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Combining B-spline least-square schemes with different weight functions to solve the generalized regularized long wave equation

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Abstract

For solving differential equations, a variety of numerical methods are available, accuracy, performance, and application are all different. In this article, we proposed new numerical techniques for solving the generalized regularized long wave equation(GRLWE) that are based on types M and M-1 of B-splines-least-square method (BSLSM) and weight function of B-splines respectively, which were proposed previously for solving integro-differential equations [2] where $M \in N$. We investigated linear stability using a Fourier method.

Keywords: B-Spline method, Petrov-Galerkin method, Least-Square method, Fourier method, generalized regularized long wave equation.

1. Introduction

Consider GRLWE has the form

$$u_t + u_x + \alpha u^p u_x - \mu u_{xxt} = 0. \tag{1.1}$$

The regularized long wave equation (RLWE) is a particular instance of (1.1) for $p = 1$, and it is used to describe a wide range of issues in numerous fields of sciences. The equation was first used to describe the growth of undular bore[21]. The RLWE's exact solution for some conditions may be

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found in ([3], [5]). Finite difference methods were used to solve it numerically ([11], [21]). Consider the modified RLWE (MRLWE), which is a special case of (1.1) for $p = 2$.

$$u_t + u_x + \alpha u^2 u_x - \mu u_{xxt} = 0, \tag{1.2}$$

subject to the boundary conditions $u \to 0 \ \ as \ \ x \to \pm\infty$.
The following boundary criteria will be considered

$$u(a, t) = u(b, t) = 0, \tag{1.3}$$

and achieve a unique B-spline solution, will be applied the boundary conditions

$$u_x(a, t) = u_x(b, t) = 0, \qquad u_{xx}(a, t) = u_{xx}(b, t) = 0, \tag{1.4}$$

the initial condition is taken as

$$u(x, 0) = f(x), \quad a < x < b \tag{1.5}$$

where $f(x)$ a localized disruption that happens inside $[a, b]$. For the numerical solution of MRLWE, several approaches have been utilized, such as cubic B-spline finite element method (FEM) [12], finite difference method [16], Adomian decomposition method [17] and collocation method [18]. The numerical solutions for the GRLWE are based on quartic B-spline functions, cubic B-spline Galerkin FEM, and cubic-quadratic B-spline Petrov-Galerkin technique, as mentioned in [14], [15] and [4]. We will employ BSLSM with change weight functions to solve (1.2)-(1.5) in this study by introducing an approximate simulation of five different varieties of the suggested approach. This method was inspired by a prior articles that combined B-spline Galerkin algorithms with change weight functions and combined B-spline least-squares algorithms with change weight functions [1] and [2] respectively.

Definition 1.1. *[13] Knots are places where the spline function can change form from one polynomial to another, whereas nodes are points where the spline function's values are defined*

Definition 1.2 ([8] , [22]). *Given m real values x_i , called knots, with $x_0 \le x_1 \le \ldots \le x_{m-1}$, a B-spline of degree n by using the Cox-de Boor recursion formula , given by the relations*

$$B_{j,0} = \begin{cases} 1 & if \ x_j \le x \le x_{j+1} \\ 0 & otherwise \end{cases}$$

$$B_{j,0} = \frac{x - x_j}{x_{j+n} - x_j} B_{j,n-1}(x) + \frac{x_{j+n+1} - x}{x_{j+n+1} - x_{j+1}} B_{j+1,n-1}(x), \quad j = 0, \ldots, m - n - 2$$

Note that $j + n + 1$ cannot exceed m-1, which limits both j and n .

The B-spline functions are employed as basis functions in numerical techniques for the approximate solutions of BVPs encountered in range of scientific applications, such as FEM, collocation, Galerkin, and least-square approaches [7].

2. Approximation of the MRLWE by B-Spline Least-Square Methods with Change Weight Function

The schemes that are dependent on type M of the BSLSM are now applied as follows:

2.1. Quadratic B-Spline Least-Square Method with Linear Weight Function

Outside of the interval $[x_{m-1}, x_{m+2}]$, the quadratic B-spline $B_m(x)$ and its fundamental derivative vanish [22].

$$\delta \int_0^t \int_a^b (u_t + u_x + \alpha u^2 u_x - \mu u_{xxt})^2 \, dx \, dt = 0 \tag{2.1}$$

is obtained by applying the least-squares formula to (1.2),

$$h\eta = x - x_m, \quad 0 \le \eta \le 1, \tag{2.2}$$

a linear B-spline shape function (BSSF) in terms of η over each element $[x_m, x_{m+1}]$ may be defined by using local transformation [24],

$$A_m = 1 - \eta, \quad A_{m+1} = \eta, \tag{2.3}$$

all splines a part from A_m and A_{m+1} are vanish over $[x_m, x_{m+1}]$. The function $u(\eta, t)$ variation's can be approximated by:

$$u_N(\eta, t) = \sum_{j_1=m}^{m+1} A_{j_1}(\eta) w_{j_1}(t), \tag{2.4}$$

where $w_m(t)$ and $w_{m+1}(t)$ represent element parameters and B-spline $A_m(\eta)$ and $A_{m+1}(\eta)$ represent element shape functions. We transfer the local coordinate ξ, onto each time interval $[t^n, t^{n+1}]$ where, $\Delta t = t^{n+1} - t^n$ and

$$t = \xi \Delta t + t^n, \quad 0 \le \xi \le 1, \tag{2.5}$$

using (2.2) and (2.5) in (2.1), we obtain

$$\delta \int_0^1 \int_0^1 (u_\xi + u_\eta + \frac{\alpha \Delta t}{h} \hat{u}^2 u_\eta - \frac{\mu}{h^2} u_{\eta\eta\xi})^2 d\eta d\xi = 0, \tag{2.6}$$

with the change in u over all element $[x_m, x_{m+1}]$ the integral equation takes its minimum value. Applying variational principle equation (2.6) becomes:

$$\int_0^1 \int_0^1 (u_\xi + u_\eta + \lambda u_\eta - \beta u_{\eta\eta\xi}) \delta(u_\xi + u_\eta + \lambda u_\eta - \beta u_{\eta\eta\xi}) d\eta d\xi = 0, \tag{2.7}$$

where, $\lambda = \frac{\alpha \Delta t}{h} \hat{u}^2$ and $\beta = \frac{\mu}{h^2}$.

To apply the least-square method (LSM) which 14ns into Petrov-Galerkin method ([6] , [10]) by (2.7), let , $\delta(u_\xi + u_\eta + \lambda u_\eta - \beta u_{\eta\eta\xi})$ be the weight function.

By using (2.2), (2.4) and (2.7) approximate the variation of the function $u_N(x, t)$ over the typical element $[x_m, x_{m+1}]$ by [9]

$$u_N(\eta, \xi) = \sum_{i_1=m}^{m+1} A_{i_1}(\eta)(w_{i_1}^n + \xi \Delta w_{i_1}^n), \tag{2.8}$$

where w_m^n and w_{m+1}^n are nodal parameters at the beginning of the time step Δt. Δw_m^n and Δw_{m+1}^n are the incremarkent of this parameters in Δt. We write the weight function as

$$\delta w1 = \sum_{i_1=m}^{m+1} w_{i_1}\Delta w_{i_1} = \delta(u_\xi + u_\eta + \lambda u_\eta),$$

by using (2.8) such that

$$\delta u_N(\eta,\xi) = \sum_{i_1=m}^{m+1} \xi A_{i_1}(\eta)\Delta w_{i_1}^n.$$

Now, we get

$$w1 = \delta(u_\xi + u_\eta + \lambda u_\eta - \beta u_{\eta\eta\xi}) = A_{i_1}(\eta) + \xi \acute{A}_{i_1}(\eta) + \lambda\xi \acute{A}_{i_1}(\eta), \tag{2.9}$$

substituting (2.9) into (2.7) gives:

$$\int_0^1\int_0^1 (u_\xi + u_\eta + \lambda u_\eta - \beta u_{\eta\eta\xi})(A_{i_1}(\eta) + \xi\acute{A}_{i_1}(\eta) + \lambda\xi)d\eta d\xi = 0, \tag{2.10}$$

using (2.2) and (2.5), we get:

$$u_N(\eta,\xi) = \sum_{i_2=m-1}^{m+1} B_{i_2}(\eta)(\gamma_{i_2}^n + \xi\Delta\gamma_{i_2}^n) \tag{2.11}$$

where, $B_{m-1}(\eta), B_m(\eta)$ and $B_{m+1}(\eta)$ are BSSFs, $\gamma_{m-1}^n, \gamma_m^n$ and γ_{m+1}^n are nodal parameters at the initial time steps, $\Delta\gamma_{m-1}^n, \Delta\gamma_m^n$ and $\Delta\gamma_{m+1}^n$ are the incremarkent of this parameters in Δt. A quadratic BSSF in terms of η over the element $[x_m, x_{m+1}]$ can be defined as

$$B_{m-1} = (1-\eta)^2, \qquad B_m = 1 + 2\eta - 2\eta^2, \qquad B_{m+1} = \eta^2, \tag{2.12}$$

all spline a part from B_{m-1}, B_m and B_{m+1} are zero over $[x_m, x_{m+1}]$. Substituting (2.11) in (2.10), integration with respect to ξ and integration by part as required leads to the following system of equations for each individual element

$$\sum_{i_2=m-1}^{m+1}\left\{\int_0^1\left[A_{i_1}B_{i_2} + \frac{(1+\lambda)}{2}(A_{i_1}\acute{B}_{i_2} + \acute{A}_{i_1}B_{i_2}) + (\frac{(1+\lambda)^2}{3} + \beta)\acute{A}_{i_1}\acute{B}_{i_2}\right]d\eta - \beta A_{i_1}\acute{B}_{i_2}|_0^1\right\}\Delta\gamma_{i_2}^n$$
$$+\sum_{i_2=m-1}^{m+1}\left\{\int_0^1\left[(1+\lambda)A_{i_1}\acute{B}_{i_2} + \frac{(1+\lambda)^2}{2}\acute{A}_{i_1}\acute{B}_{i_2}\right]d\eta\right\}\gamma_{i_2}^n = 0$$

which can be written in matrix form as follows

$$\left[X_1^e + \frac{(1+\lambda)}{2}(Q_1^e + (Q_1^e)^T) + (\frac{(1+\lambda)^2}{3} + \beta)Y_1^e - \beta Z_1^e\right]\Delta\gamma^e + \left[(1+\lambda)Q_1^e + \frac{(1+\lambda)^2}{2}Y_1^e\right]\gamma^e = 0$$

where, $\gamma^e = (\gamma_{m-1}^n, \gamma_m^n, \gamma_{m+1}^n)^T$ is element parameter and the element matrices X_1^e, Q_1^e, Y_1^e and Z_1^e are rectangular 2×3 given as:

$$X_1^e = \int_0^1 A_{i_1}B_{i_2}d\eta = \frac{1}{12}\begin{pmatrix}3 & 8 & 1\\ 1 & 8 & 3\end{pmatrix}, \quad Y_1^e = \int_0^1 \acute{A}_{i_1}\acute{B}_{i_2}d\eta = \begin{pmatrix}1 & 0 & -1\\ -1 & 0 & 1\end{pmatrix},$$
$$Q_1^e = \int_0^1 A_{i_1}\acute{B}_{i_2}d\eta = \frac{1}{3}\begin{pmatrix}-2 & 1 & 1\\ -1 & -1 & 2\end{pmatrix}, \quad Z_1^e = A_{i_1}\acute{B}_{i_2}|_0^1 = \begin{pmatrix}2 & -2 & 0\\ 0 & -2 & 2\end{pmatrix},$$

where suffices i_1 take only the value 1 and 2 and i_2 takes values $m-1, m$ and $m+1$. Assembling together contributions from all element, yields the global system of matrix equation:

$$\left[X_1 + \frac{(1+\lambda)}{2}(Q_1 + (Q_1)^T) + (\frac{(1+\lambda)^2}{3} + \beta)Y_1 - \beta Z_1\right] \Delta\gamma + \left[(1+\lambda)Q_1 + \frac{(1+\lambda)^2}{2}Y_1\right]\gamma = 0, \tag{2.13}$$

where, $\gamma = (\gamma_{-1}, \gamma_0, \gamma_1, \ldots, \gamma_N)^T$ is a global element parameter. Identifying $\gamma = \gamma^n$ and $\Delta\gamma = \gamma^{n+1} - \gamma^n$ in (2.13) obtain the $(N+1) \times (N+2)$ matrix system.

$$\begin{aligned} &\left[X_1 + \frac{(1+\lambda)}{2}(Q_1 + (Q_1)^T) + (\frac{(1+\lambda)^2}{3} + \beta)Y_1 - \beta Z_1\right]\gamma^{n+1} \\ &= \left[X_1 + \frac{(1+\lambda)}{2}(-Q_1 + (Q_1)^T) - (\frac{(1+\lambda)^2}{6} + \beta)Y_1 - \beta Z_1\right]\gamma^n, \end{aligned} \tag{2.14}$$

The matrices X_1, Y_1 and Q_1 are penta-diagonal rectangular matrices with the following row format:

$$X_1 = \frac{1}{12}(1, 11, 11, 1, 0), \quad Y_1 = (-1, 1, 1, -1, 0), \quad Q_1 = \frac{1}{3}(-1, -3, 3, 1, 0).$$

Over the element $[x_m, x_{m+1}]$, the element constant is given by:

$$\lambda = \frac{3\Delta t}{h}(\gamma^n_{m-1} + \gamma^n_m)^2.$$

We apply the boundary conditions (1.3) and (1.4) for the system (2.14) to make the matrix equation square, and therefore $\gamma^n_{-1} = \gamma^n_0$, that is, the variable γ^n_{-1} , may be remarkoved from this system.

Remark 2.1. *To iterate system* (2.14), *the initial vector of parameter* $\gamma^0 = (\gamma^0_{-1}, \gamma^0_0, \ldots, \gamma^0_N)$ *must be found. The approximation*

$$u_N(x,t) = \sum_{i_2=-1}^{N} B_{i_2}(x)\gamma_{i_2}(t)$$

is rewritten across the interval $[a,b]$ *at time* $t = 0$ *as follows*

$$u_N(x,0) = \sum_{i_2=-1}^{N} B_{i_2}\gamma^0_{i_2} .\square$$

The following conditions at the mesh points x_{i2}

$$u_N(x_{i2}, 0) = u(x_{i2}, 0), \quad i_2 = 0, \ldots, N, \quad \acute{u}_N(x_0, 0) = \acute{u}(x_N, 0) = 0, \quad \acute{\acute{u}}_N(x_0, 0) = \acute{\acute{u}}(x_N, 0) = 0,$$

lead to

$$\begin{pmatrix} 2 & -2 & & & \\ 1 & 1 & & & \\ & \ddots & \ddots & & \\ & & & 1 & 1 \\ & & & 2 & -2 \end{pmatrix} \begin{pmatrix} \gamma^0_{-1} \\ \gamma^0_0 \\ \vdots \\ \gamma^0_{N-1} \\ \gamma^0_N \end{pmatrix} = \begin{pmatrix} 0 \\ u(x_0) \\ \vdots \\ u(x_{N-1}) \\ u(x_N) \end{pmatrix}$$

to solve this system, must be convert to a tridiagonal matrix by remarkoving the first row, and then use the Thomas procedure[20].

2.2. Cubic B-Spline Least-Square Method with Quadratic Weight Function

At all element $[x_m, x_{m+1}]$ using local transformation (2.2), the cubic B-spline, shape functions in term of η over $[x_m, x_{m+1}]$ can be described as [22]:

$$C_{m-1} = (1-\eta)^3, \qquad C_m = 1 + 3(1-\eta) + 3(1-\eta)^2 - 3(1-\eta)^3,$$
$$C_{m+1} = 1 + 3\eta + 3\eta^2 - 3\eta^3, \qquad C_{m+2} = \eta^3,$$

all splines except from C_{m-1}, C_m, C_{m+1} and C_{m+2} vanish over $[x_m, x_{m+1}]$. Change of the function $u(\eta, t)$ over this element approximated by:

$$u_N(\eta, t) = \sum_{i_3=m-1}^{m+2} C_{i_3}(\eta)\sigma_{i_3}(t). \tag{2.15}$$

The spline $C_m(x)$ vanishes except at $[x_{m-2}, x_{m+2}]$.

The variation of the function $u_N(x, t)$ over the usual element can be approximated by utilizing (2.2), (2.5) and (2.15) $[x_m, x_{m+1}]$ by [6]

$$u_N(\eta, \xi) = \sum_{i_3=m-1}^{m+2} C_{i_3}(\eta)(\sigma^n_{i_3} + \xi\Delta\sigma^n_{i_3}), \tag{2.16}$$

where $C_{m-1}(\eta), C_m(\eta), C_{m+1}(\eta)$ and $C_{m+2}(\eta)$ are BSSFs, $\sigma^n_{m-1}, \sigma^n_m, \sigma^n_{m+1}$ and σ^n_{m+2} are nodal parameters in the start of the time steps Δt , $\Delta\sigma^n_{m-1}, \Delta\sigma^n_m, \Delta\sigma^n_{m+1}$ and $\Delta\sigma^n_{m+2}$ are the incremarkents of this parameters at each Δt .

We can write the weight function (2.7) as $w2(x)$ quadratic B-spline

$$\delta w2 = \sum_{i_2=m-1}^{m+1} w2_{i_2}\Delta\gamma_{i_2} = \delta(u_\xi + (1+\lambda)u_\eta - \beta u_{\eta\eta\xi}),$$

using (2.11) such that

$$\delta u_N(\eta, \xi) = \sum_{i_2=m-1}^{m+1} \xi B_{i_2}(\eta)\Delta\gamma^n_{i_2}$$

Now, we get

$$w2 = \delta(u_\xi + (1+\lambda)u_\eta - \beta u_{\eta\eta\xi}) = B_{i_2}(\eta) + (1+\lambda)\xi \acute{B}_{i_2}(\eta) - \beta \acute{\acute{B}}_{i_2}(\eta),$$

by inserting the previous equation in (2.7) yields

$$\int_0^1\int_0^1 [u_\xi + (1+\lambda)u_\eta - \beta u_{\eta\eta\xi}][B_{i_2}(\eta) + (1+\lambda)\xi\acute{B}_{i_2}(\eta) - \beta\acute{\acute{B}}_{i_2}(\eta)]d\eta d\xi = 0, \tag{2.17}$$

the following system of equations for each individual element is obtained by inserting (2.16) in (2.17), integrating with respect to ξ , and integrating by part as required:

$$\sum_{i_3=m-1}^{m+2} \left\{ \int_0^1 [B_{i_2}C_{i_3} + \frac{(1+\lambda)}{2}(B_{i_2}\acute{C}_{i_3} + \acute{B}_{i_2}C_{i_3}) + (\frac{(1+\lambda)^2}{3} + 2\beta)\acute{B}_{i_2}\acute{C}_{i_3} - \frac{\beta(1+\lambda)}{2}(\acute{B}_{i_2}\acute{\acute{C}}_{i_3} + \acute{\acute{B}}_{i_2}\acute{C}_{i_3}) \right.$$
$$\left. +\beta^2\acute{\acute{B}}_{i_2}\acute{\acute{C}}_{i_3}]d\eta - \beta(B_{i_2}\acute{C}_{i_3} + \acute{B}_{i_2}C_{i_3})|_0^1 \right\} \Delta\sigma^n_{i_3} + \sum_{i_3=m-1}^{m+2} \left\{ \int_0^1 [(1+\lambda)B_{i_2}\acute{C}_{i_3} + \frac{(1+\lambda)^2}{2}\acute{B}_{i_2}\acute{C}_{i_3} \right.$$
$$\left. -(1+\lambda)\beta\acute{\acute{B}}_{i_2}\acute{C}_{i_3}]d\eta \right\} \sigma^n_{i_3} = 0$$

which can be written in matrix form as follows

$$\left[X_2^e + \frac{(1+\lambda)}{2}(Q_2^e + (Q_2^e)^T) + (\frac{(1+\lambda)^2}{3} + 2\beta)Y_2^e - \frac{\beta(1+\lambda)}{2}(G_2^e + (G_2^e)^T) + \beta^2 M_2^e - \beta Z_2^e\right]\Delta\sigma^e$$
$$+\left[(1+\lambda)Q_2^e + \frac{(1+\lambda)^2}{2}Y_2^e - (1+\lambda)\beta(G_2^e)^T\right]\sigma^e = 0$$

where, $\sigma^e = (\sigma_{m-1}^n, \sigma_m^n, \sigma_{m+1}^n, \sigma_{m+2}^n)^T$ is element parameter and the element matrices $X_2^e, Q_2^e, Y_2^e, G_2^e$ and Z_2^e are rectangular 3×4 given as:

$$X_2^e = \int_0^1 B_{i_2} C_{i_3} d\eta = \frac{1}{60}\begin{pmatrix} 10 & 71 & 38 & 1 \\ 19 & 221 & 221 & 19 \\ 1 & 38 & 71 & 10 \end{pmatrix}, \; Y_2^e = \int_0^1 \acute{B}_{i_2}\acute{C}_{i_3} d\eta = \frac{1}{2}\begin{pmatrix} 3 & 5 & -7 & -1 \\ -2 & 2 & 2 & -2 \\ -1 & -7 & 5 & 3 \end{pmatrix},$$

$$Q_2^e = \int_0^1 B_{i_2}\acute{C}_{i_3} d\eta = \frac{1}{10}\begin{pmatrix} -6 & -7 & 12 & 1 \\ -13 & -41 & 41 & 13 \\ -1 & -12 & 7 & 6 \end{pmatrix}, \; G_2^e = \int_0^1 \acute{B}_{i_2}\acute{\acute{C}}_{i_3} d\eta = \begin{pmatrix} -4 & 6 & 0 & -2 \\ 2 & -6 & 6 & -2 \\ 2 & 0 & -6 & 4 \end{pmatrix},$$

$$Z_2^e = (B_{i_2}\acute{C}_{i_3} + \acute{B}_{i_2}C_{i_3})|_0^1 = \begin{pmatrix} 5 & 8 & -1 & 0 \\ 1 & -13 & -13 & 1 \\ 0 & -1 & 8 & 5 \end{pmatrix}, \; M_2^e = \int_0^1 \acute{\acute{B}}_{i_2}\acute{\acute{C}}_{i_3} d\eta = \begin{pmatrix} 6 & -6 & -6 & 6 \\ -12 & 12 & 12 & -12 \\ 6 & -6 & -6 & 6 \end{pmatrix},$$

where it is sufficient only for the numbers 1, 2, and 3 are used in i_2. For the usual element $[x_m, x_{m+1}]$, i_3 accepts $m-1, m, m+1$ and $m+2$. The global system of matrix equations is obtained by adding the contributions of all elements:

$$\left[X_2 + \frac{(1+\lambda)}{2}(Q_2 + (Q_2)^T) + (\frac{(1+\lambda)^2}{3} + 2\beta)Y_2 - \frac{\beta(1+\lambda)}{2}(G_2 + (G_2)^T) + \beta^2 M_2 - \beta Z_2\right]\Delta\sigma$$
$$+\left[(1+\lambda)Q_2 + \frac{(1+\lambda)^2}{2}Y_2 - (1+\lambda)\beta(G_2)^T\right]\sigma = 0, \tag{2.18}$$

where, a global element parameter is $\sigma = (\sigma_{-1}, \sigma_0, \sigma_1, \ldots, \sigma_{N+2})^T$. Identifying $\sigma = \sigma^n$ and $\Delta\sigma = \sigma^{n+1} - \sigma^n$ in the following equation to get $(N+2) \times (N+3)$ matrix system.

$$\left[X_2 + \frac{(1+\lambda)}{2}(Q_2 + (Q_2)^T) + (\frac{(1+\lambda)^2}{3} + 2\beta)Y_2 - \frac{\beta(1+\lambda)}{2}(G_2 + (G_2)^T) + \beta^2 M_2 - \beta Z_2\right]\sigma^{n+1}$$
$$= \left[X_2 + \frac{(1+\lambda)}{2}(-Q_2 + (Q_2)^T) - (\frac{(1+\lambda)^2}{6} - 2\beta)Y_2 - \frac{\beta(1+\lambda)}{2}(G_2 - (G_2)^T) + \beta^2 M_2 - \beta Z_2\right]\sigma^n, \tag{2.19}$$

The matrices X_2, Y_2, Q_2 and Z_2 are septa-diagonal rectangular matrices, and each row has the following form:

$$X_2 = \frac{1}{60}(1, 57, 302, 302, 57, 1, 0), \qquad Y_2 = \frac{1}{2}(-1, -9, 10, 10, -9, -1, 0),$$
$$Q_2 = \frac{1}{10}(-1, -25, -40, 40, 25, 1, 0), \qquad G_2 = (2, 2, -16, 16, -2, -2, 0)$$
$$M_2 = \frac{1}{10}(6, -18, 12, 12, -18, 6, 0), \qquad Z_2 = (0, 0, 0, 0, 0, 0, 0).$$

The element constant for λ over the element $[x_m, x_{m+1}]$ is given by:

$$\lambda = \frac{3\Delta t}{h}(\sigma^n_{m-1} + 4\sigma^n_m + \sigma^n_{m+1})^2.$$

We apply the boundary conditions (1.3) and (1.4) to the system (2.19), resulting in $\sigma_{-1} = \sigma_1$, $\sigma_{N+1} = \sigma_{N-1}$, which means the variables σ_{-1} and σ_{N+1} can be remarkoved from the equation. The initial vector of the parameter $\sigma 0$ is determined by remarkark(2.1) as follows:

$$\begin{pmatrix} 3 & 0 & -3 & & & \\ 1 & 4 & 1 & & & \\ & & \ddots & \ddots & \ddots & \\ & & & 1 & 4 & 1 \\ & & & 3 & 0 & -3 \end{pmatrix} \begin{pmatrix} \sigma^0_{-1} \\ \sigma^0_0 \\ \vdots \\ \sigma^0_N \\ \sigma^0_{N+1} \end{pmatrix} = \begin{pmatrix} 0 \\ u(x_0) \\ \vdots \\ u(x_N) \\ 0 \end{pmatrix}. \tag{2.20}$$

To solve this system, first convert it to tridiagonal form by deleting the first and last equations, and then use the Thomas procedure to solve it.

2.3. Quartic B-Spline Least-Square Method with Cubic Weight Function

The quartic B-spline, which uses local transformation (2.2) to define shape functions in terms of η over every $[x_m, x_{m+1}]$, may be given by [22]

$$D_{m-2} = (1-\eta)^4, \qquad D_{m-1} = (2-\eta)^4 - 5(1-\eta)^4,$$
$$D_m = (3-\eta)^4 - 5(2-\eta)^4 + 10(1-\eta)^4, \qquad D_{m+1} = (1+\eta)^4 - 5\eta^4, \quad D_{m+2} = \eta^4,$$

all splines a part from $D_{m-2}, D_{m-1}, D_m, D_{m+1}$ and D_{m+2} are zero over $[x_m, x_{m+1}]$. The variation of the function $u(\eta, t)$ over $[x_m, x_{m+1}]$ is approximated by:

$$u_N(\eta, t) = \sum_{i_4=m-2}^{m+2} D_{i_4}(\eta)\rho_{i_4}(t). \tag{2.21}$$

The variation of the function $u_N(x, t)$ over $[x_m, x_{m+1}]$ is approximated by [6] by utilizing (2.2), (2.5) and (2.21) receptively.

$$u_N(\eta, \xi) = \sum_{i_4=m-2}^{m+2} D_{i_4}(\eta)(\rho^n_{i_4} + \xi\Delta\rho^n_{i_4}), \tag{2.22}$$

where $D_{m-2}(\eta), D_{m-1}(\eta), D_m(\eta), D_{m+1}(\eta)$ and $D_{m+2}(\eta)$ are BSSFs, $\rho^n_{m-2}, \rho^n_{m-1}, \rho^n_m$ ρ^n_{m+1} and ρ^n_{m+2} are nodal parameters at the start of each time steps Δt , $\Delta\rho^n_{m-2}, \Delta\rho^n_{m-1}, \Delta\rho^n_m, \Delta\rho^n_{m+1}$ and $\Delta\rho^n_{m+2}$ are the incremarkents of this parameters at each Δt .

The weight function $w3(x)$ for cubic B-spline by (2.7) can be expressed as follows

$$\delta w3 = \sum_{i_3=m-2}^{m+1} w3_{i_4}\Delta\rho_{i_3} = \delta(u_\xi + (1+\lambda)u_\eta - \beta u_{\eta\eta\xi}),$$

Using (2.7) such that

$$\delta u_N(\eta, \xi) = \sum_{i_3=m-2}^{m+1} \xi C_{i_3}(\eta)\Delta\rho^n_{i_3}$$

Now, we get

$$w3=\delta(u_\xi+(1+\lambda)u_\eta-\beta u_{\eta\eta\xi})=C_{i_3}(\eta)+(1+\lambda)\xi\acute{C}_{i_3}(\eta)-\beta\acute{\acute{C}}_{i_3}(\eta),$$

substituting the previous equation in (2.7) yields

$$\int_0^1\int_0^1(u_\xi+(1+\lambda)u_\eta-\beta u_{\eta\eta\xi})(C_{i_3}(\eta)+(1+\lambda)\xi\acute{C}_{i_3}(\eta)-\beta\acute{\acute{C}}_{i_3}(\eta))d\eta d\xi=0, \tag{2.23}$$

The following system of equations for each individual element is obtained by putting (2.22) in (2.23), integrating with regard to ξ , and integrating by part as required:

$$\sum_{i_4=m-2}^{m+2}\left\{\int_0^1\left[C_{i_3}D_{i_4}+\frac{(1+\lambda)}{2}(C_{i_3}\acute{D}_{i_4}+\acute{C}_{i_3}D_{i_4})+(\frac{(1+\lambda)^2}{3}+2\beta)\acute{C}_{i_3}\acute{D}_{i_4})-\frac{\beta(1+\lambda)}{2}(\acute{C}_{i_3}\acute{\acute{D}}_{i_4}+\acute{\acute{C}}_{i_3}\acute{D}_{i_4})\right.\right.$$
$$\left.\left.+\beta^2\acute{\acute{C}}_{i_3}\acute{\acute{D}}_{i_4}\right]d\eta-\beta(C_{i_3}\acute{D}_{i_4}+\acute{C}_{i_3}D_{i_4})|_0^1\right\}\Delta\rho_{i_4}^n+\sum_{i_4=m-2}^{m+2}\left\{\int_0^1\left[(1+\lambda)C_{i_3}\acute{D}_{i_4}+\frac{(1+\lambda)^2}{2}\acute{C}_{i_3}\acute{D}_{i_4}-\beta(1+\lambda)\right.\right.$$
$$\left.\left.\acute{\acute{C}}_{i_3}\acute{D}_{i_4}\right]d\eta\right\}\rho_{i_4}^n=0$$

which can be written in matrix form as follows

$$\left[X_3^e+\frac{(1+\lambda)}{2}(Q_3^e+(Q_3^e)^T)+(\frac{(1+\lambda)^2}{3}+2\beta)Y_3^e-\frac{\beta(1+\lambda)}{2}(G_3^e+(G_3^e)^T)+\beta^2M_3^e-\beta Z_3^e\right]\Delta\rho^e$$
$$+\left[(1+\lambda)Q_3^e+\frac{(1+\lambda)^2}{2}Y_3^e-\beta(1+\lambda)M_3^T\right]\rho^e=0$$

where, $\rho^e=(\rho_{m-2}^n,\rho_{m-1}^n,\rho_m^n,\rho_{m+1}^n,\rho_{m+2}^n)^T$ is element parameter and the element matrices $X_3^e,Q_3^e,Y_3^e,G_3^e,M_3^e$ and Z_3^e are rectangular 4×5 given as:

$$X_3^e=\int_0^1 C_{i_3}D_{i_4}d\eta=\frac{1}{280}\begin{pmatrix}35&594&892&158&1\\211&4794&10196&3190&89\\89&3190&10196&4794&211\\1&158&892&594&35\end{pmatrix},$$

$$Y_3^e=\int_0^1\acute{C}_{i_3}\acute{D}_{i_4}d\eta=\frac{1}{5}\begin{pmatrix}10&61&-33&-37&-1\\9&141&33&-165&-18\\-18&-165&33&141&9\\-1&-37&-33&61&10\end{pmatrix},$$

$$Q_3^e=\int_0^1 C_{i_3}\acute{D}_{i_4}d\eta=\frac{1}{35}\begin{pmatrix}-20&-109&69&59&1\\-129&-1059&255&873&60\\-60&-873&-255&1059&129\\-1&-59&-69&109&20\end{pmatrix},$$

$$G_3^e = \int_0^1 \acute{C}_{i_3} \acute{\acute{D}}_{i_4} d\eta = \frac{1}{5}\begin{pmatrix} -36 & -6 & 114 & -66 & -6 \\ -42 & -162 & 378 & -102 & -72 \\ 72 & 102 & -378 & 162 & 42 \\ 6 & 66 & -114 & 6 & 36 \end{pmatrix},$$

$$M_3^e = \int_0^1 \acute{\acute{C}}_{i_3} \acute{\acute{D}}_{i_4} d\eta = \begin{pmatrix} 18 & 12 & -72 & 36 & 6 \\ -30 & 12 & 72 & -60 & 6 \\ 6 & -60 & 72 & 12 & -30 \\ 6 & 36 & -72 & 12 & 18 \end{pmatrix},$$

$$Z_3^e = (C_{i_3} \acute{\acute{D}}_{i_4} + \acute{C}_{i_3} D_{i_4})|_0^1 = \begin{pmatrix} -7 & 45 & 21 & -1 & 0 \\ 16 & 41 & -93 & -37 & 1 \\ 1 & -37 & -93 & 41 & 16 \\ 0 & -1 & 21 & 45 & 7 \end{pmatrix},$$

where it is sufficient i_3 only accepts the values 1, 2, 3, and 4, while i_4 accepts the values $m-2, m-1, m, m+1$ and $m+2$ for $[x_m, x_{m+1}]$. The global system of matrix equations is obtained by adding all contributions from all elements:

$$\left[X_3 + \frac{(1+\lambda)}{2}(Q_3 + (Q_3)^T) + (\frac{(1+\lambda)^2}{3} + 2\beta)Y_3 - \frac{\beta(1+\lambda)^2}{2}(G_3 + (G_3)^T) + \beta^2 M_3 - \beta Z_3\right]\Delta\rho + \left[(1+\lambda)Q_3 + \frac{(1+\lambda)^2}{2}Y_3 - \beta(1+\lambda)M_3^T\right]\rho = 0, \tag{2.24}$$

where, $\rho = (\rho_{-2}, \rho_{-1}, \rho_0, \rho_1, \ldots, \rho_{N+2})^T$ is a global element parameter. Identifying $\rho = \rho^n$ and $\Delta\rho = \rho^{n+1} - \rho^n$ in (2.30) obtain the $(N+3)\times(N+4)$ matrix system.

$$\left[X_3 + \frac{(1+\lambda)}{2}(Q_3 + (Q_3)^T) + (\frac{(1+\lambda)^2}{3} + 2\beta)Y_3 - \frac{\beta(1+\lambda)}{2}(G_3 + (G_3)^T) + \beta^2 M_3 - \beta Z_3\right]\rho^{n+1} = \left[X_3 + \frac{(1+\lambda)}{2}(-Q_3 + (Q_3)^T) - (\frac{(1+\lambda)^2}{6} + 2\beta)Y_3 - \frac{\beta(1+\lambda)}{2}(G_3 + (G_3)^T) + \beta(\beta M_3 + (1+\lambda)M_3^T) - \beta Z_3\right]\rho^n, \tag{2.25}$$

The rectangular nonic-diagonal matrices X_3, Y_3, Q_3, G_3, M_3 and Z_3 have the following row form:

$$X_3 = \frac{1}{280}(1, 247, 4293, 15619, 15619, 4293, 247, 1, 0), \quad Y_3 = \frac{1}{5}(-1, -55, -189, 245, 245, -189, -55, -1, 0),$$

$$Q_3 = \frac{1}{35}(-1, -119, -1071, -1225, 1225, 1071, 119, 1, 0), \quad G_3 = \frac{1}{5}(6, 138, -54, -570, 570, 54, -138, -6, 0),$$

$$M_3 = (6, 42, -162, 114, 114, -162, 42, 6, 0).$$

The element constant for λ over $[x_m, x_{m+1}]$ is given by:

$$\lambda = \frac{3\Delta t}{h}(\rho_{m-2}^n + 11\rho_{m-1}^n + 11\rho_m^n + \rho_{m+1}^n)^2.$$

To make matrix equation be square we applying the boundary condition (1.3) and (1.4) to the system (2.25), so, $\rho_{-2}^n = -\rho_{-1}^n$, $\rho_{-1}^n = \frac{1}{3}\rho_1^n$, $\rho_{N+1}^n = 3\rho_{N-1}^n$, that is mean the variables ρ_{-2}^n, ρ_{-1}^n and ρ_{N+1}^n

can be eliminated from the system (2.25). By remarkark(2.1) the initial vector of parameter ρ^0 is then determined as:

$$\begin{pmatrix} 12 & -12 & -12 & 12 & & & & \\ 4 & 12 & -12 & 4 & & & & \\ 1 & 11 & 11 & 1 & & & & \\ & & & \ddots & & & & \\ & & & & 1 & 11 & 11 & 1 \\ & & & & 4 & 12 & -12 & 4 \\ & & & & 12 & -12 & -12 & 12 \end{pmatrix} \begin{pmatrix} \rho^0_{-2} \\ \rho^0_{-1} \\ \rho^0_0 \\ \vdots \\ \rho^0_N \\ \rho^0_{N+1} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ u(x_0) \\ \vdots \\ u(x_N) \\ 0 \end{pmatrix},$$

To solve this system, first reduce it to four-diagonal form by remarkoving the first pair and last equation, then use the Thomas procedure to solve it.

2.4. Quintic B-Spline Least-Square Method with Quartic Weight Function

The quintic B-spline, which uses local transformation (2.2) to shape functions in terms of η over $[x_m, x_{m+1}]$, may be defined as [22]

$$\begin{aligned} E_{m-2} &= (1-\eta)^5, \\ E_{m-1} &= (2-\eta)^5 - 6(1-\eta)^5, \\ E_m &= (3-\eta)^5 - 6(2-\eta)^5 + 15(1-\eta)^5, \\ E_{m+1} &= (4-\eta)^5 - 6(3-\eta)^5 + 15(2-\eta)^5 - 20(1-\eta)^5, \\ E_{m+2} &= (5-\eta)^5 - 6(4-\eta)^5 + 15(3-\eta)^5 - 20(2-\eta)^5 + 15(1-\eta)^5, \\ E_{m+3} &= \eta^5. \end{aligned}$$

The variation of the function $u(\eta, t)$ over $[x_m, x_{m+1}]$ is approximated by:

$$u_N(\eta, t) = \sum_{i_5=m-2}^{m+3} E_{i_5}(\eta) g_{i_5}(t), \tag{2.26}$$

which may be approximated by utilizing (2.2) and (2.5)

$$u_N(\eta, \xi) = \sum_{i_5=m-2}^{m+3} E_{i_5}(\eta)(g^n_{i_5} + \xi \Delta g^n_{i_5}), \tag{2.27}$$

where $E_{m-2}(\eta), E_{m-1}(\eta), E_m(\eta), E_{m+1}(\eta), E_{m+2}(\eta)$ and $E_{m+3}(\eta)$ are BSSFs, $g^n_{m-2}, g^n_{m-1}, g^n_m$ g^n_{m+1}, g^n_{m+2} and g^n_{m+3} are nodal parameters at the initial of the time steps Δt , $\Delta g^n_{m-2}, \Delta g^n_{m-1}, \Delta g^n_m$, $\Delta g^n_{m+1}, \Delta g^n_{m+2}$ and Δg^n_{m+3} are the incremarkents of the nodal parameters in Δt .

We can write the weight function $w4(x)$ quartic B-spline using (2.7)as

$$\delta w4 = \sum_{i_4=m-2}^{m+2} w4_{i_4} \Delta g_{i_3} = \delta(u_\xi + (1+\lambda)u_\eta - \beta u_{\eta\eta\xi}),$$

using (2.22) such that

$$\delta u_N(\eta, \xi) = \sum_{i_4=m-2}^{m+3} \xi D_{i_4}(\eta) \Delta g^n_{i_4}$$

Now, we get

$$w4 = \delta(u_\xi + (1+\lambda)u_\eta - \beta u_{\eta\eta\xi}) = D_{i_4}(\eta) + (1+\lambda)\xi \acute{D}_{i_4}(\eta) - \beta \acute{\acute{D}}_{i_4}(\eta),$$

substituting above equation into (2.7) gives

$$\int_0^1 \int_0^1 (u_\xi + (1+\lambda)u_\eta - \beta u_{\eta\eta\xi})(D_{i_4}(\eta) + (1+\lambda)\xi \acute{D}_{i_4}(\eta) - \beta \acute{\acute{D}}_{i_4}(\eta)) d\eta d\xi = 0, \tag{2.28}$$

The following system of equations for each individual element is obtained by replacing (2.27) in (2.28), integrating with respect to ξ , and integrating by part when required:

$$\sum_{i_5=m-2}^{m+3} \left\{ \int_0^1 \left[D_{i_4}E_{i_5} + \frac{(1+\lambda)}{2}(D_{i_4}\acute{E}_{i_5} + \acute{D}_{i_4}E_{i_5}) + (\frac{(1+\lambda)^2}{3} + 2\beta \acute{D}_{i_4}\acute{E}_{i_5}) - \frac{\beta(1+\lambda)}{2}(\acute{D}_{i_4}\acute{\acute{E}}_{i_5} + \acute{\acute{D}}_{i_4}\acute{E}_{i_5}) + \beta^2 \acute{\acute{D}}_{i_4}\acute{\acute{E}}_{i_5} \right] d\eta - \beta(D_{i_4}\acute{E}_{i_5} + \acute{D}_{i_4}E_{i_5})|_0^1 \right\} \Delta g_{i_5}^n + \sum_{i_5=m-2}^{m+3} \left\{ \int_0^1 \left[(1+\lambda)D_{i_4}\acute{E}_{i_5} + \frac{(1+\lambda)^2}{2}\acute{D}_{i_4}\acute{E}_{i_5} - \beta(1+\lambda) \acute{\acute{D}}_{i_4}\acute{E}_{i_5} \right] d\eta \right\} g_{i_5}^n = 0$$

which can be written in matrix form as follows

$$\left[X_4^e + \frac{(1+\lambda)}{2}(Q_4^e + (Q_4^e)^T) + (\frac{(1+\lambda)^2}{3} + 2\beta)Y_4^e - \frac{\beta(1+\lambda)}{2}(G_4^e + (G_4^e)^T) + \beta^2 M_4^e - \beta Z_4^e \right] \Delta g^e + \left[(1+\lambda)Q_4^e + \frac{(1+\lambda)^2}{2}Y_4^e - \beta(1+\lambda)M_4^T \right] g^e = 0$$

where, $g^e = (g_{m-2}^n, g_{m-1}^n, g_m^n, g_{m+1}^n, g_{m+2}^n, g_{m+3}^n)^T$ is element parameter and the element matrices $X_4^e, Q_4^e, Y_4^e, G_4^e, M_4^e$ and Z_4^e are rectangular 5×6 given as:

$$X_4^e = \int_0^1 D_{i_4}E_{i_5} d\eta = \frac{1}{1260} \begin{pmatrix} 126 & 4747 & 15962 & 8772 & 632 & 1 \\ 1931 & 89797 & 376002 & 281662 & 36467 & 381 \\ 2601 & 155637 & 839682 & 839682 & 155637 & 2601 \\ 381 & 36467 & 281662 & 376002 & 89797 & 1931 \\ 1 & 632 & 8772 & 15962 & 4747 & 126 \end{pmatrix},$$

$$Y_4^e = \int_0^1 \acute{D}_{i_4}\acute{E}_{i_5} d\eta = \frac{1}{14} \begin{pmatrix} 35 & 559 & 298 & -734 & -157 & -1 \\ 176 & 4024 & 5104 & -6272 & -2944 & -88 \\ -122 & -1482 & 1604 & 1604 & -1482 & -122 \\ -88 & -2944 & -6272 & 5104 & 4024 & 176 \\ -1 & -157 & -734 & 298 & 559 & 35 \end{pmatrix},$$

$$Q_4^e = \int_0^1 D_{i_4}\acute{E}_{i_5} d\eta = \frac{1}{126} \begin{pmatrix} -70 & -1051 & -460 & 1330 & 250 & 1 \\ -1121 & -21689 & -20186 & 31550 & 11195 & 251 \\ -1581 & -41415 & -67434 & 67434 & 41415 & 1581 \\ -251 & -11195 & -31550 & 20186 & 21689 & 1121 \\ -1 & -250 & -1330 & 460 & 1051 & 70 \end{pmatrix},$$

$$G_4^e = \int_0^1 \acute{D}_{i_4}\acute{\acute{E}}_{i_5} d\eta = \frac{4}{7} \begin{pmatrix} -20 & -89 & 178 & -10 & -58 & -1 \\ -109 & -841 & 1136 & 628 & -755 & -59 \\ 69 & 117 & -696 & 696 & -117 & -69 \\ 59 & 755 & -628 & -1136 & 841 & 109 \\ 1 & 58 & 10 & -178 & 89 & 20 \end{pmatrix},$$

$$M_4^e = \int_0^1 \acute{D}_{i_4} \acute{E}_{i_5} d\eta = 4 \begin{pmatrix} 10 & 51 & -94 & -4 & 36 & 1 \\ -1 & 81 & -14 & -194 & 111 & 17 \\ -27 & -279 & 306 & 306 & -279 & -27 \\ 17 & 111 & -194 & -14 & 81 & -1 \\ 1 & 36 & -4 & -94 & 51 & 10 \end{pmatrix},$$

$$Z_4^e = (D_{i_4} \acute{E}_{i_5} + \acute{D}_{i_4} E_{i_5})|_0^1 = \begin{pmatrix} 9 & 154 & 264 & 54 & -1 & 0 \\ 67 & 853 & 638 & -502 & -97 & 1 \\ 43 & 171 & -1654 & -1654 & 171 & 43 \\ 1 & -97 & -502 & 638 & 853 & 67 \\ 0 & -1 & 54 & 264 & 154 & 9 \end{pmatrix},$$

where it is sufficient i_4 only accepts the values 1, 2, 3, 4, and 5, while i_5 accepts the values $m-2, m-1, m, m+1, m+2$ and $m+3$ for the typical element $[x_m, x_{m+1}]$. The global system of matrix equation is obtained by combining contributions from all elements:

$$\left[X_4 + \frac{(1+\lambda)}{2}(Q_4 + (Q_4)^T) + (\frac{(1+\lambda)^2}{3} + 2\beta)Y_4 - \frac{\beta(1+\lambda)^2}{2}(G_4 + (G_4)^T) + \beta^2 M_4 - \beta Z_4\right] \Delta g +$$
$$\left[(1+\lambda)Q_4 + \frac{(1+\lambda)^2}{2} Y_4 - \beta(1+\lambda) M_4^T\right] g = 0, \tag{2.29}$$

where, $g = (g_1, \ldots, g_{N+2})^T$ is a global element parameter. Identifying $g = g^n$ and $\Delta g = g^{n+1} - g^n$ in (2.29) obtain the $(N+4) \times (N+5)$ matrix system.

$$\left[X_4 + \frac{(1+\lambda)}{2}(Q_4 + (Q_4)^T) + (\frac{(1+\lambda)^2}{3} + 2\beta)Y_4 - \frac{\beta(1+\lambda)^2}{2}(G_4 + (G_4)^T) + \beta^2 M_4 - \beta Z_4\right] g^{n+1} =$$
$$\left[X_4 + \frac{(1+\lambda)}{2}(-Q_4 + (Q_4)^T) - (\frac{(1+\lambda)^2}{6} + 2\beta)Y_4 - \frac{\beta(1+\lambda)^2}{2}(G_4 + (G_4)^T) + \beta(\beta M_4 + (1+\lambda)M_4^T)\right.$$
$$\left. -\beta Z_4\right] g^n, \tag{2.30}$$

the matrices X_4, Y_4, Q_4, G_4, M_4 and Z_4 are rectangular 11-diagonal and row of each has the following form:

$$X_4 = \frac{1}{1260}(1, 1013, 47840, 455172, 13103540, 13103540, 455192, 47840, 1013, 1, 0),$$
$$Y_4 = \frac{1}{14}(-1, -245, -3800, -7280, 11326, 11326, -7280, -3800, -245, -1, 0),$$
$$Q_4 = \frac{1}{126}(-1, -501, -14106, -73626, -67956, 67956, 73626, 14106, 501, 1, 0),$$
$$G_4 = \frac{4}{7}(1, 117, 834, -798, -2604, 2604, 798, -834, -117, -1, 0),$$
$$M_4 = 4(1, 53, 80, -568, 434, 434, -568, 80, 53, 1, 0).$$

The element constant for λ over $[x_m, x_{m+1}]$ is given by:

$$\lambda = \frac{3\Delta t}{h}(g_{m-2}^n + 26g_{m-1}^n + 66g_m^n + 26g_{m+1}^n + g_{m+2}^n)^2.$$

Applying the boundary conditions (1.3) and (1.4) to the system (2.30), we can make the matrix equation square, so, $g_{-2}^n = 12g_0^n - g_2^n$, $g_{-1}^n = -3g_0^n - g_1^n$, $g_{N+1}^n = -g_{N-1}^n - 3g_N^n$, and $g_{N+2}^n = -g_{N-2}^n + 12g_N^n$, that is mean the variable g_{-2}^n, g_{-1}^n, g_{N+1}^n and g_{N+2}^n, can be eliminated from this system. By remarkark (2.1) the initial vector of parameter g^0 is then determined as:

$$\begin{pmatrix} 20 & 40 & -120 & 40 & 20 & & & & \\ 5 & 50 & 0 & -50 & -5 & & & & \\ 1 & 26 & 66 & 26 & 1 & & & & \\ & & \ddots & & & & & & \\ & & & 1 & 26 & 66 & 26 & 1 \\ & & & 5 & 50 & 0 & -50 & -5 \\ & & & 20 & 40 & -120 & 40 & 20 \end{pmatrix} \begin{pmatrix} g_{-2}^0 \\ g_{-1}^0 \\ g_0^0 \\ \vdots \\ g_N^0 \\ g_{N+1}^0 \\ g_{N+2}^0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ u(x_0) \\ \vdots \\ u(x_N) \\ 0 \\ 0 \end{pmatrix},$$

To solve this system, first convert it to penta-diagonal form by remarkoving the first and last pair of equations, and then use the Thomas procedure.

2.5. Sextic B-Spline Least-Square Method with Quintic Weight Function

By employing local transformation (2.2), the sextic B-spline, shape functions in terms of η over $[x_m, x_{m+1}]$, may be defined as [22]

$$\begin{aligned} F_{m-3} &= (1-\eta)^6, \qquad F_{m-2} = (2-\eta)^6 - 7(1-\eta)^6, \\ F_{m-1} &= (3-\eta)^6 - 7(2-\eta)^6 + 21(1-\eta)^6, \\ F_m &= (4-\eta)^6 - 7(3-\eta)^6 + 21(2-\eta)^6 - 35(1-\eta)^6, \\ F_{m+1} &= (-2-\eta)^6 - 7(-1-\eta)^6 + 21(-\eta)^6, \\ F_{m+2} &= (-1-\eta)^6 - 7(-\eta)^6, \\ F_{m+3} &= (-\eta)^6. \end{aligned}$$

The function $u(\eta, t)$ variation over $[x_m, x_{m+1}]$ and is approximated by:

$$u_N(\eta, t) = \sum_{i_6=m-3}^{m+3} F_{i_6}(\eta)\tau_{i_6}(t). \tag{2.31}$$

Outside the interval $[x_{m-3}, x_{m+4}]$, the spline $F_m(x)$ vanishes. The variation of the function $u_N(x,t)$ over $[x_m, x_{m+1}]$ may be approximated by utilizing (2.2), (2.5) and (2.31)

$$u_N(\eta, \xi) = \sum_{i_6=m-3}^{m+3} F_{i_6}(\eta)(\tau_{i_6}^n + \xi \Delta\tau_{i_6}^n), \tag{2.32}$$

where $F_{m-3}(\eta), F_{m-2}(\eta), F_{m-1}(\eta), F_m(\eta), F_{m+1}(\eta), F_{m+2}(\eta)$ and $F_{m+3}(\eta)$ are BSSFs, $\tau_{m-3}^n, \tau_{m-2}^n, \tau_{m-1}^n, \tau_m^n, \tau_{m+1}^n$, τ_{m+2}^n and τ_{m+3}^n are nodal parameters at the beginning of the time steps Δt , $\Delta\tau_{m-3}^n, \Delta\tau_{m-2}^n, \Delta\tau_{m-1}^n$, $\Delta\tau_m^n$, $\Delta\tau_{m+1}^n, \Delta\tau_{m+2}^n$ and $\Delta\tau_{m+3}^n$ The nodal parameter increases with each Δt . The weight function $w5(x)$ is employed, which for a quintic B-spline can be expressed as

$$\delta w5 = \sum_{i_5=m-2}^{m+3} w5_{i_5} \Delta g_{i_5} = \delta(u_\xi + (1+\lambda)u_\eta - \beta u_{\eta\eta\xi}),$$

using (2.27) such that

$$\delta u_N(\eta,\xi) = \sum_{i_5=m-2}^{m+3} \xi E_{i_5}(\eta)\Delta g_{i_5}^n$$

Now, we get

$$w5 = \delta(u_\xi + (1+\lambda)u_\eta - \beta u_{\eta\eta\xi}) = E_{i_5}(\eta) + (1+\lambda)\xi \acute{E}_{i_5}(\eta) - \beta \acute{\acute{E}}_{i_5}(\eta),$$

substituting above equation into (2.7) gives

$$\int_0^1\int_0^1 (u_\xi + (1+\lambda)u_\eta - \beta u_{\eta\eta\xi})(E_{i_5}(\eta) + (1+\lambda)\xi\acute{E}_{i_5}(\eta) - \beta\acute{\acute{E}}_{i_5}(\eta))d\eta d\xi = 0, \tag{2.33}$$

For each individual element, substituting (2.32) in (2.33), integration with respect to ξ, and integration by part as required results in the following system of equations:

$$\sum_{i_6=m-3}^{m+3}\{\int_0^1[E_{i_5}F_{i_6} + \frac{(1+\lambda)}{2}(E_{i_5}\acute{F}_{i_6} + \acute{E}_{i_5}F_{i_6}) + (\frac{(1+\lambda)^2}{3} + 2\beta)\acute{E}_{i_5}\acute{F}_{i_6}) - \frac{\beta(1+\lambda)}{2}(\acute{E}_{i_5}\acute{\acute{F}}_{i_6} + \acute{\acute{E}}_{i_5}\acute{F}_{i_6}) +$$
$$\beta^2\acute{\acute{E}}_{i_5}\acute{\acute{F}}_{i_6}]d\eta - \beta(E_{i_5}\acute{F}_{i_6} + \acute{E}_{i_5}F_{i_6})|_0^1\}\Delta\tau_{i_6}^n + \sum_{i_6=m-3}^{m+3}\{\int_0^1[(1+\lambda)E_{i_5}\acute{F}_{i_6} + \frac{(1+\lambda)^2}{2}\acute{E}_{i_5}\acute{F}_{i_6} - \beta(1+\lambda)$$
$$\acute{\acute{E}}_{i_5}\acute{F}_{i_6}]d\eta\}\tau_{i_6}^n = 0$$

It can be represented as follows in matrix form

$$[X_5^e + \frac{(1+\lambda)}{2}(Q_5^e + (Q_5^e)^T) + (\frac{(1+\lambda)^2}{3} + 2\beta)Y_5^e - \frac{\beta(1+\lambda)}{2}(G_5^e + (G_5^e)^T) + \beta^2 M_5^e - \beta Z_5^e]\Delta\tau^e +$$
$$[(1+\lambda)Q_5^e + \frac{(1+\lambda)^2}{2}Y_5^e - \beta(1+\lambda)M_5^T]\tau^e = 0$$

the element parameter is $\tau^e = (\tau_{m-3}^n, \tau_{m-2}^n, \tau_{m-1}^n, \tau_m^n, \tau_{m+1}^n, \tau_{m+2}^n, \tau_{m+3}^n)^T$. The element matrices $X_5^e, Q_5^e, Y_5^e, G_5^e, M_5^e$ and Z_5^e are rectangular 6×7 are written as follows:

$$X_5^e = \int_0^1 E_{i_5}F_{i_6}d\eta = \frac{1}{5544}\begin{pmatrix} 462 & 36959 & 244205 & 304250 & 76900 & 2503 & 1 \\ 16171 & 1537535 & 11886590 & 17975130 & 6128395 & 375559 & 1580 \\ 51014 & 5748218 & 52521800 & 96528940 & 42334750 & 3704026 & 25812 \\ 25812 & 3704026 & 42334750 & 96528940 & 5251800 & 5748218 & 51014 \\ 1580 & 375559 & 6128395 & 17975130 & 11886590 & 1537535 & 16171 \\ 1 & 2503 & 76900 & 304250 & 244205 & 36959 & 462 \end{pmatrix},$$

$$Y_5^e = \int_0^1 \acute{E}_{i_5}\acute{F}_{i_6}d\eta = \frac{1}{42}\begin{pmatrix} 162 & 4621 & 11215 & -7190 & -8140 & -631 & -1 \\ 1805 & 83245 & 274990 & -87150 & -237055 & -35455 & -380 \\ 670 & 65170 & 397840 & 94340 & -438850 & -116950 & -2220 \\ -2220 & -116950 & -438850 & 94340 & 397840 & 65170 & 670 \\ -380 & -35455 & -237055 & -87150 & 274990 & 83245 & 1805 \\ -1 & -631 & -8140 & -7190 & 11215 & 4621 & 126 \end{pmatrix},$$

$$Q_5^e = \int_0^1 E_{i_5}\acute{F}_{i_6}d\eta = \frac{1}{962}\begin{pmatrix} -252 & -8861 & -20445 & 14060 & 14480 & 1017 & -1 \\ -9113 & -388303 & -1161290 & 486520 & 950545 & 120623 & 1018 \\ -29558 & -1529148 & -5905750 & 861980 & 5530290 & 1056688 & 15498 \\ -15498 & -1056688 & -5530290 & -861980 & 5905750 & 1529148 & 29558 \\ -1018 & -120623 & -950545 & -486520 & 1161290 & 388303 & 9113 \\ -1 & -1017 & -14480 & -14060 & 20445 & 8861 & 252 \end{pmatrix},$$

$$G_5^e = \int_0^1 \acute{E}_{i_5}\acute{\acute{F}}_{i_6}d\eta = \frac{5}{21}\begin{pmatrix} -70 & -981 & 591 & 1790 & -1080 & -249 & -1 \\ -1051 & -19587 & 912 & 49946 & -19275 & -10695 & -250 \\ -460 & -19266 & -27522 & 83132 & -5664 & -28890 & -1330 \\ 1330 & 28890 & 5664 & -83132 & 27522 & 19266 & 460 \\ 250 & 10695 & 19275 & -49946 & -912 & 19587 & 1051 \\ 1 & 249 & 1080 & -1790 & -591 & 981 & 70 \end{pmatrix},$$

$$M_5^e = \int_0^1 \acute{\acute{E}}_{i_5}\acute{\acute{F}}_{i_6}d\eta = \frac{15}{7}\begin{pmatrix} 35 & 524 & -261 & -1032 & 577 & 156 & 1 \\ 141 & 3324 & 1341 & -10344 & 2751 & 2700 & 87 \\ -298 & -5208 & 2006 & 11376 & -6014 & -1496 & 34 \\ 34 & -1496 & -6414 & 11376 & 2006 & -5208 & -289 \\ 87 & 2700 & 2751 & -10344 & 1341 & 3324 & 141 \\ 1 & 156 & 577 & -1032 & -261 & 524 & 35 \end{pmatrix},$$

$$Z_5^e = (E_{i_5}\acute{F}_{i_6} + \acute{E}_{i_5}F_{i_6})|_0^1 = \begin{pmatrix} 11 & 435 & 1750 & 1270 & 135 & -1 & 0 \\ 206 & 6739 & 20905 & 7110 & -2320 & -241 & 1 \\ 396 & 9694 & 9090 & -37180 & -18760 & 654 & 106 \\ 106 & 654 & -18760 & -37180 & 9090 & 9694 & 396 \\ 1 & -24 & -2320 & 7110 & 20905 & 6739 & 206 \\ 0 & -1 & 135 & 1270 & 1750 & 435 & 11 \end{pmatrix},$$

where it is sufficient i_5 only accepts the values 1, 2, 3, 4, and 5, while i_6 accepts the values $m-3, m-2, m-1, m, m+1, m+2$ and $m+3$ for the typical element $[x_m, x_{m+1}]$. The global system of matrix equation is the sum of all contributions from all elements.

$$[X_5 + \frac{(1+\lambda)}{2}(Q_5 + (Q_5)^T) + (\frac{(1+\lambda)^2}{3} + 2\beta)Y_5 - \frac{\beta(1+\lambda)^2}{2}(G_5 + (G_5)^T) + \beta^2 M_5 - \beta Z_5]\Delta\tau +$$
$$[(1+\lambda)Q_5 + \frac{(1+\lambda)^2}{2}Y_5 - \beta(1+\lambda)M_5^T]\tau = 0, \tag{2.34}$$

where, $\tau = (\tau_{-3}, \ldots, \tau_{N+3})^T$ is a global element parameter. Identifying $\tau = \tau^n$ and $\Delta\tau = \tau^{n+1} - \tau^n$ in (2.34) obtain the $(N+5)\times(N+6)$ matrix system.

$$[X_5 + \frac{(1+\lambda)}{2}(Q_5 + (Q_5)^T) + (\frac{(1+\lambda)^2}{3} + 2\beta)Y_5 - \frac{\beta(1+\lambda)^2}{2}(G_5 + (G_5)^T) + \beta^2 M_5 - \beta Z_5]\tau^{n+1} = [X_5 +$$
$$\frac{(1+\lambda)}{2}(-Q_5 + (Q_5)^T) - (\frac{(1+\lambda)^2}{6} - 2\beta)Y_5 - \frac{\beta(1+\lambda)^2}{2}(G_5 + (G_5)^T) + \beta(\beta M_5 + (1+\lambda)M_5^T) - \beta Z_5]\tau^n, \tag{2.35}$$

the matrices $X_5,\ Y_5,\ Q_5,\ G_5,\ M_5$ and Z_5 are rectangular 13-diagonal and row of each has the following form:

$$\begin{aligned}
X_5 &= \frac{1}{5544}(1, 4083, 478271, 10187685, 66318474, 162512286, 162512286, 66318414, 10187685, 478271, 4083, 1, 0)\\
Y_5 &= \frac{1}{42}(-1, -1011, -45815, -360525, -447810, 855162, 855162, -447810, -360525, -45815, -1011, -1, 0)\\
Q_5 &= \frac{1}{462}(-1, -2035, -150601, -2050851, -7534626, -5986134, 5986134, 7534626, 2050851, 150601, 2035, 1, 0)\\
G_5 &= \frac{21}{5}(1, 499, 13105, 45915, -65190, -130242, 130242, 65190, -45915, -13105, -499, -1, 0)\\
M_5 &= \frac{7}{15}(1, 243, 3311, -75, -22086, 18606, 18606, -22086, -75, 3311, 243, 1, 0).
\end{aligned}$$

The element constant for λ over the element $[x_m, x_{m+1}]$ is given by:

$$\lambda = \frac{3\Delta t}{h}(\tau^n_{m-3} + 57\tau^n_{m-2} + 302\tau^n_{m-1} + 302\tau^n_m + 57\tau^n_{m+1} + \tau^n_{m+2})^2.$$

We apply the boundary conditions (1.3) and (1.4) to the system (2.35), $\tau^n_{-3} = \frac{2800}{54}\tau^n_0 + \frac{2425}{54}\tau^n_1 + \frac{151}{4}\tau^n_2$, $\tau^n_{-2} = \frac{520}{54}\tau^n_0 + \frac{271}{54}\tau^n_1 + \frac{13}{54}\tau^n_2$, $\tau^n_{-1} = \frac{134}{54}\tau^n_0 - \frac{50}{54}\tau^n_1 - \frac{2}{54}\tau^n_2$, $\tau^n_{N+1} = -\frac{1}{16}\tau^n_{N-3} - \frac{41}{16}\tau^n_{N-2} - \frac{171}{16}\tau^n_{N-1} - \frac{131}{16}\tau^n_N$, and $\tau^n_{N+2} = -\frac{7}{16}\tau^n_{N-3} + \frac{225}{16}\tau^n_{N-2} + \frac{1699}{16}\tau^n_{N-1} + \frac{291}{16}\tau^n_N$, making the matrix equation square, that is mean the variable $\tau^n_{-3},\ \tau^n_{-2},\ \tau^n_{-1},\ \tau^n_{N+1}$ and τ^n_{N+2}, can be taken out of this system. The initial vector of the parameter τ^0 is determined as follows by remarkark (2.1):

$$\begin{pmatrix}
30 & 270 & -300 & -300 & 270 & 30 & & & & & & \\
6 & 150 & 240 & -240 & -150 & -60 & & & & & & \\
1 & 57 & 302 & 302 & 57 & 1 & & & & & & \\
 & & & & \ddots & \ddots & \ddots & & & & & \\
 & & & & & & 1 & 57 & 302 & 302 & 57 & 1\\
 & & & & & & 6 & 150 & 240 & -240 & -150 & -6\\
 & & & & & & 30 & 270 & -300 & -300 & 270 & 30
\end{pmatrix}
\begin{pmatrix} \tau^0_{-3}\\ \tau^0_{-2}\\ \tau^0_{-1}\\ \tau^0_0\\ \vdots\\ \tau^0_N\\ \tau^0_{N+1}\\ \tau^0_{N+2} \end{pmatrix}
=
\begin{pmatrix} 0\\ 0\\ 0\\ u(x_0)\\ \vdots\\ u(x_N)\\ 0\\ 0 \end{pmatrix},$$

to solve this system, first reduce it to six-diagonal form by eliminating the first three and last pair of equations and then apply Thomas algorithm.

3. Stability Analysis of Least-Square B-Spline Methods with Different Weight Function

We apply the Von Neumann stability method for the stability of all schemes in this paper, since this method is applicable to linear schemes, the nonlinear term u^2u_x is linearized by taking u as locally constant value [19].

3.1. Stability of Quadratic B-Spline Least-Square Method with Linear B-Spline as a Weight Function

A typical member of the matrix system (2.14) can be written in terms of the nodal parameters γ^n_m as

$$a_1\gamma^{n+1}_{m-2} + a_2\gamma^{n+1}_{m-1} + a_2\gamma^{n+1}_m + a_1\gamma^{n+1}_{m+1} = a_3\gamma^n_{m-2} + a_4\gamma^n_{m-1} + a_4\gamma^n_m + a_3\gamma^n_{m+1},$$

where,

$$a_1 = \frac{1}{12} - \frac{(1+\lambda)^2}{3} - \beta,\quad a_2 = \frac{11}{12} + \frac{(1+\lambda)^2}{3} + \beta,\quad a_3 = \frac{1}{12} - \frac{(1+\lambda)^2}{6} + \beta,\quad a_4 = \frac{11}{12} - \frac{(1+\lambda)^2}{6} - \beta,$$

substitution of $\gamma_m^n = Y_1^n e^{irmh}$, where r is the mode number and h is size of the element , leads to

$$Y_1(a_1 e^{-2irh} + a_2 e^{-irh} + a_2 + a_1 e^{irh}) = a_3 e^{-2irh} + a_4 e^{-irh} + a_4 + a_3 e^{irh},$$

then, $Y_1 = \frac{M_1 + iN_1}{K_1 + iQ_1}$, where,

$$\begin{aligned}
M_1 &= \frac{1}{12} - \frac{(1+\lambda)^2}{6} - \beta + cosh(rh) + (\frac{1}{12} + \frac{(1+\lambda)^2}{6} + \beta)cos(2rh),\\
N_1 &= (-\frac{10}{12} + \frac{2(1+\lambda)^2}{6} + 2\beta)sin(rh) + (-\frac{1}{12} - \frac{(1+\lambda)^2}{6} - \beta)sin(2rh),\\
K_1 &= \frac{11}{12} + \frac{(1+\lambda)^2}{3} + \beta + cos(rh) + (\frac{1}{12} - \frac{(1+\lambda)^2}{3} - \beta)cos(2rh),\\
Q_1 &= (-\frac{10}{12} - \frac{2(1+\lambda)^2}{3} - 2\beta)sin(rh) + (-\frac{1}{12} - \frac{(1+\lambda)^2}{3} + \beta)sin(2rh),
\end{aligned}$$

after simplification, we obtain that $|Y_1| = 1$ and the linearized numerical scheme for the MRLWE is unconditionally stable.

Remark 3.1. *In like manner we can prove that all other numerical schemes for the MRLWE are unconditionally stable.*□

4. Conclusion

We developed a new B-spline least-square technique for solving the generalized regularized long wave equation with change weight functions in this paper, which provided new approximate simulations of five different types of the proposed scheme. These strategies were based on previous researchs [1] and [2] that combined B-spline Galerkin algorithms with change weight functions, as well as a B-spline least-squares algorithm with change weight functions.

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