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On the location of zeros of generalized derivative

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Abstract

Let $P(z) = \prod_{v=1}^{n}(z - z_v)$, be a monic polynomial of degree n, then, $G_\gamma[P(z)] = \sum_{k=1}^{n} \gamma_k \prod_{v=1, v\neq k}^{n} (z - z_v)$, where $\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_n)$ is a n-tuple of positive real numbers with $\sum_{k=1}^{n} \gamma_k = n$, be its generalized derivative. The classical Gauss-Lucas Theorem on the location of critical points have been extended to the class of generalized derivative[4]. In this paper, we extend the Specht Theorem and the results proved by A.Aziz [1] on the location of critical points to the class of generalized derivative.

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1. Introduction

Let P_n denote the linear space of all polynomials of degree n over the field C of complex numbers, $\delta\wp_n$ denotes the collection of all monic polynomials of degree n in P_n, where n is positive integer and R_+^n be the set of all n tuples $\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_n)$ of positive real numbers with $\sum_{v=1}^{n} \gamma_v = n$. Let a polynomial $P(z) \in P_n$ of degree n having all its zeros in $D = \{z \in C : |z| \leq 1\}$. The Gauss-Lucas Theorem [3, p.71], asserts that all its critical points (i.e. zeros of $P'(z)$) also lie in $D = \{z \in C : |z| \leq 1\}$. N.A.Rather et al[4], extended the Gauss-Lucas Theorem [3, p.71], to the class of generalized derivatives. Also if z* is a simple zero of $P(z)$ and $P'(z^*) \neq 0$, thus, by continuity their exists a neighbourhood of z^* which does not contain a critical point of $P(z)$. A method of constructing such a neighbourhood is given by Krawtchouk, who presented it as a refinement of Gauss-Lucas Theorem. Infact he proved,

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Theorem 1.1. *Let $z_1, z_2, \ldots, z_n$, be n , not necessarily distinct , be the zeros of $P(z)$ with a simple zero at z_1. Define*

$$\omega_\nu = \frac{z_\nu + (n-1)z_1}{n}, \qquad \nu = 2, \ldots, n.$$

If M is an open disc or open half plane that contains z_1 but none of the points $\omega_2, \omega_3, \ldots, \omega_n$. Then M is devoid of critical points of the polynomial P(z).

Moreover, Specht [3] using Theorem 1.1 and proved a result on the relationship between zeros and critical points which is also a refinement of Gauss-Lucas Theorem. Infact he proved,

Theorem 1.2. *Let $P(z)$ be a polynomial of degree n with zeros $z_1, z_2, \ldots, z_n$, then the convex hull $K(P(z))$ of $n^2 - n$ points*

$$\omega_{\nu\mu} = \frac{z_\nu + (n-1)z_\mu}{n}, \qquad \nu, \mu \in (1, 2, \ldots, n), \nu \neq \mu.$$

contains all critical points of the polynomial $P(z)$.

Definition: If $z_1, z_2, \ldots, z_n$ be the zeros of polynomial $P(z) \in \delta\wp$ repeated as per their multiplicity, then P(z) $= \prod_{v=1}^{n}(z - z_v)$. For $\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_n)$ is a n-tuple of positive real numbers, we define

$$G_\gamma[P(z)] = \sum_{k=1}^{n} \gamma_k \prod_{v=1, v\neq k}^{n} (z - z_v),$$

with $\sum_{k=1}^{n} \gamma_k = n$, as a generalized derivative[4] of P(z). For $\gamma = (1, 1, \ldots, 1)$, we get

$$G_\gamma[P(z)] = P'(z).$$

N.A.Rather et al[4], extended the Gauss-Lucas Theorem [3, p.71], to the class of generalized derivatives. Infact they proved

Theorem 1.3. *Every convex set containing all the zeros of the zeros of P(z) also contains the zeros $G_\gamma[P(z)]$, for all $\gamma \in R_+^n$.*

Let $P(z^*) = 0$, then the famous Sendov's conjecture [3, p.224], says that the closed disk $|z - z^*| \leq 1$ contains a critical point of $P(z)$. The conjecture has been proved for the polynomials of degree atmost eight[2]. Also, the conjecture is true for some special class of polynomials such as the polynomials having a zero at the origin and the polynomials having all their zeros on $|z| = 1$, as shown in [2]. However, the general version is still unproved. A.Aziz[1], proved the following results regarding the relationship between the zeros and critical points of a polynomial.

Theorem 1.4. *If $P(z)$ is a polynomial of degree n and ω is a zero of $P'(z)$, then for every given real or complex number α, $P(z)$ has at least one zero in the region*

$$\left|\omega - \frac{\alpha + z}{2}\right| \leq \left|\frac{\alpha - z}{2}\right|.$$

Taking $\alpha = 0$ in Theorem 1.1 and noting that

$$\left|\omega - \frac{z}{2}\right| \leq \left|\frac{z}{2}\right|.$$

$$\Rightarrow |\omega - z| \leq |z|.$$

He [1] proved the following corollary.

Corollary 1.5. *If all the zeros of a polynomial $P(z)$ of degree n lie in $D = \{z \in C : |z| \leq 1\}$, and ω is zero of $P'(z)$, then $P(z)$ has at least one zero in both the circles*

$$\left|\omega - \frac{z}{2}\right| \leq \frac{1}{2} \qquad \text{and} \qquad |\omega - z| \leq 1.$$

2. Main Results

In this paper, we extend the Theorem 1.1 and Theorem 1.2 to the class of generalized derivative which is infact the refinement of Theorem 1.3. The first result we prove as

Theorem 2.1. *Let $z_1, z_2, \ldots, z_n$, be n , not necessarily distinct , be the zeros of $P(z)$ with a simple zero at z_1. Define*

$$\omega_\nu = \frac{z_\nu + (n - \gamma_1)z_1}{n}, \qquad \nu = 2, \ldots, n. \tag{2.1}$$

If M is an open disc or open half plane that contains z_1 but none of the points $\omega_2, \omega_3, \ldots, \omega_n$. Then M is free from the zeros of $G_\gamma[P(z)]$, for all $\gamma \in R^n_+$.

Next, we prove the following result

Theorem 2.2. *Let $P(z)$ be a polynomial of degree n with zeros $z_1, z_2, \ldots, z_n$, then the convex hull $K(P(z))$ of $n^2 - n$ points*

$$\omega_{\nu\mu} = \frac{z_\nu + (n - \gamma_1)z_\mu}{n}, \qquad \nu, \mu \in (1, 2, \ldots, n), \nu \neq \mu. \tag{2.2}$$

contains all zeros of $G_\gamma[P(z)]$, for all $\gamma \in R^n_+$.

Remark 2.3. *For a n-tuple $\gamma = (1, 1, \ldots, 1)$, Theorem 2.1 and Theorem 2.2 reduces to Theorem 1.1 and Theorem 1.2.*

Next, we prove the following result

Theorem 2.4. *Let $P(z)$ be a polynomial of degree n and $G_\gamma[P(w)] = 0$, for all $\gamma \in R^n_+$, then for every given real or complex number α, $P(z)$ has atleast one zero in the region*

$$\left|w - \frac{\alpha + z}{2}\right| \leq \left|\frac{\alpha - z}{2}\right|.$$

Taking $\alpha = 0$ in Theorem 2.4 and noting that,

$$\left|\omega - \frac{z}{2}\right| \leq \left|\frac{z}{2}\right|.$$

$$\Rightarrow |\omega - z| \leq |z|.$$

We get the following result.

Corollary 2.5. *If all the zeros of a polynomial $P(z)$ of degree n lie in $D = \{z \in C : |z| \leq 1\}$, and ω is zero of $G_\gamma[P(z)]$, for all $\gamma \in R^n_+$, then $P(z)$ has at least one zero in both the circles*

$$\left|\omega - \frac{z}{2}\right| \leq \frac{1}{2} \qquad \textit{and} \qquad |\omega - z| \leq 1.$$

Remark 2.6. *For a n-tuple $\gamma = (1, 1, \ldots, 1)$, Theorem 2.4 and Corollary 2.5 reduce to Theorem 1.4 and Corollary 1.5.*

3. Proofs

Proof of Theorem 2.1. If possible suppose that ζ be any zero of $G_\gamma[P(z)]$ that lie in M, consider the bilinear transformation

$$\eta : z \longrightarrow \frac{1}{\gamma_1\zeta - (n - \gamma_1)z_1 - nz}. \tag{3.1}$$

The denominator vanishes at

$$z = \frac{\gamma_1\zeta}{n} + \frac{(n - \gamma_1)z_1}{n}, \tag{3.2}$$

this is a point of M, because M is a convex set containing ζ and z_1. Hence η maps the complement of M onto a closed unit disc D.
By hypothesis $\omega_\nu \notin M$, which implies the point $\eta(\omega_\nu) \in D$ and by convexity of D,

$$\omega^* = \frac{\gamma_2\eta(\omega_2) + \gamma_3\eta(\omega_3) + \cdots + \gamma_n\eta(\omega_n)}{n - \gamma_1} \in D.$$

So there exists $\gamma^* \notin M$ such that,

$$\eta(\gamma^*) = \omega^*.$$

That is,

$$\eta(\gamma^*) = \frac{\gamma_2\eta(\omega_2) + \gamma_3\eta(\omega_3) + \cdots + \gamma_n\eta(\omega_n)}{n - \gamma_1}$$

$$\Rightarrow \eta(\gamma^*) = \frac{\sum_{\nu=2}^{n} \gamma_\nu\eta(\omega_\nu)}{n - \gamma_1}$$

$$\Rightarrow (n-\gamma_1)\eta(\gamma^*) = \sum_{\nu=2}^{n} \gamma_\nu \eta(\omega_\nu)$$

$$\Rightarrow \frac{n-\gamma_1}{\gamma_v\zeta + (n-\gamma_1)z_1 - n\gamma^*} = \sum_{\nu=2}^{n} \frac{\gamma_\nu}{\gamma_1\zeta + (n-\gamma_v)z_1 - n\omega_\nu}$$

$$\Rightarrow \frac{n-\gamma_1}{\gamma_1\zeta + (n-\gamma_1)z_1 - n\gamma^*} = \sum_{\nu=2}^{n} \frac{\gamma_\nu}{\gamma_1\zeta + (n-\gamma_1)z_1 - \gamma_1 z_\nu - (n-\gamma_1)z_1}$$

This gives,

$$\Rightarrow \frac{n-\gamma_1}{\gamma_1\zeta + (n-\gamma_1)z_1 - n\gamma^*} = \sum_{\nu=2}^{n} \frac{\gamma_\nu}{\gamma_1(\zeta - z_\nu)}$$

$$\Rightarrow \frac{\gamma_1(n-\gamma_1)}{\gamma_1\zeta + (n-\gamma_1)z_1 - n\gamma^*} = \sum_{\nu=2}^{n} \frac{\gamma_\nu}{(\zeta - z_\nu)}. \tag{3.3}$$

Since ζ is a zero of $G_\gamma[P(z)]$, therefore we have

$$0 = \frac{G_\gamma[P(\zeta)]}{P(\zeta)} = \sum_{\nu=1}^{n} \frac{\gamma_\nu}{(\zeta - z_\nu)}.$$

$$= \frac{\gamma_1}{\zeta - z_1} + \sum_{\nu=2}^{n} \frac{\gamma_\nu}{(\zeta - z_\nu)}.$$

Using this in (3.3), we obtain

$$\frac{\gamma_1}{\zeta - z_1} + \frac{\gamma_1(n-\gamma_1)}{\gamma_1\zeta - (n-\gamma_1)z_1 - n\gamma^*} = 0$$

$$\Rightarrow \gamma^* = \zeta.$$

This is contradiction to the fact that $\gamma^* \notin M$.
Thus, the theorem is proved. □

Proof of theorem 2.2. It is enough to show that any closed half plane H containing the points $\omega_{\nu\mu}$ also contains all the zeros of $G_\gamma[P(z)]$.
Let $\omega_{\nu\mu} \in H$, where $\nu, \mu \in (1, 2, \ldots, n), \nu \neq \mu$, If all the zeros $z_1, z_2, \ldots, z_n \in H$, by Theorem 1.3. result is true. If one zero say $z_1 \notin H$, then z_1 must be a simple one, otherwise z_1 would coincide with one of the points $\omega_{\nu\mu}$ and can not be outside H.
Then put $\omega_\nu = \omega_{\gamma_1}$ for $\nu = 2, 3, \ldots, n$, and $M = H^c$ = complement of H. By Theorem 2.1. H^c is devoid of zeros of $G_\gamma[P(z)]$. That is, all the zeros of $G_\gamma[P(z)]$ lie in H.
This proves the Theorem 2.2. □

Proof of theorem 2.4. Let $z_1, z_2, \dots, z_n$ be the zeros of P(z) and let ω be a zero of $G_\gamma[P(z)]$ respectively. If $\omega = \alpha$ or $\omega = z_v$ for some $v = 1, 2, \dots, n$, then the result is prove. Thus, we assume that $\omega \neq \alpha$ and $\omega \neq z_v$ for any $v = 1, 2, \dots, n$. Since ω is a zero of $G_\gamma[P(z)]$ and $P(\omega) \neq 0$, we have

$$\sum_{v=1}^{n} \frac{\gamma_v}{\omega - z_v} = \frac{G_\gamma[P(\omega)]}{P(\omega)} = 0$$

Which implies,

$$\sum_{v=1}^{n} \gamma_v \Big\{ \frac{(\omega - z_v) - (\alpha - z_v)}{\omega - z_v} \Big\} = \sum_{v=1}^{n} \frac{\gamma_v(\omega - \alpha)}{(\omega - z_v)} = 0$$

Now, we have

$$\sum_{v=1}^{n} \frac{\gamma_v(\alpha - z_v)}{(\omega - z_v)} = n.$$

This implies,

$$n = \sum_{v=1}^{n} Re\Big\{ \frac{\gamma_v(\alpha - z_v)}{(\omega - z_v)} \Big\}$$

$$= \sum_{v=1}^{n} \gamma_v Re\Big\{ \frac{\alpha - z_v}{\omega - z_v} \Big\}$$

$$\leq n \max_{1 \leq v \leq n} Re\Big\{ \frac{\alpha - z_v}{\omega - z_v} \Big\}.$$

which shows that for at least one $v = 1, 2, \dots, n$.

$$Re\Big\{ \frac{\alpha - z_v}{\omega - z_v} \Big\} \geq 1.$$

Thus for at least one $v = 1, 2, \dots, n$,

$$\Big| 1 - \frac{\alpha - z_v}{2(\omega - z_v)} \Big| \leq \Big| \frac{\alpha - z_v}{2(\omega - z_v)} \Big|.$$

From this we obtain, for one $v = 1, 2, \dots, n$,

$$\Big| \omega - \frac{\alpha - z_v}{2} \Big| \leq \Big| \frac{\alpha - z_v}{2} \Big|.$$

For at least one $v = 1, 2, \dots, n$, which completes the proof of Theorem 2.4. □

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