alpha(s)} h(s,r,u(\phi(r)))dr, \int_0^{\beta(s)} f(s,r,u(\varphi(r)))dr\right)\right| \\
&+\left| q\left(s, v(\theta(s)), \nu(s, u(\zeta(s)), \int_0^{\alpha(s)} h(s,r,u(\phi(r)))dr, \int_0^{\beta(s)} f(s,r,u(\varphi(r)))dr\right)\right. \\
&\left. -q\left(s, v(\theta(s)), \nu(s, v(\zeta(s)), \int_0^{\alpha(s)} h(s,r,u(\phi(r)))dr, \int_0^{\beta(s)} f(s,r,u(\varphi(r)))dr\right)\right| \\
&+\left| q\left(s, v(\theta(s)), \nu(s, v(\zeta(s)) \int_0^{\alpha(s)} h(s,r,u(\phi(r)))dr, \int_0^{\beta(s)} f(s,r,u(\varphi(r)))dr\right)\right. \\
&\left. -q\left(s, v(\theta(s)), \nu(s, v(\zeta(s)), \int_0^{\alpha(s)} h(s,r,v(\phi(r)))dr, \int_0^{\beta(s)} f(s,r,u(\varphi(r)))dr\right)\right| \\
&+\left| q\left(s, v(\theta(s)), \nu(s, v(\zeta(s)) \int_0^{\alpha(s)} h(s,r,v(\phi(r)))dr, \int_0^{\beta(s)} f(s,r,u(\varphi(r)))dr\right)\right. \\
&\left. -q\left(s, v(\theta(s)), \nu(s, v(\zeta(s)), \int_0^{\alpha(s)} h(s,r,v(\phi(r)))dr, \int_0^{\beta(s)} f(s,r,v(\varphi(r)))dr\right)\right| \\
\leq\ & d_1|u(\theta(s)) - v(\theta(s))| + d_2|\nu(s, u(\zeta(s)) - \nu(s, v(\zeta(s))| + d_3 \int_0^{\alpha(s)} |h(s,r,u(\phi(r))) - \\
& h(s,r,v(\phi(r)))|dr + d_4 \int_0^{\beta(s)} |f(s,r,v(\varphi(r))) - f(s,r,v(\varphi(r)))|dr \\
\leq\ & d_1|u(\theta(s)) - v(\theta(s))| + d_2 d_5|u(\zeta(s)) - v(\zeta(s))| + d_3 A_1 \omega(h,\rho) + d_4 A_2 \omega(f,\rho) \\
\leq\ & (d_1 + d_2 d_5)\|u - v\| + d_3 A_1 \omega(h,\rho) + d_4 A_2 \omega(f,\rho)
\end{aligned}$$

where,

$$\omega(h,\rho) = sup\{|h(s,r,u) - h(s,r,v)| : s \in I_c, r \in [0, A_1], u, v \in [-\sigma, \sigma], \|u - v\| \leq \rho\},$$

$$\omega(f,\rho) = sup\{|f(s,r,u) - f(s,r,v)| : s \in I_c, r \in [0, A_2], u, v \in [-\sigma, \sigma], \|u - v\| \le \rho\}.$$

Since $h(s,r,u)$ and $f(s,r,u)$ are uniform continuous on $I_c \times [0, A_1] \times \mathbb{R}$ and $I_c \times [0, A_2] \times \mathbb{R}$, we conclude that $\omega(h,\rho)$ and $\omega(f,\rho)$ as $\rho \to 0$. Hence, F is continuous on B_σ. Again, we show that F satisfies the densifying map. Fixed a arbitrary $\rho > 0$ and take $u \in \Omega$, where Ω is bounded subset of Δ, $s_1, s_2 \in I_c$ with $s_1 \le s_2$ such that $s_1 - s_2 \le \rho$, we get

$$\begin{aligned}
&|(Fu)(s_2) - (Fu)(s_1)| = \\
&\left| q\left(s_2, u(\theta(s_2)), \nu(s_2, u(\zeta(s_2)), \int_0^{\alpha(s_2)} h(s_2, r, u(\phi(r)))dr, \int_0^{\beta(s_2)} f(s_2, r, u(\varphi(r)))dr\right)\right. \\
&\left. -q\left(s_1, u(\theta(s_1)), \nu(s_1, u(\zeta(s_1)), \int_0^{\alpha(s_1)} h(s_1, r, u(\phi(r)))dr, \int_0^{\beta(s_1)} f(s_1, r, u(\varphi(r)))dr\right)\right| \\
\le\ &\left| q\left(s_2, u(\theta(s_2)), \nu(s_2, u(\zeta(s_2)), \int_0^{\alpha(s_2)} h(s_2, r, u(\phi(r)))dr, \int_0^{\beta(s_2)} f(s_2, r, u(\varphi(r)))dr\right)\right. \\
&\left. -q\left(s_2, u(\theta(s_2)), \nu(s_2, u(\zeta(s_2)), \int_0^{\alpha(s_2)} h(s_2, r, u(\phi(r)))dr, \int_0^{\beta(s_1)} f(s_1, r, u(\varphi(r)))dr\right)\right| \\
&+\left| q\left(s_2, u(\theta(s_2)), \nu(s_2, u(\zeta(s_2)), \int_0^{\alpha(s_2)} h(s_2, r, u(\phi(r)))dr, \int_0^{\beta(s_1)} f(s_1, r, u(\varphi(r)))dr\right)\right. \\
&\left. -q\left(s_2, u(\theta(s_2)), \nu(s_2, u(\zeta(s_2)), \int_0^{\alpha(s_1)} h(s_1, r, u(\phi(r)))dr, \int_0^{\beta(s_1)} f(s_1, r, u(\varphi(r)))dr\right)\right| \\
&+\left| q\left(s_2, u(\theta(s_2)), \nu(s_2, u(\zeta(s_2)), \int_0^{\alpha(s_1)} h(s_1, r, u(\phi(r)))dr, \int_0^{\beta(s_1)} f(s_1, r, u(\varphi(r)))dr\right)\right. \\
&\left. -q\left(s_2, u(\theta(s_2)), \nu(s_1, u(\zeta(s_1)), \int_0^{\alpha(s_1)} h(s_1, r, u(\phi(r)))dr, \int_0^{\beta(s_1)} f(s_1, r, u(\varphi(r)))dr\right)\right| \\
&+\left| q\left(s_2, u(\theta(s_2)), \nu(s_1, u(\zeta(s_1)) \int_0^{\alpha(s_1)} h(s_1, r, u(\phi(r)))dr, \int_0^{\beta(s_1)} f(s_1, r, u(\varphi(r)))dr\right)\right. \\
&\left. -q\left(s_2, u(\theta(s_1)), \nu(s_1, u(\zeta(s_1)) \int_0^{\alpha(s_1)} h(s_1, r, u(\phi(r)))dr, \int_0^{\beta(s_1)} f(s_1, r, u(\varphi(r)))dr\right)\right| \\
&+\left| q\left(s_2, u(\theta(s_1)), \nu(s_1, u(\zeta(s_1)) \int_0^{\alpha(s_1)} h(s_1, r, u(\phi(r)))dr, \int_0^{\beta(s_1)} f(s_1, r, u(\varphi(r)))dr\right)\right. \\
&\left. -q\left(s_1, u(\theta(s_1)), \nu(s_1, u(\zeta(s_1)) \int_0^{\alpha(s_1)} h(s_1, r, u(\phi(r)))dr, \int_0^{\beta(s_1)} f(s_1, r, u(\varphi(r)))dr\right)\right| \\
\le\ & d_1|u(\theta(s_2)) - u(\theta(s_1))| + d_2|\nu(s_2, u(\zeta(s_2)) - \nu(s_2, u(\zeta(s_1))| \\
&+d_2|\nu(s_2, u(\zeta(s_1)) - \nu(s_1, u(\zeta(s_1))| + d_3 \int_0^{\alpha(s_1)} |(h(s_2, r, u(\phi(r)) - h(s_1, r, u(\phi(r)))|dr \\
&+d_3 \int_{\alpha(s_1)}^{\alpha(s_2)} |h(s_2, r, u(\phi(r))))|dr + d_4 \int_0^{\beta(s_1)} |f(s_2, r, u(\varphi(r))) - f(s_1, r, u(\varphi(r)))|dr
\end{aligned}$$

$$+ \int_{\beta(s_1)}^{\beta(s_2)} |f(s_2, r, u(\varphi(r)))|dr + \omega_q(I_c, \rho).$$

For simplify,

$$\begin{aligned}
&\omega_q(I_c, \rho) = sup\{|q(s, v_1, v_2, v_3, v_4) - q(\hat{s}, v_1, v_2, v_3, v_4)| : |s - \hat{s}| \leq \rho, s \in I_c, v_1, v_2 \in [-\sigma, \sigma], \\
&\qquad v_3 \in [-A_1N_1, A_1N_1], v_4 \in [-A_2N_2, A_2N_2\}. \\
&\omega_h(I_c, \rho) = sup\{|h(s, r, u) - h(\hat{s}, r, u)| : |s - \hat{s}| \leq \rho, s \in I_c, r \in [0, A_1], u \in [-\sigma, \sigma]\}, \\
&\omega_f(I_c, \rho) = sup\{|f(s, r, u) - f(\hat{s}, r, u)| : |s - \hat{s}| \leq \rho, s \in I_c, r \in [0, A_2], u \in [-\sigma, \sigma]\}, \\
&\omega_\nu(I_c, \rho) = sup\{|\nu(s, v_1) - \nu(\hat{s}, v_1)| : |s - \hat{s}| \leq \rho, s \in I_c, v_1 \in [-\sigma, \sigma]\}, \\
&\omega(\alpha, \rho) = sup\{|\alpha(s) - \alpha(\hat{s})| : |s - \hat{s}| \leq \rho, s, \hat{s} \in I_c\}.
\end{aligned}$$

From above relation, we get

$$\begin{aligned}
|(Fu)(s_2) - (Fu)(s_1)| &\leq d_1|u(\theta(s_2) - u(\theta(s_1)| + d_2d_5|u(\zeta(s_2) - u(\zeta(s_1)| + d_2\omega_\nu(I_c, \rho) \\
&+ d_3A_1\omega_h(I_c, \rho) + d_3N_1\omega(\alpha, \rho) + d_4A_2\omega_f(I_c, \rho) + d_4N_2\omega(\beta, \rho) + \omega_q(I_c, \rho).
\end{aligned}$$

Taking limit as $\rho \to 0$, we get $\omega(Fu, \rho) \leq (d_1 + d_2d_5)\omega(u, \rho)$, this gives the relation $\chi(F\Omega) \leq (d_1 + d_2d_5)\chi(\omega)$, then F is a condensing map. Let $u \in \partial B_\sigma$ and if $Fu = ku$ then we obtain $||Fu|| = k||u|| = k\sigma$ and by (T_3),

$$|Fu(s)| = \left| q\left(s, u(\theta(s)), \nu(s, u(\zeta(s)), \int_0^{\alpha(s)} h(s, r, u(\phi(r)))dr, \int_0^{\beta(s)} f(s, r, u(\varphi(r)))dr\right)\right| \leq \sigma$$

$\forall s \in I_c$, hence $||Fu|| \leq \sigma$ i.e $k \leq 1$. □ Second, we will study the Eq.(0.2) under the following assumptions,

(B_1) $q \in C(I_c \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$, $\nu \in C(I_c \times \mathbb{R}, \mathbb{R})$, $h \in C(I_c \times [0, A_1] \times \mathbb{R}, \mathbb{R})$, $f \in C(I_c \times [0, A_2] \times \mathbb{R}, \mathbb{R})$, and $\alpha, \beta : I_c \to \mathbb{R}^+$, $\phi : [0, A_1] \to I_d$, $\varphi : [0, A_2] \to I_c$, $\theta, \zeta : I_c \to I_c$, are continuous so $\alpha(s) \leq A_1, \beta(s) \leq A_2 \ \ \forall \ s \in I_c$;

(B_2) $\exists$ non-negative constants $d_1d_4 + d_3A_2N_2 < 1$ so

$$|q(s, v_1, v_2, v_3) - q(s, x_1, x_2, x_3| \leq d_1|v_1 - x_1| + d_2|v_2 - x_2| + d_3|v_3 - x_3|;$$

$$|\nu(s, v_1) - \nu(s, v_2)| \leq d_4|v_1 - x_1|.$$

(B_3) $\exists$ a $\sigma > 0$ of the inequality

$$sup\{|q(s, v_1, v_2, v_3)| : s \in I_c, v_1 \in [-\sigma, \sigma], v_2 \in [-A_1N_1, A_1N_1], v_3 \in [-A_2N_2, A_2N_2]\} \leq \sigma,$$

where, $\quad N_1 = sup\{|h(s, r_1, u)| : \forall s \in I_c, r_1 \in [0, A_1] \ \ and \ \ u \in [-\sigma, \sigma]\},$

$$N_2 = sup\{|f(s, r_2, u)| : \forall s \in I_c, r_2 \in [0, A_2] \ \ and \ \ u \in [-\sigma, \sigma]\}.$$

Then Eq.(0.2) has at least one solution in I_c.
Proof . The proof is relevant to the Theorem 2.1 and leave this parts. □ Third, we will study the Eq.(0.3) under the following assumptions,

(G_1) $q \in C(I_c \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$, $h \in C(I_c \times I_c \times \mathbb{R}, \mathbb{R})$, $f \in C(I_c \times [0, A] \times \mathbb{R}, \mathbb{R})$, and $\beta : I_c \to \mathbb{R}^+$, $\varphi : [0, A] \to I_c$, $\theta, \zeta : I_c \to I_c$, are continuous so $\beta(s) \le A \;\; \forall\, s \in I_c$.

(G_2) $\exists$ non-negative constants $d_1 + d_2 c N_1 < 1$, so

$$|q(s, v_1, v_2, v_3) - q(s, x_1, x_2, x_3| \le d_1|v_1 - x_1| + d_2|v_2 - x_2| + d_3|v_3 - x_3|.$$

(G_3) $\exists$ a $\sigma > 0$ of the inequality

$$sup\{|q(s, v_1, v_2, v_3)| : s \in I_c, v_1 \in [-\sigma, \sigma], v_2 \in [-cN_1, cN_1], v_3 \in [-AN_2, AN_2]\} \le \sigma,$$

where,

$$N_1 = sup\{|h(s, r_1, u)| : \forall s, r_1 \in I_c, \quad and \quad u \in [-\sigma, \sigma]\},$$
$$N_2 = sup\{|f(s, r_2, u)| : \forall s \in I_c, r_2 \in [0, A] \quad and \quad u \in [-\sigma, \sigma]\}.$$

Then Eq.(0.3) has at least one solution in I_c.
Proof . The proof is relevant to the Theorem 2.1 and leave this parts. □

Corollary 2.2. *[3] Suppose that*

(D_1) $q \in C(I_c \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$, $h \in C(I_c \times [0, A_1] \times \mathbb{R}, \mathbb{R})$, *and* $\alpha : I_c \to \mathbb{R}^+$, $\phi : [0, A_1] \to I_c$, $\theta : I_c \to I_c$, *are continuous such that* $\alpha(s) \le A_1 \;\; \forall\, s \in I_c$;

(D_2) *There exists non-negative constant* $d \in (0, 1)$ *such that*

$$|q(s, v_1, v_2) - q(s, x_1, x_2| \le d(|v_1 - x_1| + |v_2 - x_2|);$$

and there exists non-negative constants b_1 *such that;* $|q(s, 0, 0)| \le b_1$

(D_3) *There exists constants* c_1 *and* c_2 *such that;* $|h(s, r, u)| \le c_1 + c_2|u|$ *,*
Moreover $d + A_1 d c_2 < 1$.

Then the following equation has at least one solution in I_c.

$$u(s) = q\left(s, u(\theta(s)), \int_0^{\alpha(s)} h(s, r, u(\phi(r)))dr\right), s \in I_c = [0, c]. \tag{2.1}$$

Proof . Let $\sigma = \frac{F_2}{1-F_1}$, where $F_1 = d + A_1 d c_2, F_2 = A_1 d c_1 + b_1$, and $q(s, v_1, v_2, v_3, v_4) = q(s, v_1, v_3)$, also $v_1 = u(\theta(s))$ and $v_2 = \int_0^{\alpha(s)} h(s, r, u(\phi(r))dr$. We see that (T_2) is led by (D_2). Again, we show that D_3 is also fulfill, then

$$\begin{aligned}
|u(s)| &= \left|q\left(s, u(\theta(s)), \int_0^{\alpha(s)} h(s, r, u(\phi(r)))dr\right)\right|, \\
&\le d|u(\theta(s))| + d\left|\int_0^{\alpha(s)} h(s, r, u(\phi(r)))dr\right| + |q(s, 0, 0)|, \\
&\le (d + A_1 d c_2)||u|| + A_1 d c_1 + b_1, \; for\ all\ s \in I_c,
\end{aligned}$$

consequently, $sup|q(s, v_1, v_3)| \le F_1\sigma + F_2 = F_1\frac{F_2}{1-F_1} + F_2 = \sigma$.

Corollary 2.3. *Suppose that*

(E_1) *$q \in C(I_c \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$, $K \in C(I_c \times \mathbb{R}, \mathbb{R})$, $h \in C(I_c \times [0, A_1] \times \mathbb{R}, \mathbb{R})$, $f \in C(I_c \times [0, A_2] \times \mathbb{R}, \mathbb{R})$, and $\alpha, \beta : I_c \to \mathbb{R}^+$, $\phi : [0, A_1] \to I_c$, $\varphi : [0, A_2] \to I_c$, $\theta : I_c \to I_c$, are continuous such that $\alpha(s) \leq A_1, \beta(s) \leq A_2 \ \ \forall \ s \in I_c$;*

(E_2) *There exists non-negative constants $d \in (0, 1)$ and b_1, b_2 such that*

$$|q(s, v_1, v_2) - q(s, x_1, x_2| \leq d(|v_1 - x_1| + |v_2 - x_2|), \ |K(s, v_1) - K(s, x_1| \leq d|v_1 - x_1|;$$

$$|K(s, 0)| \leq b_1, \quad |q(s, 0, 0)| \leq b_2.$$

(E_3) *There exists the constants c_1, c_2, c_3 and c_4 such that*

$$|h(s, r, u)| \leq c_1 + c_2|u|, \ |f(s, r, u)| \leq c_3 + c_4|u|,$$

Moreover, $d + c_2A_1d + c_4A_2d < 1$.

Then the following equation has at least one solution in $I_c = [0, c]$.

$$u(s) = K(s, u(\theta(s)) + q\left(s, \int_0^{\alpha(s)} h(s, r, u(\phi(r)))dr, \int_0^{\beta(s)} f(s, r, u(\varphi(r)))dr\right), \tag{2.2}$$

Proof . Let $\sigma = \frac{H_2}{1-H_1}$, where $H_1 = d + c_2A_1d + c_4A_2d, \quad H_2 = b_1 + c_1A_1d + c_3A_2d + b_2$, and

$$q(s, v_1, v_2, v_3, v_4) = K(s, v_1) + q(s, v_3, v_4),$$

where $v_1 = K(s, u(\theta(t)), v_2 = \int_0^{\alpha(s)} h(s, r, u(\phi(r)))dr$ and $v_3 = \int_0^{\beta(s)} f(s, r, u(\varphi(r)))dr$. We see that ($T_2$) is led by ($E_2$). Now, we show that E_3 is also fulfill, then

$$\begin{aligned}
|u(s)| &= \left|K(s, u(\theta(s)) + q\left(s, \int_0^{\alpha(s)} h(s, r, u(\phi(r)))dr, \int_0^{\beta(s)} f(s, r, u(\varphi(r)))dr\right)\right|, \\
&\leq |K(s, u(\theta(s)) - K(s, 0)| + |K(s, 0)| + \Big|q(s, \int_0^{\alpha(s)} h(s, r, u(\phi(r)))dr, \\
&\quad \int_0^{\beta(s)} f(s, r, u(\varphi(r)))dr - q(s, 0, 0)\Big| + |q(s, 0, 0)|, \\
&\leq d||u|| + b_1 + A_1d(c_1 + c_2||u||) + A_2d(c_3 + c_4||u||) + b_2, \\
&\leq (d + c_2A_1d + c_4A_2d)||u|| + b_1 + c_1A_1d + c_3A_2d + b_2,
\end{aligned}$$

for all $s \in I_c$, consequently $sup|q(s, v_1, v_2, v_3, v_4)| \leq H_1\sigma + H_2 = H_1\frac{H_2}{1-H_1} + H_2 = \sigma$. □

Corollary 2.4. *[19] Let*

(J_1) *$h, r \in C(I \times I \times \mathbb{R}, \mathbb{R})$, $q \in C(I \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$, $\forall \ s \in I = [0, 1]$,*

(J_2) *There exists constant $d \in (0, 1)$ such that*

$$|q(s, v_1, v_2, v_3) - q(s, x_1, x_2, x_3| \leq d(|v_1 - x_1| + |v_2 - x_2| + |v_3 - x_3|);$$

and there exists non-negative constants b_1 such that, $|q(s, 0, 0)| \leq b_1$;

(J_3) *There exists the non-negative constants* c_1, c_2, c_3 *and* c_4 *such that*

$$|h(s,r,u)| \leq c_1 + c_2|u|, |f(s,r,u)| \leq c_3 + c_4|u|, \textit{and moreover}, d + dc_2 + dc_4 < 1.$$

Then the following equation has at least one solution in $I = [0,1]$.

$$u(s) = q\left(s, u(s), \int_0^1 h(s,r,u(r))dr, \int_0^s f(s,r,u(r))dr\right), s \in I. \tag{2.3}$$

Proof . Let $\sigma = \frac{Q_1}{1-Q_2}$, where $Q_1 = (d + dc_2 + dc_4), Q_2 = dc_1 + dc_3 + b_1$, and $q(s, v_1, v_2, v_3, v_4) = q(s, v_1, v_3, v_4)$, in here $\phi(r) = \varphi(r) = r, \theta(s) = s, \alpha(s) = 1, \beta(s) = s$, $v_2 = \int_0^1 h(s,r,u(r))dr$ and $v_3 = \int_0^s f(s,r,u(r))dr$. Thus (T_2) is conducted by (J_2). We show that (J_3) is holds

$$\begin{aligned}
|u(s)| &= \left|q\left(s, u(s), \int_0^1 h(s,r,u(r))dr, \int_0^s f(s,r,u(r))dr\right)\right|, \\
&\leq d|u(s)| + d\left|\int_0^1 h(s,r,u(r))dr\right| + d\left|\int_0^s f(s,r,u(r))dr\right| + |q(s,0,0)|, \\
&\leq d||u|| + d(c_1 + c_2||u||) + d(c_3 + c_4||u||) + b_1, \\
&\leq (d + dc_2 + dc_4)||u|| + dc_1 + dc_3 + b_1,
\end{aligned}$$

for all $s \in I = [0,1]$, consequently $sup|q(s, v_1, v_3, v_4)| \leq Q_1\sigma + Q_2 = \sigma$. □

Corollary 2.5. *Putting* $q(s, v_1, v_2, v_3, v_4) = q(s, v_2, v_3, v_4)$ *and* $A_1 = A_2 = A$, *Eq.*(1) *reduces to following FIE studied in [15]*

$$u(s) = q\left(s, \nu(s, u(\zeta(s)), \int_0^{\alpha(s)} h(s,r,u(\phi(r)))dr, \int_0^{\beta(s)} f(s,r,u(\varphi(r)))dr\right). \tag{2.4}$$

Corollary 2.6. *Replacing* $q(s, v_1, v_2, v_3, v_4) = q(s, v_1, v_2, v_3), \nu(s, v_1) = v_1$ *and* $A_2 = A$, *Eq.*(1) *reduces to following FIE studied in [22]*

$$u(s) = q\left(s, u(\theta(s)), u(\zeta(s)), \int_0^{\alpha(s)} h(s,r,u(\phi(r)))dr\right). \tag{2.5}$$

Corollary 2.7. *Putting* $q(s, v_1, v_2, v_3, v_4) = \tilde{q}(s, v_1) + v_2 v_3$, *and* $A_1 = A$, *Eq.*(1) *reduces to following FIE studied in [28]*

$$u(s) = \tilde{q}(s, u(\theta(s))) + \nu(s, u(\zeta(s)) \int_0^{\alpha(s)} h(s,r,u(\phi(r)))dr. \tag{2.6}$$

Corollary 2.8. *Substituting* $q(s, v_1, v_2, v_3, v_4) = 1 + v_1 v_3, \theta(s) = \alpha(s) = \phi(s) = s$, *and* $h(s,r,u) = \frac{s}{s+r}\psi(r)u$. *Then Eq.*(1) *reduces to Chandrasekhar integral equation in radiative transfer [11].*

$$u(s) = 1 + u(s) \int_0^s \frac{s}{s+r}\psi(r)u(r)dr, s \in I_c = [0,c]. \tag{2.7}$$

Corollary 2.9. *Putting* $q(s, v_1, v_2, v_3, v_4) = a(s) + v_3, \alpha(s) = s$ *and* $\phi(r) = r$, *Eq.*(1) *reduces to Volterra Urysohn integral equation*

$$u(s) = a(s) + \int_0^s h(s,r,u(r))dr. \tag{2.8}$$

Corollary 2.10. *Putting* $q(s, v_1, v_2, v_3, v_4) = b(s) + v_4, \beta(t) = c$ *and* $\varphi(r) = r$, *Eq.*(1) *reduces to Urysohn integral equation*

$$u(s) = b(s) + \int_0^c f(s,r,u(r))dr. \tag{2.9}$$

3. An Example

Example 3.1. *Let the following Volterra non-linear FIE:*

$$\begin{aligned} u(s) = \frac{1}{3}(\frac{s^2}{1+s^2})arctan(|u(\sqrt{s})|) + \frac{1}{3}\int_0^s \frac{e^{-2s^2}r\ sin(u(r))}{4+|sin(u(r))|}dr \\ + \frac{1}{3}\int_0^{s^2} e^{-3r}(e^{\sqrt{r}} + \sqrt{s}\ cos(r) + \frac{1}{2}u(r^2))dr, \qquad s \in [0,1] \end{aligned} \tag{3.1}$$

Put, $\theta(s) = \sqrt{s}, \alpha(s) = s, \phi(s) = s, \beta(s) = s^2, \varphi(s) = s^2, \forall\ s \in [0,1]$,

$$q(s, v_1, v_2, v_3, v_4) = q_1(s, v_1, v_2) + q_2(s, v_3, v_4),$$

where, $q_1(s, v_1, v_2) = 0v_1 + \frac{1}{3}v_2,\ q_2(s, v_3, v_4) = \frac{v_3}{3} + \frac{v_4}{3},\ v_2 = \big(\frac{s^2}{1+s^2}\big)arctan(|u(\sqrt{s})|)$,

$$v_3 = \int_0^{\alpha(s)} h(s, r, u(\phi(r)))dr,\ v_4 = \int_0^{\beta(s)} f(s, r, u(\varphi(r)))dr,$$

and

$$h(s, r, u(\phi(r))) = \frac{e^{-2s^2}r\ sin(u(r))}{4+|sin(u(r))|}, f(s, r, u(\phi(r))) = e^{-3r}(e^{\sqrt{r}} + \sqrt{s}\ sin(r) + \frac{1}{2}u(r^2)).$$

It is seen that these functions holds (T_1) and (T_2). Now, we check that (T_3) also holds. Choose $\sigma = \frac{5}{6} + \frac{2}{3}e$ then $N_1 \leq \frac{1}{4}, N_2 \leq \frac{4e}{3} + \frac{17}{12}$ and

$$sup\{|q(s, v_1, v_2, v_3, v_4)| : s \in [0,1], v_1, v_2 \in [-\sigma, \sigma], v_3 \in [-AN_1, AN_1]\}, v_4 \in [-AN_2, AN_2]\}$$

$$\leq sup\{|\frac{1}{3}(u(s) + v_2 + v_3)|; s \in [0,1], -\frac{1}{4} \leq v_2 \leq \frac{1}{4}, -(\frac{4e}{3} + \frac{17}{12}) \leq v_3 \leq (\frac{4e}{3} + \frac{17}{12})\} \leq \frac{5}{6} + \frac{2}{3}e$$

Hence, from Theorem 2.1 Eq.(3.1) has at least one solution in $C[0,1]$. □

4. An iterative algorithm to solve Example4.1

Here, we utilize a sequence of MHP and Adomian decomposition method to solve Eq.(3.1). Homotopy perturbation is a powerful concept in perturbations theory and topology[9, 20]. Any modifications of MHP and Adomian decomposition method can be viewed in [13, 30, 31] and [2, 32] respectively. In this proposed method, we discreet a non-linear functional equation to some smaller difficulties and to free of non-linearity, we apply a linear combination of Adomian polynomials. Therefore we present an iterative algorithm to solve the above problem.
Now, take the general form of Eq.(3.1) as,

$$M(s, u(s)) - g(s, u(s)) = 0, \quad s \in [0,1] \tag{4.1}$$

where M is a non-linear integral operator and g is a function. According to [30, 31], divide the operator M to some nonlinear or linear operators as M_1 and M_2. Also g converts to g_1 and g_2 . Thus (4.1) can be represented by $M_1(u) - g_1(s) + M_2(u) - g_2(s, u(s)) = 0$. Therefore we can define a MHP in this form:

$$H(\vartheta, p) = M_1(\vartheta) - g_1(s) + p(M_2(\vartheta) - g_2(s, \vartheta(s))) = 0, \quad p \in [0,1] \tag{4.2}$$

$$u(s) \simeq \vartheta(s) = \vartheta_0(s) + p\vartheta_1(s) + p^2\vartheta_2(s) + p^3\vartheta_3(s) + \dots, \tag{4.3}$$

here p is an embedding parameter. Putting parameter $p = 0$ to $p = 1$ we can get $M_1(\vartheta) = g_1(s)$ to $M(\vartheta) = g(s, u(s))$. In this way, we achieve the solution of (4.1) for $p = 1$ and $u(s) \simeq \lim_{p\to 1} \vartheta(s)$. To introduce operators M_1, M_2 and functions g_1, g_2 , we consider to the non-linear functional Volterra integral equation (3.1) as follows,

$$
\begin{aligned}
&u(s) + \int_0^s k_1(s,r)\frac{sin(u(r))}{4+|sin(u(r))|}dr + \int_0^{s^2} k_2(s,r)(2e^{\sqrt{r}} + 2\sqrt{s}\ cos(r) + u(r^2))ds \\
&\qquad - \frac{1}{3}(\frac{s^2}{1+s^2})arctan(|u(\sqrt{s})|) = 0, \qquad s \in [0,1], \qquad (4.4)\\
&k_1(s,r) = -\frac{1}{3}re^{-2s^2}, \quad k_2(s,r) = -\frac{1}{6}e^{-3r}.
\end{aligned}
$$

Then, we have

$$
\begin{aligned}
&M_1(u(s)) = u(s),\\
&M_2(u(s)) = \int_0^s \frac{k_1(s,r)\ sin(u(r))}{4+|sin(u(r))|}dr + \int_0^{s^2} k_2(s,r)(2e^{\sqrt{r}} + 2\sqrt{s}\ cos(r) + u(r^2))dr,\\
&g_1(s) = 0, \quad g_2(s,u(s)) = \frac{1}{3}(\frac{s^2}{1+s^2})arctan(|u(\sqrt{s})|). \qquad (4.5)
\end{aligned}
$$

Substituting (4.5) and (4.3) in (4.2) leads to,

$$
\begin{aligned}
&\Big(\sum_{i=0}^{\infty} p^i\vartheta_i(s)) - g_1(s)\Big) + p\Big(\int_0^s k_1(s,r)\frac{sin(\sum_{i=0}^{\infty} p^i\vartheta_i(r))}{4+|sin(\sum_{i=0}^{\infty} p^i\vartheta_i(r))|}dr \qquad (4.6)\\
&\quad + \int_0^{s^2} k_2(s,r)(2e^{\sqrt{r}} + 2\sqrt{s}\ cos(r) + \sum_{i=0}^{\infty} p^i\vartheta_i(r^2))dr - g_2(s, \sum_{i=0}^{\infty} p^i\vartheta_i(s))\Big) = 0.
\end{aligned}
$$

For relief, operator M_2 is converted to operators $\widehat{M_2}$ and $\widehat{\widehat{M_2}}$ and we use Adomian polynomials for approximate nonlinear terms,

$$
\begin{aligned}
&\widehat{M_2}(\sum_{i=0}^{\infty} p^i\vartheta_i(r)) = \frac{sin(\sum_{i=0}^{\infty} p^i\vartheta_i(r))}{4+|sin(\sum_{i=0}^{\infty} p^i\vartheta_i(r))|} = \sum_{i=0}^{\infty} p^i\widehat{A}_i(r)\\
&\widehat{\widehat{M_2}}(\sum_{i=0}^{\infty} p^i\vartheta_i(r)) = \sum_{i=0}^{\infty} p^i\vartheta_i(r^2) = \sum_{i=0}^{\infty} p^i\widehat{\widehat{A}}_i(r) \qquad (4.7)\\
&g_2(s, \sum_{i=0}^{\infty} p^i\vartheta_i(s)) = \sum_{i=0}^{\infty} p^iA_i(s). \qquad p \in [0,\ 1].
\end{aligned}
$$

In which Adomian polynomials are given as,

$$
\begin{aligned}
&\widehat{A}_k(r) = \frac{1}{k!}\big(\frac{d^k}{dp^k}\frac{sin(\sum_{i=0}^{\infty} p^i\vartheta_i(r))}{4+|sin(\sum_{i=0}^{\infty} p^i\vartheta_i(r))|}\big)_{p=0},\\
&\widehat{\widehat{A}}_k(r) = \frac{1}{k!}\big(\frac{d^k}{dp^k}\sum_{i=0}^{\infty} p^i\vartheta_i(r^2)\big)_{p=0}, \qquad (4.8)\\
&A_k(s) = \frac{1}{k!}\big(\frac{d^k}{dp^k}g_2(s, \sum_{i=0}^{\infty} p^i\vartheta_i(s))\big)_{p=0}.
\end{aligned}
$$

Substituting (4.7) in (4.6) implies that,

$$\begin{aligned}&\big(\vartheta_0(s)+p\vartheta_1(s)+p^2\vartheta_2(s)+\cdots-g_1(s)\big)+p\Big(\int_0^s k_1(s,r)\sum_{i=0}^{\infty}p^i\widehat{A}_i(r)dr\\&+\int_0^{s^2}k_2(s,r)(2e^{\sqrt{r}}+2\sqrt{s}\ cos(r)+\sum_{i=0}^{\infty}p^i\widehat{\widehat{A}}_i(r))dr-\sum_{i=0}^{\infty}p^iA_i(s)\Big)=0,\end{aligned}\tag{4.9}$$

with rearranging (4.9) in terms of p powers and taking of the coefficients of p powers equal to zero, we approach an iterative algorithm for numerical solution of (3.1).

Algorithm:

$$\begin{aligned}&\vartheta_0(s)=g_1(s),\\&\vartheta_{k+1}(s)=-\int_0^s k_1(s,r)\widehat{A}_k(r)dr-\int_0^{s^2}k_2(s,r)(2e^{\sqrt{r}}+2\sqrt{s}\ cos(r)+\widehat{\widehat{A}}_k(r))dr\\&\qquad+A_k(s),\qquad k=0,1,2,\cdots\end{aligned}\tag{4.10}$$

For convergence of these kinds of algorithm see [19]. Since in (3.1), $u(0)=0$, and solution space is $C[0,1]$, then a simple choice for start point in algorithm (4.10)is $\vartheta_0(s)=g_1(s)=0$ or s. Therefore from (4.8) Adomian polynomials are,

$$\widehat{A}_0(r)=\frac{sin(\vartheta_0(r))}{4+|sin(\vartheta_0(r)|},\quad \widehat{\widehat{A}}_0(r)=\vartheta_0(r^2),\quad A_0(s)=g_2(s,\vartheta_0(s)).\tag{4.11}$$

Thus in algorithm (4.10) we can get

$$\begin{aligned}&\vartheta_0(s)=g_1(s)=0\\&\vartheta_1(s)=-\int_0^s k_1(s,r)\widehat{A}_0(r)dr-\int_0^{s^2}k_2(s,r)(2e^{\sqrt{r}}+2\sqrt{s}\ cos(r)+\widehat{\widehat{A}}_0(r))dr+A_0(s)\\&\quad=\frac{1}{9}-\frac{1}{9}e^{-3s^2+s}+\frac{\sqrt{s}}{10}-\frac{1}{10}e^{-3s^2}\sqrt{s}\ cos(s^2)+\frac{e^{1/12}\sqrt{\pi}}{18\sqrt{3}}(Erf(\frac{1}{2\sqrt{3}})-Erf(\frac{1-6s}{2\sqrt{3}}))\\&\quad\ +\frac{1}{30}e^{-3s^2}\sqrt{s}\ sin(s^2).\end{aligned}$$

We keep some approximations of the solution (3.1) as,

$$\begin{aligned}u(s)\simeq\sum_{i=0}^{1}\vartheta_i(s)&=\frac{1}{9}-\frac{1}{9}e^{-3s^2+s}+\frac{\sqrt{s}}{10}-\frac{1}{10}e^{-3s^2}\sqrt{s}\ cos(s^2)\\&+\frac{e^{1/12}\sqrt{\pi}}{18\sqrt{3}}(Erf(\frac{1}{2\sqrt{3}})-Erf(\frac{1-6s}{2\sqrt{3}}))+\frac{1}{30}e^{-3s^2}\sqrt{s}\ sin(s^2).\end{aligned}\tag{4.12}$$

By considering Figure.1, the approximate solution (4.12) is in space $C[0,1]$. For validity of this numerical result, we replace (4.12) in (3.1) and comparing both sides of it, the absolute errors in some points are given in table 1. It's axiomatic that by increasing the number of iterations in algorithm (4.10), we can improve the accuracy in the approximation.

5. Conclusions

In the current work, we studied the existence of a solution for functional non-linear Volterra integral equation. For illustrating the efficiency and applicability of our results we gave some corollary and an example respectively. Also, we offer an iterative algorithm to find the solution of the above problem with acceptable accuracy.

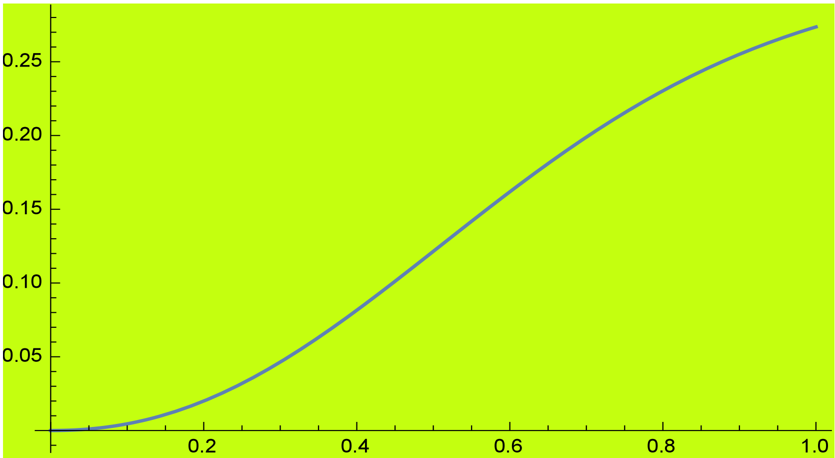

Figure 1: $u(s)$:The sum of the first two terms of the series(4.3)

Table 1: Absolute errors

s	Absolute errors for $u(s)$
0.0	0
0.1	1.7×10^{-4}
0.2	1.2×10^{-3}
0.3	3.9×10^{-3}
0.4	8.1×10^{-3}
0.5	1.3×10^{-2}
0.6	2.0×10^{-2}
0.7	2.6×10^{-2}
0.8	3.3×10^{-2}
0.9	4.0×10^{-2}
1.0	4.6×10^{-2}

References

[1] M. A. Abdou, *On the solution of linear and non-linear integral equation*, Appl. Math. Comput. 146(2003), 857-871.

[2] G. Adomian, *Solving Frontier Problem of Physics: The Decomposition Method; Kluwer Academic Press: Dordrecht the Netherlands*, 1994.

[3] A. Aghajani, Y. Jalilian, *Existence and global attractivity of solutions of non-linear functional integral equation*, Commun. Non. Sci. Number. Simul. 15(2010), 3306-3312.

[4] I. K. Argyros, *Quadratic equations and applications to Chandrasekhar and related equations* , Bull. Aus. Math. Soc. 32(1985), 275-292.

[5] J. Banas, K. Goebel, *Measures of Noncompactness in Banach Spaces*, Marcel Dekker, New York, (1980).

[6] J. Banas, M. Lecko, *Fixed points of the product of operators in Banach algebra*, Panamer. Math. J. 12 (2002), 101-109.

[7] J. Banas, B. Rzepka, *On existence and asymptotic stability of solutions of a nonlinear integral equation*, J. Math. Anal. Appl. 284 (2003), 165-173.

[8] J. Banas, K. Sadarangani, *Solutions of some functional integral equations in Banach algebra*, Math. Comput. Modelling 38 (2003), 245-250.

[9] J. Biazar, H. Ghazvini, *He homotopy perturbation method for solving systems of Volterra integral equations of the second kind*, Cha. Solit. Frac. 39(2009), 770-777.

[10] J. Caballero, A. B. Mingarelli, K. Sadarangani, *Existence of solutions of an integral equation of Chandrasekhar type in the theory of radiative transfer*, Elect. J. Diff. Eqs. 57(2006), 1-11.

[11] S. Chandrasekhar, *Radiative Thznsfer*, Oxford Univ. Press, London, 1950.

[12] C. Corduneanu, *Integral Equations and Applications*, Cambridge University Press, New York, 1990.

[13] A Das, M Rabbani, B Hazarika, R Arab, *Solvability of infinite system of nonlinear singular integral equations in the C(I $\times$ I,c) space and modified semi-analytic method to find a closed-form of solution.*, Int. J. Non. Anal.

Appl. 10(1)(2019), 63-76.

[14] A. Deep, Deepmala, J. R. Roshan, K. S. Nisar, T. Abdeljawad, *An extension of Darbo's fixed point theorem for a class of system of nonlinear integral equations*, Adv. Diff. Eqs, 2020:483(2020).

[15] A. Deep, Deepmala, R. Ezzati, *Application of Petryshyn's fixed point theorem to solvability for functional integral equations*, Appl. Math. Comput. 395(2021), 125878.

[16] Deepmala, H.K.Pathak, *A study on some problems on existence of solutions for some nonlinear functional-integral equations*, Acta. Math. Sci.33(5)(2013),1305-1313.

[17] K. Deimling, *Nonlinear Functional Analysis*, Springer-Verlag, 1985.

[18] B. C. Dhage, *On α-condensing mappings in Banach algebras*, The Math. Student 63 (1994), 146-152.

[19] B. Hazarika, H.M.Srivastava, R. Arab, M. Rabbani, *Application of simulation function and measure of noncompactness for solvability of nonlinear functional integral equations and introduction to an iteration algorithm to find solution*, Appl. Math. Comput. 360(2019), 131-146.

[20] J. H. He, *A new approach to nonlinear partial differential equations.*, Comm. Non Linear Sci. Number. Simulation 2(4)(1997), 230-235.

[21] S. Hu, M. Khavanin, W. Zhuang, *Integral equations arising in the kinetic theory of gases*, Appl. Anal. 34 (1989), 261-266.

[22] M. Kazemi, R. Ezzati, *Existence of solutions for some nonlinear Volterra integral equations via Petryshyn's fixed point theorem*, Int. J. Anal. Appl. 9(2018).

[23] K. Maleknejad, K. Nouri, R. Mollapourasl, *Existence of solutions for some nonlinear integral equations*, Commun. Nonlinear Sci. Numer. Simulat. 14 (2009), 2559-2564.

[24] S.A. Mohiuddine, H.M. Srivastava, A. Alotaibi, *Applications of measures of noncompactness to the infinite system of second order differential equations in l_p spaces*, Adv. Differ. Equ. 2016(2016), 1-13.

[25] M. Mursaleen, S.A. Mohiuddine, *Applications of noncompactness to the infinite system of differential equations in l_p spaces*, Nonlinear Anal. 4(2012), 2111-2115.

[26] R.D.Nussbaum, *The fixed point index and fixed point theorem for k set contractions*, Proquest LLC, Ann Arbor, MI, 1969. Thesis(Ph.D)- The University of Chicago.

[27] D.O'Regan, *Existence results for nonlinear integral equations*, J. Math. Anal. Appl. 192(1995), 705-726.

[28] I. Ozdemir, U. Cakan, B. Iihan, *On the existence of the solution for some non;inear Volterra integral equations*, Abstr. Appl. Anal. 5(2013).

[29] W.V. Petryshyn, *Structure of the fixed points sets of k-set-contractions*, Arch. Rational Mech. Anal., 40(1970-1971), 312-328.

[30] M. Rabbani, *An iterative algorithm to find a closed form of solution for Hammerstein nonlinear integral equation constructed by the concept of cosm-rs*, Mathematical Sciences 13 (3), (2019)299-305.

[31] B. Hazarik, M. Rabbani, RP. Agarwal, A. Das, R. Arab, *Existence of Solution for Infinite System of Nonlinear Singular Integral Equations and Semi-Analytic Method to Solve it.*, Iranian Journal of Science and Technology, Transactions A: Science 45(2021), 235-245.

[32] M. Rabbani, A. Gelayeri, *Computation Adomian Method for Solving Non-linear Differential Equation in the Fluid Dynamic.*, Int. J. Mechat., Elect. Compu. Techn. 5(14)(2015), 2039-2043.

[33] J. R. Roshan, *Existence of solutions for a class of system of functional integral equation via measure of noncompactness* J. Comput. Appl. Math. 313(2017), 129-141.