Int. J. Nonlinear Anal. Appl. 13 (2022) No. 1, 75-84
ISSN: 2008-6822 (electronic)
http://dx.doi.org/10.22075/ijnaa.2020.20984.2220

A subgradient extragradient method for equilibrium problems on Hadamard manifolds

Mahdi Ali-Akbari

Department of Computer Engineering, University of Torbat Heydarieh, Torbat Heydarieh, Iran

(Communicated by Madjid Eshaghi Gordji)

Abstract

It is generalized the subgradient extragradient algorithm from linear spaces to nonlinear cases. This algorithm introduces a method for solving equilibrium problems on Hadamard manifolds. The global convergence of the algorithm is presented for pseudomonotone and Lipschitz-type continuous bifunctions.

Keywords: Equilibrium problem, Hadamard manifold, Subgradient extragradient algorithm.
2010 MSC: Primary 90C33; Secondary 49M37.

1. Introduction

Equilibrium problem (abbreviated by EP) is a fundamental problem and arises in an extensive variety of application areas. It is a valuable mathematical tool in the study of optimization and control problems, traffic network problems, Nash equilibrium problems, the problems of finding zeros of operators, etc. These important applications employ many authors to have widely studied it in recent years (see e.g. [1, 7, 12, 18, 28], and the references therein). It is well known (see e.g. [8, 9, 20]) that various classes of mathematical programming problems such as variational inequality problems (abbreviated by VIPs), fixed point problems, and minimax problems can be formulated in the form of EP. Many methods have been extensively studied for approximating solutions of the equilibrium problem under suitable conditions, in which two general approaches are regularized methods and projection methods. Our attention is focused on the second method.

The gradient method is the simplest projection method for variational inequality problems in which iterations are generated by one projection on the feasible set. In [21], Korpelevich extended this method by calculating the second projection onto the feasible set in each iteration to reduce the

Email address: mahdialiakbari@torbath.ac.ir (Mahdi Ali-Akbari)

Received: July 2020 *Accepted:* September 2020

assumptions of convergencity. Korpelevich's method (named extragradient method and abbreviated EGM) presented for saddle point problems, and then, this method has been studied and improved in both Euclidean spaces and Hilbert spaces by many authors[13, 14, 15, 16]. In the case, when the feasible set has a simple structure, then the projection can be computed easily. However, if the feasible set is any closed convex set, the computation of projections, in general, is complicated. As the remarks of the authors in [5], this can affect the efficiency of the used method. Recently, inspired by EGM, Y. Censor et al. [4] have introduced an algorithm, which is called the subgradient extragradient method (SEGM), for solving VIPs on Hilbert spaces. In this method, they have replaced the second projection in EGM by a projection onto a specially constructed half-space and allow a clear computation. The projection on a half-space is inherently explicit, and so, the SEGM can be considered as an improvement of EGM over each computational step. At the same time, these methods have been developed or inherited for equilibrium problems (see [13, 26, 27, 28]). In [13], the author introduced SEGM for equilibrium problems by replacing two projections of SEGM in VIPs by two optimization programs.

Extensions of ideas and techniques for optimization methods from Euclidean spaces to Riemannian manifolds have some extraordinary advantages. From the Riemannian manifold point of view, it is possible to transform the nonconvex problem in linear context to a convex problem by endowing the space with an appropriate Riemannian metric (see e.g. [3, 10, 29, 30]). Due to this, it can be solved some (nonconvex) constrained optimization problems or problems with nonconvex objective functions. Actually, in recent years some methods have been retrieved from the Hilbert spaces framework to the more general setting Riemannian manifolds to solve nonlinear or nonconvex cases (see e.g. [11, 17, 22]). In particular, Hadamard manifolds supply an appropriate environment for the development of optimization methods as presented in [6, 19, 25, 23, 33]. Motivated and inspired by the research works mentioned above, we present a subgradient extragradient method for equilibrium problems on Hadamard manifolds whose iterative process retrieves the proposed in [13].

The paper is organized as follows: In Section 3, we recall some basic concepts and important results on Riemannian manifolds and convexities for further use. In Section 4, we present a subgradient extragradient method for equilibrium problems on Hadamard manifolds and prove strong convergence result of the generated sequence of iterates to the solution of the equilibrium problem under some mild condition.

2. Preliminaries

In this section, we review some basic definitions and useful properties of Riemannian manifolds, which can be found in any textbook on Riemannian geometry, see [2, 31, 32].

Let M be a connected finite-dimensional manifold. For $p \in M$, we denote T_pM the tangent space of M at p which is a vector space of the same dimension as M, and by $TM = \bigcup_{p\in M} T_pM$ the tangent bundle of M, which is naturally a manifold. We suppose that M can be endowed with a Riemannian metric $\langle .,. \rangle$, which the corresponding norm denoted by $||.||$, to become a Riemannian manifold. By using the metric, for a piecewise smooth curve $\gamma : [a, b] \to M$ joining x to y, we can define the length of γ as $L(\gamma) = \int_a^b ||\gamma^{'}(t)||dt$. Then, for any $x, y \in M$, the Riemannian distance $d(x, y)$ is defined by minimizing this length over the set of all such curves joining x to y, which induces the original topology on M.

Let ∇ be the Levi-Civita connection associated with $(M, \langle , \rangle)$. Let γ be a smooth curve in M. A vector field X is said to be *parallel* along γ if $\nabla_{\gamma^{'}} X = 0$. γ is called a *geodesic*, if $\gamma^{'}$ be parallel along γ. A geodesic joining x to y in M is said to be *minimal* if its length equals $d(x, y)$. A Riemannian manifold is *complete* iff the geodesics are defined for any values of t. By Hopf-Rinow theorem, we know that if M is complete then any pair of points in M can be joined by a minimal geodesic. Moreover, (M, d) is a complete metric space, and bounded closed subsets are compact.

Assuming that M is complete, the *exponential map* $\exp_x : T_xM \to M$ at $x \in M$ is defined by $\exp_x \nu = \gamma_\nu(1, x)$, where $\gamma(.) = \gamma_\nu(., x)$ is the geodesic starting at x with the velocity ν, i.e. $\gamma_\nu(0, x) = x$ and $\gamma_\nu^{'}(0, x) = \nu$. Then $\exp_x t\nu = \gamma_\nu(t, x)$ for each real number t. Note that for any $x \in M$, the map $\exp_x$ is differentiable on T_xM.

A *Hadamard manifold* is a complete simply connected Riemannian manifold of nonpositive sectional curvature. Throughout the remainder of the paper, we assume that M is an $n-$dimensional Hadamard manifold. The following result is well known and will be useful.

Proposition 2.1. *Let M be a Hadamard manifold and $x \in M$. Then $\exp_x : T_xM \to M$ is a diffeomorphism, and for any $x, y \in M$, there exists a unique normalized geodesic joining x to y, which is minimal.*

This property shows that M is diffeomorphic to the Euclidean space R^n. Hence, M has the same topological, differential structure, and some geometrical properties as R^n. Some of them are mentioned in the following. Recall that a *geodesic triangle* $\triangle(x_1, x_2, x_3)$ of a Riemannian manifold is a set consisting of three points x_1, x_2, x_3 and three minimal geodesics joining these points.

Proposition 2.2. *(Comparison theorem for triangle) Let $\triangle(x_1, x_2, x_3)$ be a geodesic triangle. Denote, for each $i = 1, 2, 3(mod3)$, by $\gamma_i : [0, l_i] \to M$ the geodesic joining x_i to x_{i+1}, and set $\alpha_i := \angle(\gamma_i^{'}(0), -\gamma_{i-1}^{'}(l_{i-1}))$, the angle between the vector $\gamma_i^{'}(0)$ and $-\gamma_{i-1}^{'}(l_{i-1})$, and $l_i := L(\gamma_i)$. Then*

$$\alpha_1 + \alpha_2 + \alpha_3 \leq \pi,$$
$$l_i^2 + l_{i+1}^2 - 2l_il_{i+1}\cos\alpha_{i+1} \leq l_{i-1}^2. \tag{2.1}$$

In term of the distance and the exponential map, inequality (2.1) can be rewritten as

$$d^2(x_i, x_{i+1}) + d^2(x_{i+1}, x_{i+2}) - 2\langle \exp_{x_{i+1}}^{-1} x_i, \exp_{x_{i+1}}^{-1} x_{i+2}\rangle \leq d^2(x_{i-1}, x_i), \tag{2.2}$$

since

$$\langle \exp_{x_{i+1}}^{-1} x_i, \exp_{x_{i+1}}^{-1} x_{i+2}\rangle = d(x_i, x_{i+1})d(x_{i+1}, x_{i+2})\cos\alpha_{i+1}. \tag{2.3}$$

Let $x_{i+2} = x_i$ in relation (2.3), we obtain

$$||\exp_{x_{i+1}}^{-1} x_i||^2 = \langle \exp_{x_{i+1}}^{-1} x_i, \exp_{x_{i+1}}^{-1} x_i\rangle = d^2(x_i, x_{i+1}).$$

Note that, the inequality (2.2) is an extension of the "law of cosies" in Euclidean space to Hadamard manifold, which be useful in the next.

The following lemma was proved by applying the properties of the exponential map.

Lemma 2.3. *Let $x_0 \in M$ and $\{x_n\} \subset M$ such that $x_n \to x_0$. Then the following assertion hold.*

1. *For any $y \in M$,*
$$\exp_{x_n}^{-1} y \to \exp_{x_0}^{-1} y \quad and \quad \exp_y^{-1} x_n \to \exp_y^{-1} x_0$$
2. *If $\{v_n\}$ is a sequence such that $v_n \in T_{x_n}M$ and $v_n \to v_0$, then $v_0 \in T_{x_0}M$.*
3. *Given the sequences $\{u_n\}$ and $\{v_n\}$ satisfying $u_n, v_n \in T_{x_n}M$, if $u_n \to u_0$ and $v_n \to v_0$ with $u_0, v_0 \in T_{x_0}M$. then*
$$\langle u_n, v_n\rangle \to \langle u_0, v_0\rangle.$$

A subset $K \subseteq M$ is said to be *convex* if for any two points x and y in K, the geodesic joining x to y is contained in K, i.e., if $\gamma : [a,b] \to M$ is a geodesic such that $x = \gamma(a)$ and $y = \gamma(b)$, then $\gamma(ta + (1-t)b) \in K$ for all $t \in [0,1]$. A function $f : K \to \mathbb{R}$ is said to be *convex* iff for any geodesic segment $\gamma : [a,b] \to K$ the composition $f \circ \gamma : [a,b] \to \mathbb{R}$ is convex. The *subdifferential* of a function $f : M \to \mathbb{R}$ at $x \in M$ is defined by

$$\partial f(x) = \{v \in T_x M : f(y) - f(x) \geq \langle v, \exp_x^{-1} y\rangle,\ \forall y \in M\},$$

and its elements are called *subgradient* of f at x. The subdifferential $\partial f(x)$ at $x \in M$ is a closed convex set. It is known that if f is convex and M is a Hadamard manifold, then $\partial f(x)$ is a nonempty set, for each $x \in M$. The nonempty set $N_K(x)$ defined by

$$N_K(x) = \{v \in T_x M; \langle v, \exp_x^{-1} y\rangle \leq 0, \forall y \in K\}$$

is called the *normal cone* of K at $x \in K$.

The Riemannian distance and its square play a fundamental role in the following. We proceed now stating some results which we will go to use.

Proposition 2.4. *Let M be a Hadamard manifold and $d : M \times M \to \mathbb{R}$ be the distance function. Then d is a convex function with respect to the product Riemannian metric; that is, given any pair of geodesics $\gamma_1 : [0,1] \to M$ and $\gamma_2 : [0,1] \to M$ the following inequality holds for all $t \in [0,1]$:*

$$d(\gamma_1(t), \gamma_2(t)) \leq (1-t)d(\gamma_1(0), \gamma_2(0)) + td(\gamma_1(1), \gamma_2(1)).$$

In particular, for each $y \in M$, the function $d(.,y) : M \to \mathbb{R}$ is a convex function.

Proposition 2.5. *Let M be a Hadamard manifold and $x \in M$. The map $\Phi_x(y) = d^2(x,y)$ satisfying the following;*

1. *Φ_x is convex. Indeed, for any geodesic $\gamma : [0,1] \to M$ the following inequality holds for all $t \in [0,1]$:*

$$d^2(x, \gamma(t)) \leq (1-t)d^2(x, \gamma(0)) + td^2(x, \gamma(1)) - t(1-t)d^2(\gamma(0), \gamma(1)).$$

2. *Φ_x is smooth. Moreover, $\partial\Phi_x(y) = -2\exp_y^{-1} x$*

Proposition 2.6. *Let K be a nonempty convex subset of a Hadamard manifold M and $g : K \to \mathbb{R}$ be a convex subdifferentiable and lower semicontinuous function on K. Then, x^* is a solution to the following convex problem*

$$\min\{g(x) : x \in K\}$$

if and only if $0 \in \partial g(x^*) + N_K(x^*)$.

Proof . The idea of proof can be found in [24]. □

3. Subgradient extragradient algorithm

Let M be a Hadamard manifold, K a nonempty closed and convex subset of M and $f : K\times K \to \mathbb{R}$ a bifunction with $f(x,x) = 0$ for all $x \in K$. The *equilibrium problem* for f is to find $x^* \in K$ such that

$$f(x^*, x) \geq 0, \quad \forall x \in K. \tag{3.1}$$

We denoted the solution set of equilibrium problem (3.1) by $EP(f,K)$.

Definition 3.1. A bifunction $f : M \times M \to \mathbb{R}$ is said to be

1. *monotone* on K if
$$f(x, y) + f(y, x) \leq 0, \quad \forall x, y \in K;$$
2. *pseudomonotone* on K if
$$f(x, y) \geq 0 \quad \Rightarrow \quad f(y, x) \leq 0, \quad \forall x, y \in K;$$
3. *Lipschitz-type continuous* on K if there exist two positive constants α_1 and α_2 such that
$$f(x, y) + f(y, z) \geq f(x, z) - \alpha_1 d^2(x, y) - \alpha_2 d^2(y, z), \quad \forall x, y, z \in K$$

Next, we introduce a subgradient extragradient method (SEGM) for equilibrium problems on Hadamard manifolds whose algorithm retrieves the proposed in [13].

Algorithm 3.2. Initialization. *Choose an initial point $x_0 \in K$ and a parameter λ satisfies* $0 < \lambda < \min\{\frac{1}{2c_1}, \frac{1}{2c_2}\}$.

Iterative steps. *Assume that $x_n \in K$ and we calculate $x_{n+1} \in K$ as follows:*

Step 1. *Compute*

$$y_n = argmin\{\lambda f(x_n, y) + \frac{1}{2}d^2(x_n, y) \; : \; y \in K\}.$$

If $x_n = y_n$, then stop and x_n is a solution to EP. Otherwise,

Step 2. *Compute*

$$x_{n+1} = argmin\{\lambda f(y_n, y) + \frac{1}{2}d^2(x_n, y) \; : \; y \in T_n\},$$

where $T_n = \{m \in M \; : \langle \exp^{-1}_{y_n} x_n - \lambda v_n, \exp^{-1}_{y_n} m\rangle \leq 0\}$ and $v_n \in \partial_2 f(x_n, y_n)$.

Remark 3.3. *It is easy to see that $x_n \in EP(f,K)$ if and only if $x_n = y_n$.*

The main result of this paper is Theorem 3.8, which states the convergence of the algorithm 3.2 for a bifunction with some conditions, provided the algorithm is well defined. We first recall the notion of Fejér convergence and the following related result [11].

Definition 3.4. Let X be a complete metric space and $K \subseteq X$ be a nonempty set. A sequence $\{x_n\} \subset X$ is called *Fejér convergent* to K if for every $y \in K$,

$$d(x_{n+1}, y) \leq d(x_n, y) \qquad n = 0, 1, 2, \ldots.$$

Lemma 3.5. *Let X be a complete metric space and $K \subseteq X$ be a nonempty set. If $\{x_n\} \subset X$ be Fejér convergent to K, then $\{x_n\}$ is bounded. In addition, if an accumulation point x of $\{x_n\}$ belongs to K, then $\{x_n\}$ globally converges to x.*

Unless stated to the contrary, in the remainder of this paper we assume that $f : M \times M \to \mathbb{R}$ is a bifunction satisfying the following conditions:
$(C1)$ f is pseudomonotone on K and $f(x,x) = 0$ for all $x \in M$;
$(C2)$ f is upper semicontinuous on the first variable;
$(C3)$ f is convex and lower semicontinuous on the second variable;
$(C4)$ f is Lipschitz-type continuous on K with the constants α_1, α_2;

Lemma 3.6. *For any $y \in K$ and $\lambda > 0$,*

1. $\lambda[f(x_n, y) - f(x_n, y_n)] \geq \langle \exp_{y_n}^{-1} x_n, \exp_{y_n}^{-1} y \rangle$.
2. $\lambda[f(y_n, y) - f(y_n, x_{n+1})] \geq \langle \exp_{x_{n+1}}^{-1} x_n, \exp_{x_{n+1}}^{-1} y \rangle$.

Proof . 1) Proposition 2.6 and the definition of y_n in Algorithm 3.2 implies that

$$0 \in \partial_2[\lambda f(x_n, y) + \frac{1}{2} d^2(x_n, y)](y_n) + N_C(y_n).$$

Therefore, there exist $\overline{z} \in N_C(y_n)$ and $z \in \partial_2 f(x_n, y_n)$ such that

$$\lambda z - \exp_{y_n}^{-1} x_n + \overline{z} = 0.$$

So, for any $y \in K$,

$$\langle \exp_{y_n}^{-1} x_n, \exp_{y_n}^{-1} y \rangle = \lambda \langle z, \exp_{y_n}^{-1} y \rangle + \langle \overline{z}, \exp_{y_n}^{-1} y \rangle.$$

Now, since $\overline{z} \in N_K(y_n)$, $\langle \overline{z}, \exp_{y_n}^{-1} y \rangle \leq 0$ for any $y \in M$. It follows

$$\langle \exp_{y_n}^{-1} x_n, \exp_{y_n}^{-1} y \rangle \leq \lambda \langle z, \exp_{y_n}^{-1} y \rangle. \tag{3.2}$$

On the other hand, from $z \in \partial_2 f(x_n, y_n)$ and the definition of subdifferential, we have

$$f(x_n, y) - f(x_n, y_n) \geq \langle z, \exp_{y_n}^{-1} y \rangle \quad \forall y \in M. \tag{3.3}$$

Multiplying both sides of the inequality (3.3) by $\lambda > 0$, and using (3.2), we obtain

$$\lambda[f(x_n, y) - f(x_n, y_n)] \geq \langle \exp_{y_n}^{-1} x_n, \exp_{y_n}^{-1} y \rangle \quad \forall y \in K.$$

2) The idea of the proof is similar to part 1. □

Lemma 3.7. *For any $x^* \in EP(f, K)$ and $\lambda > 0$,*

$$d^2(x_{n+1}, x^*) \leq d^2(x_n, x^*) - (1 - 2\lambda\alpha_1) d^2(y_n, x_n) - (1 - 2\lambda\alpha_2) d^2(x_{n+1}, y_n)$$

Proof . From $x_{n+1} \in T_n$ and the definition of T_n, it follows

$$\langle \exp_{y_n}^{-1} x_n - \lambda v_n, \exp_{y_n}^{-1} x_{n+1} \rangle \leq 0$$

for some $v_n \in \partial_2 f(x_n, y_n)$. Hence,

$$\lambda \langle v_n, \exp_{y_n}^{-1} x_{n+1} \rangle \ \geq \ \langle \exp_{y_n}^{-1} x_n, \exp_{y_n}^{-1} x_{n+1} \rangle. \tag{3.4}$$

Since $v_n \in \partial_2 f(x_n, y_n)$, by the definition of subdifferential, we have

$$f(x_n, y) - f(x_n, y_n) \geq \langle v_n, \exp_{y_n}^{-1} y \rangle, \quad \forall y \in M. \tag{3.5}$$

Taking $y = x_{n+1}$ into (3.5), we conclude

$$f(x_n, x_{n+1}) - f(x_n, y_n) \geq \langle v_n, \exp_{y_n}^{-1} x_{n+1} \rangle.$$

It follows from the last inequality and relation (3.4) that

$$\lambda[f(x_n, x_{n+1}) - f(x_n, y_n)] \geq \langle \exp_{y_n}^{-1} x_n, \exp_{y_n}^{-1} x_{n+1} \rangle. \tag{3.6}$$

On the other hand, by Proposition (2.6) and the definition of x_{n+1} in Algorithm 3.2, we have

$$0 \in \partial_2[\lambda f(y_n, y) + \frac{1}{2} d^2(x_n, y)](x_{n+1}) + N_{T_n}(x_{n+1}).$$

So, there exist $z \in \partial_2 f(y_n, x_{n+1})$ and $\overline{z} \in N_{T_n}(x_{n+1})$ such that

$$\lambda z - \exp_{x_{n+1}}^{-1} x_n + \overline{z} = 0.$$

Note that $\overline{z} \in N_{T_n}(x_{n+1})$ and the definition of the normal cone imply that $\langle \overline{z}, \exp_{x_{n+1}}^{-1} y \rangle \leq 0$, for any $y \in T_n$. Hence, from the last equality, it follows

$$\langle \lambda z - \exp_{x_{n+1}}^{-1} x_n, \exp_{x_{n+1}}^{-1} y \rangle \geq 0 \quad \forall y \in T_n,$$

or equivalently,

$$\lambda \langle z, \exp_{x_{n+1}}^{-1} y \rangle \geq \langle \exp_{x_{n+1}}^{-1} x_n, \exp_{x_{n+1}}^{-1} y \rangle \quad \forall y \in T_n. \tag{3.7}$$

Now, using the definition of subdifferential for $z \in \partial_2 f(y_n, x_{n+1})$, we have

$$f(y_n, y) - f(y_n, x_{n+1}) \geq \langle z, \exp_{x_{n+1}}^{-1} y \rangle \quad \forall y \in M.$$

This together with the relation (3.7) lead to

$$\lambda[f(y_n, y) - f(y_n, x_{n+1})] \geq \langle \exp_{x_{n+1}}^{-1} x_n, \exp_{x_{n+1}}^{-1} y \rangle \quad \forall y \in T_n. \tag{3.8}$$

Taking $y = x^*$ in the relation (3.8), we get

$$\lambda[f(y_n, x^*) - f(y_n, x_{n+1})] \geq \langle \exp_{x_{n+1}}^{-1} x_n, \exp_{x_{n+1}}^{-1} x^* \rangle. \tag{3.9}$$

Because of $x^* \in EP(f, K)$, $f(x^*, y_n) \geq 0$. So, by pseudomonotonicity of f, $f(y_n, x^*) \leq 0$. Therefore, from (3.9) and $\lambda > 0$, we have

$$-\lambda f(y_n, x_{n+1}) \geq \langle \exp_{x_{n+1}}^{-1} x_n, \exp_{x_{n+1}}^{-1} x^* \rangle. \tag{3.10}$$

Using the Lipschitz-type continuity of f, we have

$$f(y_n, x_{n+1}) \geq f(x_n, x_{n+1}) - f(x_n, y_n) - \alpha_1 d^2(x_n, y_n) - \alpha_2 d^2(y_n, x_{n+1}). \tag{3.11}$$

Multiplying both two sides of the relation (3.11) by $\lambda > 0$ and combining with relations (3.6) and (3.10), respectively, leads to

$$\begin{aligned} -\langle \exp_{x_{n+1}}^{-1} x_n, \exp_{x_{n+1}}^{-1} x^* \rangle &\geq \lambda f(y_n, x_{n+1}) \\ &\geq \lambda[f(x_n, x_{n+1}) - f(x_n, y_n)] - \lambda\alpha_1 d^2(x_n, y_n) - \lambda\alpha_2 d^2(y_n, x_{n+1}) \\ &\geq \langle \exp_{y_n}^{-1} x_n, \exp_{y_n}^{-1} x_{n+1} \rangle - \lambda\alpha_1 d^2(x_n, y_n) - \lambda\alpha_2 d^2(y_n, x_{n+1}). \end{aligned}$$

Using Proposition (2.2) twice, we obtain

$$\begin{aligned} d^2(x^*, x_n) - d^2(x_n, x_{n+1}) - d^2(x^*, x_{n+1}) &\geq -2 < \exp^{-1}_{x_{n+1}} x_n, \exp^{-1}_{x_{n+1}} x^* > \\ &\geq 2 < \exp^{-1}_{y_n} x_n, \exp^{-1}_{y_n} x_{n+1} > \\ &\quad -2\lambda\alpha_1 d^2(x_n, y_n) - 2\lambda\alpha_2 d^2(y_n, x_{n+1}) \\ &\geq -d^2(x_n, x_{n+1}) + d^2(x_n, y_n) + d^2(x_{n+1}, y_n) \\ &\quad -2\lambda\alpha_1 d^2(x_n, y_n) - 2\lambda\alpha_2 d^2(y_n, x_{n+1}). \end{aligned}$$

By simplifying the above relation, it concludes the desired result. □

Theorem 3.8. *Let K be a nonempty closed convex subset of a Hadamard manifold M and $f : M \times M \to \mathbb{R}$ be a bifunction satisfying conditions C(1), C(2), C(3) and C(4). Assume that the sequences $\{x_n\}$ and $\{y_n\}$ generated by Algorithm 3.2 and $0 < \lambda < \min\{\frac{1}{2\alpha_1}, \frac{1}{2\alpha_2}\}$. Then the following statements holds:*

1. *The sequence $\{x_n\}$ is Fejér convergent to $EP(f, K)$. In addition, the sequences $\{x_n\}$ and $\{y_n\}$ are bounded.*
2. *Every limit point of $\{x_n\}$ belongs to $EP(f, K)$. That is, the sequence $\{x_n\}$ converges strongly to $\widehat{x} \in EP(f, K)$.*

Proof . 1) Let $x^* \in EP(f, K)$. By Lemma 3.7,

$$d^2(x_{n+1}, x^*) - d^2(x_n, x^*) \leq -(1 - 2\lambda\alpha_1) d^2(y_n, x_n) - (1 - 2\lambda\alpha_2) d^2(x_{n+1}, y_n). \tag{3.12}$$

Since $0 < \lambda < \min\{\frac{1}{2\alpha_1}, \frac{1}{2\alpha_2}\}$, then $d(x_{n+1}, x^*) \leq d(x_n, x^*)$. Hence, it concludes that the sequence $\{x_n\}$ is Fejér convergence to $EP(f, K)$ and the sequence $\{x_n\}$ is bounded. For boundedness of $\{y_n\}$, it sufficient to prove that the sequence $\{d(x_n, y_n)\}$ is convergent to zero. We have,

$$d(x_{n+1}, x^*) \leq d(x_n, x^*) \leq \ldots \leq d(x_0, x^*).$$

This shows that the sequence $\{d(x_n, x^*)\}$ is bounded. Furthermore, by $d(x_{n+1}, x^*) \leq d(x_n, x^*)$, it follows that the sequence $\{d(x_n, x^*)\}$ is non increasing and convergent. Also, from relation (3.12), it follows

$$(1 - 2\lambda\alpha_1) d^2(y_n, x_n) \leq d^2(x_n, x^*) - d^2(x_{n+1}, x^*).$$

This implies the desired result.

2) Let $\widehat{x}$ be a accumulation point of the sequence $\{x_n\}$. So, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ which converges to $\widehat{x}$. From $\{d(x_{n_k}, y_{n_k})\} \to 0$, it is concluded that $\{y_{n_k}\}$ converges to $\widehat{x}$. Since K is closed and $\{y_{n_k}\} \subseteq K$, then $\widehat{x} \in K$.

Now, let $y \in M$ be arbitrary. Using Lemma 3.6 (2), we get

$$\lambda f(y_n, y) \geq \lambda f(y_n, x_{n+1}) + \langle \exp^{-1}_{x_{n+1}} x_n, \exp^{-1}_{x_{n+1}} y \rangle. \tag{3.13}$$

Since f is Lipschitz-type continuous, hence

$$f(y_n, x_{n+1}) \geq f(x_n, x_{n+1}) - f(x_n, y_n) - \alpha_1 d^2(x_n, y_n) - \alpha_2 d^2(y_n, x_{n+1}). \tag{3.14}$$

By taking $y = x_{n+1}$ in Lemma 3.6 (1), we have

$$\lambda[f(x_n, x_{n+1}) - f(x_n, y_n)] \geq \langle \exp^{-1}_{y_n} x_n, \exp^{-1}_{y_n} x_{n+1} \rangle. \tag{3.15}$$

We obtain the following relation from relations (3.14) and (3.15):

$$\lambda f(y_n, x_{n+1}) \geq \langle \exp_{y_n}^{-1} x_n, \exp_{y_n}^{-1} x_{n+1} \rangle - \lambda\alpha_1 d^2(x_n, y_n) - \lambda\alpha_2 d^2(y_n, x_{n+1}). \tag{3.16}$$

Combination of relations (3.13) and (3.16) imply that

$$\begin{aligned}\lambda f(y_n, y) \geq\ & \langle \exp_{y_n}^{-1} x_n, \exp_{y_n}^{-1} x_{n+1} \rangle \\ +\ & \langle \exp_{x_{n+1}}^{-1} x_n, \exp_{x_{n+1}}^{-1} y \rangle \\ -\ & \lambda\alpha_1 d^2(x_n, y_n) - \lambda\alpha_2 d^2(y_n, x_{n+1}).\end{aligned}$$

This is also true for the subsequence n_k. Since $\{x_{n_k}\}$ and $\{y_{n_k}\}$ converge to $\widehat{x}$ and f is upper semicontinuous in the first variable, it follows

$$f(\widehat{x}, y) \geq \limsup_{k\to\infty} f(y_{n_k}, y) \geq 0.$$

It concludes that $\widehat{x} \in EP(f, K)$, because of y is arbitrary. Therefore, by Lemma 3.5, the sequence $\{x_n\}$ converges strongly to $\widehat{x} \in EP(f, K)$. □

Acknowledgements The author is very grateful to the anonymous referees for their valuable and useful comments, which helps in improving the quality of this work.

References

[1] P. N. Anh, T. N. Hai and P. M. Tuan, *On ergodic algorithms for equilibrium problems* , J. Glob. Optim. 64 (2016) 17–195.

[2] M. P. do Carmo, *Riemannian Geometry*, Birkhauser, Boston, 1992.

[3] J. X. Cruz Neto, O. P. Ferreira and L. R. Lucambio Pérez, *Contributions to the study of monotone vector fields*, Acta Math. Hungar. 94 (2002) 30–320.

[4] Y. Censor, A. Gibali and S. Reich, *The subgradient extragradientmethod for solving variational inequalities in Hilbert space*, J. Optim. Theory Appl. 148 (2011) 318–335.

[5] Y. Censor, A. Gibali and S. Reich, *Algoriyhms for the split variational inequality problem*, Numer. Alghorithms 59 (2012) 301–323.

[6] J. X. Cruz Neto, P. S. M. Santos and P. A. Soares Jr, *An extragradient method for equlibrium problems on Hadamard manifolds*, Optim. Lett. 10 (2016) 1327–1336.

[7] V. Dadashi, O. S. Iyiola and Y. Shehu, *The subgradient extragradient method for pseudomonotone equlibrium problems*, Optimization 69 (2020) 901–923.

[8] F. Facchinei and J. S. Pang, *Finite-dimensional variational inequalities and complementarity problems*, Berlin, Springer, 2002.

[9] K. Fan, *A minimax inequality and applications*, In: *Shisha O, editor. Inequality III*, pp. 103–113. New York, Academic Press, 1972.

[10] O. P. Ferreira, L. R. Lucambio Pérez and S. Z. Nemeth, *Singularities of monotone vector fields and an extragradient-type algorithm*, J. Global Optim. 31 (2005) 133–151.

[11] O. P. Ferreira and P. R. Oliveira, *Proximal point alghorithm on Riemannian manifolds*, Optimization 51 (2002) 257–270.

[12] D. V. Hieu, *New subgradient extragradient methods for common solutions to equilibrium problems*, Comput. Optim. Appl. 67 (2017) 1–24.

[13] D. V. Hieu, *Halpern subgradient extragradient method extended to equilibrium problems*, RACSAM 111 (2017) 823–840.

[14] D. V. Hieu, *New extragradient method for a class of equilibrium problems in Hilbert spaces*, Appl. Anal. 97 (2018) 811–824.

[15] D. V. Hieu, *Convergence analysis of a new algorithm for strongly pseudomontone equilibrium problems*, Numer. Algorithms 77 (2018) 983–1001.

[16] C. Izuchukwu, *Strong convergence theorem for a class of multiple-sets split variational inequality problems in Hilbert spaces*, Int. J. Nonlinear Anal. Appl. 9 (2018) 27–40.

[17] H. Khatibzadeh and S. Ranjbar, *A variational inequality in complete CAT(0) spaces*, J. Fixed Point Theory Appl. 17 (2015) 557–574.

[18] H. Khatibzadeh, V. Mohebbi and S. Ranjbar, *Convergence analysis of the proximal point algorithm for pseudo-monotone equilibrium problems*, Optimization Methods and Software 30 (2015) 1146–1163.

[19] H. Khatibzadeh, V. Mohebbi and S. Ranjbar, *New results on the proximal point algorithm in non-positive curvature metric spaces*, Optimization 66 (2017) 1191–1199.

[20] I. V. Konnov, *Equilibrium models and variational inequalities*, Amsterdam, Elsevier, 2007.

[21] G. M. Korpelevich, *The extragradient method for finding saddle points and other problems*, Ekonomikai Matematcheskie Metody 12 (1976) 747–756.

[22] A. Kristály, *Nash-type equilibria on Riemannian manifolds: a variational approach*, J. Math. Pure Appl. 101 (2014) 660–688.

[23] X. Li and N. Huang, *Generalized vector quasi-equilibrium problems on Hadamard manifolds*, Optim. Lett. 9 (2015) 155–170.

[24] C. Li, G. López and V. Martín-Márquez, *Monotone vector field and the proximal point algorithm on Hadamard manifolds*, J. Lond. Math. Soc. 79 (2009) 663–683.

[25] C. Li, G. López and V. Martín-Márquez, *Iterative algorithms for nonexpansive mappings on Hadamard manifolds*, Taiwan. J. Math. 14 (2010) 541–559.

[26] S. Moradi, M. Shokouhnia and S. Jafari, *Optimally local dense conditions for the existence of solutions for vector equilibrium problems*, Int. J. Nonlinear Anal. Appl. 10 (2019) 15–26.

[27] T. T. V. Nguyen, J. J. Strodiot and V. H. Nguyen, *The interior proximal extragradient method for solving equilibrium problems*, J. Glob. Optim. 44 (2009) 175–192.

[28] D. Q. Tran, L. D. Muu and V. H. Nguyen, *Extragradient algorithms extended to equilibrium problems*, Optimization 57 (2008) 749–776.

[29] T. Rapcsák, *Smooth nonlinear optimization in* $\mathbb{R}^n$, *Nonconvex Optimization and its Applications*, Kluwer Academic Publishers, Dordrecht, 1997.

[30] S. Ranjbar, *W-convergence of the proximal point algorithm in complete CAT(0) metric spaces*, Bull. Iranian Math. Soc. 43 (2017) 817–834.

[31] T. Sakai, *Riemannian geometry*, American Mathematical Society, Providence, RI, 1996.

[32] C. Udriste, *Convex functions and optimization methods on Riemannians manifolds*, Kluwer Academic Publishers Group, Dordrecht, 1994.

[33] J. H. Wang, G. López, V. Martín-Márquez and C. Li, *Monotone and accretive vector fields on Riemannian manifolds*, J. Optim. Theory Appl. 146 (2010) 691–708.