Z-prime gamma submodule of gamma modules

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Abstract

Let $R$ be a $\Gamma$-ring and $\partial$ be an $R\Gamma$-module. A proper $R\Gamma$-submodule $T$ of an $R\Gamma$-module $\partial$ is called $Z$-prime $R\Gamma$-submodule if for each $t \in \partial$, $\gamma \in \Gamma$ and $f \in \partial^* = \text{Hom}_{R\Gamma}(\partial, R)$, $f(t)\gamma t \in T$ implies that either $t \in T$ or $f(t) \in [T :_{R\Gamma} \partial]$. The purpose of this paper is to introduce interesting theorems and properties of $Z$-prime $R\Gamma$-submodule of $R\Gamma$-module and the relation of $Z$-prime $R\Gamma$-submodule, which represents of generalization $Z$-prime $R$-submodule of $R$-module.

Keywords: $\Gamma$-ring, $R\Gamma$-module, $R\Gamma$-submodule, and prime $R\Gamma$-submodule.

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1. Introduction

The topic of a $\Gamma$-ring was introduced in 1964 by Nobusawa\textsuperscript{[4]}. He considered a set of homeomorphisms of a module to another module, which as closed under the addition and subtraction defined naturally but has no more a structure of a ring since he cannot have defined the product. After that, Barnes in\textsuperscript{[4, 6]} weakened the generalization of Nobusawa. Then, many papers studied the $\Gamma$-ring in several algebraic structures. In\textsuperscript{[3]}, Ameri and Sadeghi presented the concept of a gamma modules in $R$ investigate at some such modules. In this regard, we investigate submodules and homomorphism of a gamma modules and give the related basic results of a gamma modules. In 2005, Tekir and Sengul\textsuperscript{[7]} presented the concept of prime $\Gamma\text{M}$-submodules of $\Gamma\text{M}$-modules and discussed some interesting and useful properties. Also, Zyarah and al-Mothafar provided the defining the semiprime $R\Gamma$-submodule of $R\Gamma$-module and the relation of semiprime $R\Gamma$-submodule. With multiplication $R\Gamma$-modules\textsuperscript{[11]}. Also, in another work\textsuperscript{[10]}, they introduced some results and properties of primary...
RT-submodule and the definition for primary radical of RT-submodule of RT-module besides some of its basic properties. In this paper, Z-prime RT-submodule of RT-module and are investigated the basic properties, some theorems, and propositions. In addition, the relation between Z-prime RT-submodule with other RT-modules is investigated.

2. Preliminaries

Definition 2.1. Let R and Γ be an additive abelian groups, so we’ll consider R is a Γ-ring R, shortly (ΓR) if there exists a mapping h : R × Γ × R → R such that for every d_1, d_2, d_3 ∈ R and γ, δ ∈ Γ, the following conditions are hold:

i. (d_1 + d_2)γd_3 = d_1γd_3 + d_2γd_3.

ii. d_1(γ + δ)d_3 = d_1γd_3 + d_1δd_3.

iii. d_1γ(d_2 + d_3) = d_1γd_2 + d_1γd_3.

iv. (d_1γd_2)δd_3 = d_1γ(d_2δd_3).

Definition 2.2. A left RT-module is an additive abelian group η together with a mapping h : R × Γ × η → η such that for all h, h_1, h_2 ∈ η and γ, γ_1, γ_2 ∈ Γ, r_1, r_2, r_3 ∈ R the following hold:

i. r_3γ(h_1 + h_2) = r_3γh_1 + r_3γh_2.

ii. (r_1 + r_2)γh = r_1γh + r_2γh.

iii. r_3(γ_1 + γ_2)h = r_3γ_1h + r_3γ_2h.

iv. r_1γ_1(r_2γ_2)h = (r_1γ_1r_2)γ_2h, aright RT-module is defined in analogous manner.

Definition 2.3. A proper RT - S. T of η is called prime RT-submodule, shortly (P.RT – S.) if for any an ideal J of ΓR and for any RT – S. H of η, JΓH ⊆ T implies H ⊆ T or J ⊆ [T : R, η].

Definition 2.4. Let T be a proper RT - S. of a RT-module η. The RT – S. T of η is called that S-prime RT – S., whenever φ(K) ⊆ T, for some K be a RT – S. of η and φ ∈ End_{RT}(η), implies that K ⊆ T or φ(η) ⊆ T.

Definition 2.5. An RT-module η is called Jacobson radical, denoted by J_{Γ}(η), by J_{Γ}(η) = \{Y | Y is RΓ – small RΓ – submodule of η\}.

Definition 2.6. An RT-module η is called RT-faithful if it’s RT-annihilator is the zero ideal of a ΓR.

Definition 2.7. An ideal A of a ΓR is called prime if for any ideals I and J of R, IΓJ ⊆ A implies, either I ⊆ A or J ⊆ A.

Definition 2.8. Let η be an RT-module. We said that η is a multiplication RT-module if any proper RT – S. T of η, then there exist any ideal I of ΓR such that T = IΓη.
3. Z-Prime $R\Gamma$-submodule of $R\Gamma$-modules

In this section, we introduced $Z - P.R\Gamma - S$. of $R\Gamma$-modules some propositions, and theorems.

Definition 3.1. A proper $R\Gamma - S$. of an $R\Gamma$-module $\partial$ is called $Z - P.R\Gamma - S$. if for each $t \in \partial$, $\gamma \in \Gamma$ and $f \in \partial^* = \text{Hom}_{R\Gamma}(\partial, R)$, $f(t)\gamma t \in T$ implies that either $t \in T$ or $f(t) \in [T : R\Gamma \partial]$.

Remark and Example 3.2. 1. Every Z-prime $R$-submodule is $Z - P.R\Gamma - S$. but the converse isn’t true in general, as in the following example:

Let $Z$ be a $Z_{2Z}$-module, $\Gamma = 2Z$ and $6Z$ be a proper $Z_{2Z} - S$. of $Z$. Then $6Z$ is $Z - P.Z_{2Z} - S$. of $Z$, since $\varphi \in Z^* = \text{Hom}_{Z_{2Z}}(Z, Z) = Z$ and $\varphi : Z \to Z$; $\varphi(a) = 3a$, $a \in Z$ and so $\varphi(a)\gamma(a) \in 6Z$ also $\varphi(a) \in [6Z : Z_{2Z} Z] = 3Z$. But $6Z$ is not Z-prime of $Z - S$. of $Z$, since $\varphi \in Z^* = \text{Hom}_{Z}(Z, Z) = Z$ and $\varphi : Z \to Z$; $\varphi(a) = 3a$, $a \in Z$ and so $\varphi(a) a \in 6Z$ also $\varphi(a) \notin 6Z = [6Z : Z]$. 

2. Every $P.R\Gamma - S$. is $Z - P.R\Gamma - S$. but the converse is not true in general, as in the following example: Let $\partial = Z_8$ be a $Z_{2Z}$-module, $\Gamma = 2Z$ and $T = \langle v \rangle$ be a proper $Z_{2Z}$-submodule of $Z_8$. Then $\langle v \rangle$ is $Z$-prime $Z_{2Z}$-submodule, since $f \in Z^* = \text{Hom}_{Z_{2Z}}(Z_8, Z) = 0$ and so $f(a)a = 0 \in \langle v \rangle$ for all $a \in Z_8$ and $0 \in \langle v \rangle$. But $\langle v \rangle$ is not prime $Z_{2Z}$-submodule, since $2 \in 2Z, 2 \in Z_8, 1 \in Z$ such that $(1)(2)(2) \in \langle v \rangle$ but $2 \notin \langle v \rangle$ and $2 \notin \langle v \rangle$.

3. Let $I$ be an ideal of a $\Gamma R$, then $I$ be a Z-prime ideal if for every $r \in R$, $f \in R^* = \text{Hom}_{R\gamma}(R, R)$ such that $f(r)\gamma r \in I$ implies that either $r \in I$ or $f(r) \in I$.

Lemma 3.3. Let $D$ and $F$ be any two $\Gamma - S$. of an $R\Gamma$-module $\partial$, if $[D : R\Gamma x]$ is a Z-prime ideal of a $\Gamma R$ for each $x \in F$, then $[D : R\Gamma F]$ is a Z-prime ideal of a $\Gamma R$.

Proof. Let $f \in R^* = \text{Hom}_{R\gamma}(R, R), b \in R$ such that $f(b)a \in [D : R\Gamma F]$ and so, $f(b)abou \in E$ for all $\alpha \in \Gamma, u \in D$, then

\[ f(b)ab \in [D : R\Gamma < u >] \] (3.1)

But $[D : R\Gamma < u >]$ is Z-prime ideal, so either $f(b) \in [D : R\Gamma < u >]$ or $b \in [D : R\Gamma < u >]$. Thus for any $\alpha \in \Gamma, u \in D$, either $f(b)au \in D$ or $bou \in D$. Suppose that $f(b) \notin [D : R\Gamma F]$ and $b \notin [D : R\Gamma F]$, there exists $v, w \in F$ such that $f(b)av \notin D$ and $bou \notin D$. Hence $f(b) \notin [D : R\Gamma < v >]$ and $b \notin [D : R\Gamma < w >]$. But by (3.1), $f(b) ab \in [D : R\Gamma < v >]$ which is a Z-prime ideal, hence $b \in [D : R\Gamma < v >]$. Thus $bou \in D$, similarly, $f(b)ab \in [D : R\Gamma < w >]$ implies that $f(b)abou \in D$. On the other hand, by (3.1), $f(b)ab \in [D : R\Gamma < v + w >]$, so either $f(b) \in [D : R\Gamma < v + w >]$ or $b \in [D : R\Gamma < v + w >]$. Hence both $f(b)\alpha v + f(b)\alpha w = d_1 \in D$ or $bou + bou = d_2 \in D$. Then either $f(b)\alpha v = d_1 = f(b)\alpha w \in D$ or $bou = d_2 = bou \in D$, which is contradiction. Therefore, either $f(b) \in [D : R\Gamma F]$ or $b \in [D : R\Gamma F]$. □

Proposition 3.4. Let $L$ be a $Z - P.R\Gamma - S$. of an $R\Gamma$-module $\partial$ and $T$ be a summand of $\partial$, then either $T \subseteq L$ or $T \cap L$ is a $Z - P.R\Gamma - S$. of $\partial$.

Proof. Let $f \in T^* = \text{Hom}_{R\gamma}(T, R)$ and $a \in T$ such that $f(a)\gamma a \in T \cap L$. Suppose that $T \notin L$, then $T \cap L$ be a proper $R\Gamma - S$. of $T$. Suppose that $a \notin T \cap L$, since $T$ be a summand of $\partial$ then there exist a projection $\rho : \partial \to T$ and $f : T \to R$ such that $f(a)\gamma a = f \circ \rho(a)\gamma a \in L, \gamma \in \Gamma$ and $a \notin L$. Then $f \circ \rho(a) \in [L : R\Gamma \partial] \subseteq [L : R\Gamma T]$, since $L$ be a $Z - P.R\Gamma - S$. of $\partial$. Thus $f(a)\Gamma T \subseteq L$ and $f(a)\Gamma T \subseteq T$, and therefore, $f(a) \in [L \cap T : R\Gamma T]$. □
Remark 3.5. Let $T$ be a $Z - P.R.\Gamma - S.$ of $R\Gamma$-module $\partial$, then $T$ is called $P$-$Z$-prime $R\Gamma - S.$, where $P = \text{rad}_R([T : R_\Gamma, \partial])$ and hence if $< 0 >$ is a $Z - P.R.\Gamma - S.$ of $\partial$, then $< 0 > = P = \text{rad}_R([0 : R_\Gamma, \partial]) = \text{rad}_R(am(R_\Gamma, \partial)) - Z - P.R.\Gamma - S.$ of $\partial$.

Proposition 3.6. Let $P$ be a $Z$-prime ideal of a $\Gamma R$ and let $n$ be a positive integer. $T_i$ be a $P - Z - P.R.\Gamma - S.$ of an $R\Gamma$-module $\partial$ such that $1 \leq i \leq n$. Then $\bigcap_{i=1}^n T_i$ is also $P - Z - P.R.\Gamma - S.$ of $\partial$.

Proof. Let $f \in \partial^* = \text{Hom}_{R_\Gamma}(\partial, R)$ and $x \in \partial$ such that $f(x)\gamma x \in \bigcap_{i=1}^n T_i$. It’s clear that $P = \text{rad}_R([\bigcap_{i=1}^n T_i : R_\Gamma, \partial])$. Suppose that $x \notin \bigcap_{i=1}^n T_i$, then there exist $m \in Z^+$ with $1 \leq m \leq n$ such that $x \notin T_m$. But $f(x)\gamma x \in T_m$ and $T_m$ is a $P - Z - P.R.\Gamma - S.$ of $\partial$. It follows that $f(x) \in P$ and hence $\bigcap_{i=1}^n T_i$ is a $P - Z - P.R.\Gamma - S.$ of $\partial$.  □

Proposition 3.7. Let $T$ be a $R.\Gamma - S.$ of an $R\Gamma$-module $\partial$ and let $P$ be a prime ideal of a $\Gamma R$. If $[T : R_\Gamma, K] \subseteq P$ for each $R.\Gamma - S.K$ of $\partial$ containing $T$ properly $P \subseteq [T : R_\Gamma, \partial]$, then $T$ be a $Z - P.R.\Gamma - S.$ of $\partial$.

Proof. Let $\xi \in \partial^* = \text{Hom}_{R_\Gamma}(\partial, R)$ and $t \in \partial$ such that $\xi(t)\gamma t \in T$. Suppose that $x \notin T$ and let $K = T^+ < t >$ and let $K$ be an $R.\Gamma - S.$ of $\partial$ properly containing $T$ properly, but $\xi(t)\gamma K = \xi(t)\gamma T + \xi(t)\gamma < t > \subseteq T$. And hence $\xi(t) \in [T : R_\Gamma, K] \subseteq P \subseteq [T : R_\Gamma, \partial]$. Thus $T$ be a $Z - P.R.\Gamma - S.$ of $\partial$. □

Proposition 3.8. Let $\partial_1$ and $\partial_2$ be two $R.\Gamma$-modules and $\partial = \partial_1 \bigoplus \partial_2$. If $T = T_1 \bigoplus T_2$ is a $Z - P.R.\Gamma - S.$ of $\partial$, then $T_1$ and $T_2$ are $Z - P.R.\Gamma - S.$ of $\partial_1$ and $\partial_2$ respectively.

Proof. To show that $\partial_1$ is a $Z - P.R.\Gamma - S.$ of $\partial_1$. Let $f \in \partial_1 = \text{Hom}_{R_\Gamma}(\partial_1, R_\Gamma)$, $t \in \partial_1$ and $\gamma \in \Gamma$ such that $(t)\gamma t \in T_1$, then $f(t)\gamma t \in T_1 \bigoplus T_2$, where $f = \text{Hom}_{R_\Gamma}(\partial_1 \bigoplus \partial_2, R_\Gamma)$. Since $T$ is a $Z - P.R.\Gamma - S.$ of $\partial$, then either $T \subseteq T_1 \bigoplus T_2$ or $f(t) \in [T_1 \bigoplus T_2 : R_\Gamma, \partial_1 \bigoplus \partial_2]$. Thus either $t \in T_1$ or $f(t) \in [T_1 : R_\Gamma, \partial_1 \bigoplus \partial_2]$ and $f(t) \in [T_1 : R_\Gamma, \partial_1 \bigoplus \partial_2]$. Therefore, $T_1$ is a $Z - P.R.\Gamma - S.$ of $\partial_1$ and similarly to prove $T_2$ is a $Z - P.R.\Gamma - S.$ of $\partial_2$. □

Proposition 3.9. Let $\partial, \partial'$ be an $R.\Gamma$-modules and $\varphi : \partial \rightarrow \partial'$ be an $R.\Gamma$-epimorphism. If $T$ is a $Z - P.R.\Gamma - S.$ of $\partial$ and $\text{Ker} \varphi \subseteq T$, then $\varphi(T)$ is a $Z - P.R.\Gamma - S.$ of $\partial'$.

Proof. To show that $\varphi(T)$ is a proper $R.\Gamma - S.$ of $\partial'$. Suppose that $\varphi(T) = \partial'$, since $\varphi$ is an $R.\Gamma$-epimorphism, then $\varphi(T) = \varphi(\partial)$ and $\partial = T + \text{Ker} \varphi$, but $\text{Ker} \varphi \subseteq T$, hence $\partial = T$ which is contradiction, since $T$ is a $Z - P.R.\Gamma - S.$ of $\partial$. Now, we define $\psi \in (\partial')^* = \text{Hom}_{R_\Gamma}(\partial', R)$ and $w \in \partial'$, let $\psi(z)\gamma w \in \varphi(T), \gamma \in \Gamma$ and, $w \notin \varphi(T)$. Since $\varphi$ is an $R.\Gamma$-epimorphism, then there exist $u \in \partial$ such that $\varphi(u) = w$ and $u \notin T$. Then $\psi(z)\gamma w = \psi(z)\gamma \varphi(u) \in \varphi(T)$ and $\varphi(\psi(z)\gamma u) \in \varphi(T)$, since $\text{Ker} \varphi \subseteq T$, then $\psi(z)\gamma u \in T$. Since $T$ is a $Z - P.R.\Gamma - S.$ of $\partial$ and $u \notin T$, then $\psi(z) \in [T : R_\Gamma, \partial]$. Thus $\varphi(\varphi(z)\gamma(\partial) \subseteq \varphi(T)$ and $\psi(z)\gamma(\partial) \subseteq \varphi(T)$, then $\psi(z) \in [\varphi(T) : R_\Gamma, \partial']$. Therefore, $\varphi(T)$ is a $Z - P.R.\Gamma - S.$ of $\partial'$. □

Proposition 3.10. Let $\partial, \partial'$ be an $R.\Gamma$-modules and $\varphi : \partial \rightarrow \partial'$ be an $R.\Gamma$-monomorphism. If $T'$ is a $Z - P.R.\Gamma - S.$ of $\partial'$ and $\varphi(\partial) \subseteq T'$, then $\varphi^{-1}(T')$ is a $Z - P.R.\Gamma - S.$ of $\partial$.

Proof. To show that $\varphi^{-1}(T')$ is a proper $R.\Gamma - S.$ of $\partial$. Suppose that $\varphi^{-1}(T') = \partial$, let $x \in \partial$ and $x \in \partial^{-1}T'$, then $\varphi(x) \subseteq T'$ which is contradiction. Now, we define $f \in (\partial')^* = \text{Hom}_{R_\Gamma}(\partial, R)$ and $w \in \partial$. Suppose that $w \notin \varphi^{-1}(T')$ and $\gamma \in \Gamma$, then $\varphi(w) \notin T'$. Let $f(w)\gamma w \in \varphi^{-1}(T')$, then $f(w)\gamma w \in T'$ and $f(w)\gamma w \in T'$. Since $\varphi$ is an $R.\Gamma$-monomorphism, we put $\varphi^{-1}(\varphi(w)) = w$, then $f(\varphi(w))\gamma w \in T'$ and $f(\varphi(w))\gamma w \in T'$ is a $Z - P.R.\Gamma - S.$ of $\partial'$ and $w \notin T'$, then $f(\varphi(w)) \in [T' : R_\Gamma, \partial']$. Thus $f(w)\gamma(\partial) \subseteq f(w)\gamma(\partial) \subseteq T'$ and $f(w)\gamma(\partial) \subseteq \varphi^{-1}(T')$, hence $f(w) \in [\varphi^{-1}(T') : R_\Gamma, \partial']$. Therefore $\varphi^{-1}(T')$ is a $Z - P.R.\Gamma - S.$ of $\partial$. □

Corollary 3.11. Let $T$ and $L$ be a two $Z - P.R.\Gamma - S.s$ of $R.\Gamma$-module $\partial$ and $L \subseteq T$, then $T$ is a $Z - P.R.\Gamma - S.$ of $\partial$ if and only if $T/L$ is a $Z - P.R.\Gamma - S.$ of $\partial/L$ [9].
4. Z-Prime $R \Gamma - S.s$ of a Faithful Multiplication $R \Gamma$-modules

We present in this section Z-prime $R \Gamma - S.s$ of multiplication $R \Gamma$-modules and also give some examples, propositions and theorems of this.

**Proposition 4.1.** Let $T$ be a proper $R \Gamma - S.$ of cyclic faithful $R \Gamma$-module $\partial$. If $T$ is a $Z - P.R \Gamma - S.$ of $\partial$, then $T$ is a $P.R \Gamma - S.$ of $\partial$.

**Proof.** Let $t \in \partial$, $k \in R$ and $\beta \in \Gamma$ such that $k \beta t \in T$ and $t \notin T$. Suppose that $\partial = < x >, x \in \partial$, then $t = x \beta r, r \in R$. Define $\eta : \partial \rightarrow R$ by $\eta(t) = \eta(k \beta x) = k$. Since $\partial$ is a faithful $R \Gamma$-module, then $\eta$ is well-defined and which implies that $\eta(t) \beta(t) \in T$ and $x \notin T$, since $T$ is a $Z - P.R \Gamma - S.$ of $\partial$, then $\eta(t) \in [T : R_{\partial}]$. Thus $k \in [T : R_{\partial}]$ and therefore $T$ is $P.R \Gamma - S.$ of $\partial$. □

**Corollary 4.2.** Let $T$ be a proper $R \Gamma - S.$ of a cyclic faithful $R \Gamma$-module $\partial$. If $T$ is a $Z - P.R \Gamma - S.$ of $\partial$, then $[T : R_{\partial}]$ is a $Z$-prime ideal of a $\Gamma R$.

**Proposition 4.3.** Let $T$ be a proper $R \Gamma - S.$ of a multiplication $R \Gamma$-module $\partial$. If $[T : R_{\partial}]$ is a $Z$-prime ideal of a $\Gamma R$, then $T$ is a $Z - P.R \Gamma - S.$ of $\partial$.

**Proof.** Let $f \in \partial = \text{Hom}_{R_{\partial}}(\partial, R), t \in \partial$ and $\gamma \in \Gamma$ such that $f(t) \gamma t \in T$, then $f(t) \Gamma < t > \subseteq T$ and $< t > = \Gamma \partial$ for some an ideal $I$ in a $\Gamma R$. Since $\partial$ is a multiplication $R \Gamma$-module and so $f(t) \Gamma \partial \subseteq T$, then $f(t) \Gamma \subseteq [T : R_{\partial}]$ and $f(t) > \Gamma \subseteq [T : R_{\partial}]$. Now, we define $g : R \rightarrow R$, it’s clear that $g \in R^*$. Now, $g(f(t) > \Gamma \subseteq [T : R_{\partial}]$. Since $[T : R_{\partial}]$ is a $Z$-prime ideal of a $\Gamma R$, then $< t > \subseteq [T : R_{\partial}]$ or $I \subseteq [T : R_{\partial}]$. If $f(t) > \Gamma \subseteq [T : R_{\partial}]$, then $f(t) \in [T : R_{\partial}]$. If $I \subseteq [T : R_{\partial}]$, then $< t > \subseteq T$ i.e., $t \in T$. Thus $T$ is a $Z - P.R \Gamma - S.$ of $\partial$. □

**Proposition 4.4.** Let $T$ be a proper $R \Gamma - S.$ of a cyclic faithful $R \Gamma$-module $\partial$. Then $[T : R_{\partial}]$ is a $Z$-prime ideal of a $\Gamma R$ if and only if $T$ is a $Z - P.R \Gamma - S.$ of $\partial$.

**Proposition 4.5.** Let $\partial$ be a finitely generated multiplication $R \Gamma$-module. If $I$ is a $Z$-prime ideal of a $\Gamma R$ such that $\text{ann}_{R_{\partial}}(\partial) \subseteq I$, then $\Gamma \partial$ is a $Z - P.R \Gamma - S.$ of $\partial$.

**Proof.** Let $f \in \partial = \text{Hom}_{R_{\partial}}(\partial, R), t \in \partial$ and $\gamma \in \Gamma$ such that $f(t) \gamma t \in \Gamma \partial$, then $f(t) \Gamma < t > \subseteq \Gamma \partial$. Since $\partial$ is a multiplication $R \Gamma$-module, then $< t > = A \Gamma \partial$ for some $A$ be an ideal in a $\Gamma R$, and $f(t) \Gamma A \partial \subseteq \Gamma \partial$. Then $f(t) \Gamma A \subseteq A + \text{ann}_{R_{\partial}}(\partial) = I$ by $[? ]$. Now, we define $g : R \rightarrow R$, it’s clear that $g \in R^*$. Now, $g(f(t) \Gamma A \subseteq I$. Since $I$ is a $Z$-prime ideal of a $\Gamma R$, then $f(t) \in I$ and $f(t) \in [\Gamma \partial : R_{\partial}]$ or $A \subseteq I$ and $A \Gamma \partial \subseteq \Gamma \partial$ also $< t > \subseteq \Gamma \partial$. Thus $f(t) \in [\Gamma \partial : R_{\partial}]$ or $t \in \Gamma \partial$ and therefore, $\Gamma \partial$ is a $Z - P.R \Gamma - S.$ of $\partial$. □

**Proposition 4.6.** Let $\partial$ be a cyclic $R \Gamma$-projective $R \Gamma$-module. If $T$ is a $Z - P.R \Gamma - S.$ of $\partial$, then $T$ is a $S - P.R \Gamma - S.$ of $\partial$.

**Proof.** Let $f \in \text{End}_{R_{\partial}}(\partial), w \in \partial$ and $\partial = R \Gamma w, \gamma \in \Gamma$ such that $f(w) \in T$ and $w \notin T$. Since $\partial$ is a cyclic $R \Gamma$-module, then there exist $h : R \rightarrow \partial$ define by $h(r) = r \gamma w$, for each $r \in R$. Since $\partial$ is projective $R \Gamma$-modules, then there exist an exist $R \Gamma$-homomorphism $\theta : \partial \rightarrow R$, such that $h \circ \theta = f$. Clearly $h \circ \theta \in \text{End}_{R_{\partial}}(\partial), f(w) = h(\theta(w)) = \theta(w) \gamma w \in T$ since $\theta \in \partial^* = \text{Hom}_{R_{\partial}}(\partial^*, R)$ and $T$ is a $Z - P.R \Gamma - S.$ of $\partial$, $w \notin T$, then $\theta(w) \Gamma \partial \subseteq T$. Now, $f(\partial) = (h \circ \theta)(\partial) = h(\theta(\partial)) = \theta(\partial) \Gamma \partial \subseteq T$ and therefore, $T$ is a $S - P.R \Gamma - S.$ of $\partial$. □

**Proposition 4.7.** Let $\partial$ be a cyclic $Rw3b_{\partial}$-projective $R \Gamma$-module and $T$ be a proper $R \Gamma - S.$ of $\partial$, then the following are equivalent:

1. $T$ is a $Z - P.R \Gamma - S.$ of $\partial$.
2. $T$ is a $S - P.R \Gamma - S.$ of $\partial$.
3. $T$ is a $P.R \Gamma - S.$ of $\partial$.
5. Conclusions

In this paper, $Z$-prime $R\Gamma$-submodule of $R\Gamma$-module and are investigated the basic properties, some theorems, and propositions. In addition, the relation between $Z$- prime $R\Gamma$-submodule with other $R\Gamma$-modules is investigated.

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