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Hadamard and Fejér type inequalities for p-convex functions via Caputo fractional derivatives

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Abstract

Here our aim is to prove the Hermite-Hadamard and Fejér inequalities for p-convex functions via Caputo fractional derivatives. We also establish some useful identities in order to find further Hadamard's and Fejér type inequalities which are generalizations of the results given in the literature cited here.

Keywords: Hermite-Hadamard inequality, Hermite-Hadamard-Fejér inequality, p-convex function, Caputo fractional derivatives.
2010 MSC: 26A51, 26D07, 26D10, 26D15.

1. Introduction

Fractional integral inequalities are helpful in establishing the uniqueness of solutions for certain fractional partial differential equations. These inequalities also provide upper as well as lower bounds for solutions of the fractional boundary value problems. These considerations have led various researchers in the field of integral inequalities to explore certain extensions and generalizations by involving fractional calculus operators.

On the other hand, convex analysis also plays important role in the advancement of inequalities and in optimization problems. Our aim is to use convex analysis and fractional calculas in order to find important inequalities which plays useful scientific role.

Let Ω be a real interval. Then a function $\zeta : \Omega \to \mathbb{R}$ is called convex if the following inequality holds:

$$\zeta(\eta h_1 + (1-\eta)h_2) \leq \eta\zeta(h_1) + (1-\eta)\zeta(h_2), \tag{1.1}$$

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for all $h_1, h_2 \in \Omega$ and $\eta \in [0,1]$. Convex functions have been used to investigate various scientific problems. Many refinements have been built for convex functions in order to study problems of pure and applied sciences. See [6, 7, 17, 18, 19].

The Hermite-Hadamard inequality [11, 10] for a convex function $\zeta : \Omega \to \mathbb{R}$ on an interval Ω is defined by

$$\zeta\left(\frac{h_1+h_2}{2}\right) \leq \frac{1}{h_2-h_1}\int_{h_1}^{h_2}\zeta(g)dg \leq \frac{\zeta(h_1)+\zeta(h_2)}{2}, \tag{1.2}$$

for all $h_1, h_2 \in \Omega$ with $h_1 < h_2$. Then Fejér [9] gave its generalization as:

$$\zeta\left(\frac{h_1+h_2}{2}\right)\int_{h_1}^{h_2}\Upsilon(g)dg \leq \frac{1}{h_2-h_1}\int_{h_1}^{h_2}\zeta(g)\Upsilon(g)dg \leq \frac{\zeta(h_1)+\zeta(h_2)}{2}\int_{h_1}^{h_2}\Upsilon(g)dg, \tag{1.3}$$

where $\Upsilon : [h_1, h_2] \to \mathbb{R}$ is nonnegative, integrable and symmetric over $(h_1+h_2)/2$, called Hermite-Hadamard-Fejér inequality.

Due to extensive applicability of Hermite-Hadamard and Hadamard-Fejér inequalities and fractional integrals, number of researchers expand their research involving generalized fractional integrals have been obtained for diverse classes of convex functions. For instance see [1, 2, 3, 4, 5, 14, 16, 20, 21, 22, 23, 24, 25] etc.

Definition 1.1 ([13]). *Let $\alpha > 0$ be a non integer real number and $m = [\alpha]+1$. Let $\zeta \in AC^m[h_1, h_2]$, the space of functions having mth derivatives absolutely continuous, then the left and right sided Caputo fractional derivatives are characterised as:*

$$({}^{C}\mathcal{D}^{\alpha}_{h_1+}\zeta)(g) = \frac{1}{\Gamma(m-\alpha)}\int_{h_1}^{g}\frac{\zeta^m(\upsilon)}{(g-\upsilon)^{\alpha-m+1}}d\upsilon, \quad g > h_1, \tag{1.4}$$

$$({}^{C}\mathcal{D}^{\alpha}_{h_2-}\zeta)(g) = \frac{(-1)^m}{\Gamma(m-\alpha)}\int_{g}^{h_2}\frac{\zeta^m(\upsilon)}{(\upsilon-g)^{\alpha-m+1}}d\upsilon, \quad g < h_2, \tag{1.5}$$

respectively. If $\alpha = m = \{1, 2, 3, \dots\}$ and $\zeta^(m)(g)$ exists then following holds

$$({}^{C}\mathcal{D}^{\alpha}_{h_1+}\zeta)(g) = \zeta^m(g) = (-1)^m({}^{C}\mathcal{D}^{\alpha}_{h_2-}\zeta)(g). \tag{1.6}$$

In simple words, we have

$$({}^{C}\mathcal{D}^{o}_{h_1+}\zeta)(g) = \zeta(g) = ({}^{C}\mathcal{D}^{o}_{h_2-}\zeta)(g), \tag{1.7}$$

where $m = 1$ and $\alpha = 0$.

Definition 1.2 ([12]). *Consider an interval $\Omega \subset (0,\infty)$, and $p \in \mathbb{R}\backslash\{0\}$. A function $\zeta : \Omega \to \mathbb{R}$ is called p-convex if*

$$\zeta\left([\eta h_1^p + (1-\eta)h_2^p]^{\frac{1}{p}}\right) \leq \eta\zeta(h_1) + (1-\eta)\zeta(h_2), \tag{1.8}$$

for all $h_1, h_2 \in \Omega$ and $\eta \in [0,1]$. If (1.8) is reversed then ζ is called p-concave.

Here we prove several Hermite-Hadamard and Fejér type inequalities for p-convex functions via Caputo fractional derivatives.

2. Hadamard's type inequalities

First we prove Hadamard's inequalities for p-convex function via Caputo fractional derivatives.

Theorem 2.1. *Let $\zeta : [h_1, h_2] \subset (0, \infty) \to \mathbb{R}$ be a function with $\zeta \in C^m[h_1, h_2]$. Further let ζ^m is positive p-convex function. Then*
(i) for $p > 0$, we have

$$\begin{aligned}
&\zeta\left(\left[\frac{h_1^p + h_2^p}{2}\right]^{1/p}\right) \\
&\leq \frac{\Gamma(m-\alpha+1)}{2(h_2^p - h_1^p)^{m-\alpha}}\left[({}^{C}\mathcal{D}^{\alpha}_{h_1^p+}\zeta)(\mu(h_2^p)) + (-1)^m({}^{C}\mathcal{D}^{\alpha}_{h_2^p-}\zeta)(\mu(h_1^p))\right] \\
&\leq \frac{\zeta^m(h_1) + \zeta^m(h_2)}{2},
\end{aligned} \tag{2.1}$$

where $\mu(s) = s^{\frac{1}{p}}$, for all $s \in [h_1^p, h_2^p]$.
(ii) for $p < 0$, we have

$$\begin{aligned}
&\zeta\left(\left[\frac{h_1^p + h_2^p}{2}\right]^{1/p}\right) \\
&\leq \frac{\Gamma(m-\alpha+1)}{2(h_1^p - h_2^p)^{m-\alpha}}\left[(-1)^m({}^{C}\mathcal{D}^{\alpha}_{h_1^p-}\zeta)(\mu(h_2^p)) + ({}^{C}\mathcal{D}^{\alpha}_{h_2^p+}\zeta)(\mu(h_1^p))\right] \\
&\leq \frac{\zeta^m(h_1) + \zeta^m(h_2)}{2},
\end{aligned} \tag{2.2}$$

where $\mu(s) = s^{\frac{1}{p}}$, $s \in [h_2^p, h_1^p]$.

Proof . From p-convexity of ζ^m, we get

$$\zeta^m\left(\left[\frac{a^p + b^p}{2}\right]^{\frac{1}{p}}\right) \leq \frac{\zeta^m(a) + \zeta^m(b)}{2}.$$

Taking $a^p = \eta h_1^p + (1-\eta)h_2^p$ and $b^p = (1-\eta)h_1^p + \eta h_2^p$ with $\eta \in [0, 1]$, we get

$$\zeta^m\left(\left[\frac{h_1^p + h_2^p}{2}\right]^{\frac{1}{p}}\right) \leq \frac{\zeta^m\left([\eta h_1^p + (1-\eta)h_2^p]^{\frac{1}{p}}\right) + \zeta^m\left([(1-\eta)h_1^p + \eta h_2^p]^{\frac{1}{p}}\right)}{2}. \tag{2.3}$$

Multiplying (2.3) by $\eta^{m-\alpha-1}$ on both sides with $\eta \in (0, 1)$, $\alpha > 0$ and then integrating along η over $\eta \in [0, 1]$, we obtain

$$\begin{aligned}
&2\zeta^m\left(\left[\frac{h_1^p + h_2^p}{2}\right]^{\frac{1}{p}}\right)\int_0^1 \eta^{m-\alpha-1} d\eta \\
&\leq \int_0^1 \eta^{m-\alpha-1}\zeta^m\left([\eta h_1^p + (1-\eta)h_2^p]^{\frac{1}{p}}\right) d\eta + \int_0^1 \eta^{m-\alpha-1}\zeta^m\left([\eta h_2^p + (1-\eta)h_1^p]^{\frac{1}{p}}\right) d\eta.
\end{aligned}$$

Then by change of variable, we get

$$\zeta^m\left(\left[\frac{h_1^p + h_2^p}{2}\right]^{\frac{1}{p}}\right) \leq \frac{\Gamma(m-\alpha+1)}{2(h_2^p - h_1^p)^{m-\alpha}}\left[({}^{C}\mathcal{D}^{\alpha}_{h_1^p+}\zeta)(\mu(h_2^p)) + (-1)^m({}^{C}\mathcal{D}^{\alpha}_{h_2^p-}\zeta)(\mu(h_1^p))\right]. \tag{2.4}$$

Now consider,

$$\zeta^m\left([\eta h_1^p+(1-\eta)h_2^p]^{\frac{1}{p}}\right)+\zeta^m\left([\eta h_2^p+(1-\eta)h_1^p]^{\frac{1}{p}}\right)\leq[\zeta^m(h_1)+\zeta^m(h_2)]\,. \tag{2.5}$$

Multiplying (2.5) by $\eta^{m-\alpha}$ on both sides with $\eta\in(0,1)$, $\alpha>0$, and then integrating over $\eta\in[0,1]$, we obtain

$$\frac{\Gamma(m-\alpha+1)}{2(h_2^p-h_1^p)^{m-\alpha}}\left[({}^C\mathcal{D}^{\alpha}_{h_1^p+}\zeta)(\mu(h_2^p))+(-1)^m({}^C\mathcal{D}^{\alpha}_{h_2^p-}\zeta)(\mu(h_1^p))\right]\leq\frac{\zeta^m(h_1)+\zeta^m(h_2)}{2}. \tag{2.6}$$

Hence from inequalities (2.4) and (2.6), we get (2.1).

(ii) The proof is analogous to (i). □

Remark 2.2. *In Theorem* 2.1 (i)*, if one takes* $p=1$ *in inequality* 2.1 *then one gets inequality* 2.2 *of Thoerem* 2.3 *in* [8].

Corollary 2.3. *Under similar assumption of Theorem* 2.1 (ii)*. If we take* $p=-1$*, then we get*

$$\begin{aligned}&\zeta\left(\frac{2h_1h_2}{h_1+h_2}\right)\\&\leq\frac{(h_1h_2)^{m-\alpha}\Gamma(m-\alpha+1)}{2(h_2-h_1)^{m-\alpha}}\left[(-1)^m({}^C\mathcal{D}^{\alpha}_{1/h_1-}\zeta)\left(\mu\left(\frac{1}{h_2}\right)\right)+({}^C\mathcal{D}^{\alpha}_{1/h_2+}\zeta)\left(\mu\left(\frac{1}{h_1}\right)\right)\right]\\&\leq\frac{\zeta(h_1)+\zeta(h_2)}{2},\end{aligned} \tag{2.7}$$

where $\mu(s)=\frac{1}{s}$, $s\in\left[\frac{1}{h_2},\frac{1}{h_1}\right]$.

Lemma 2.4. *Let* $\zeta:[h_1,h_2]\subset(0,\infty)\to\mathbb{R}$ *be a function such that* $\zeta\in C^m[h_1,h_2]$ *with* $h_1<h_2$*. If* ζ^{m+1} *is posiative, then*

(i) *for* $p>0$*, we have*

$$\begin{aligned}&\left(\frac{\zeta^m(h_1)+\zeta^m(h_2)}{2}\right)-\frac{\Gamma(m-\alpha+1)}{2(h_2^p-h_1^p)^{m-\alpha}}\left[({}^C\mathcal{D}^{\alpha}_{h_1^p+}\zeta)(\mu(h_2^p))+(-1)^m({}^C\mathcal{D}^{\alpha}_{h_2^p-}\zeta)(\mu(h_1^p))\right]\\&=\frac{(h_2^p-h_1^p)}{2p}\int_0^1\left[(1-\eta)^{m-\alpha}-\eta^{m-\alpha}\right]G_\eta^{\frac{1}{p}-1}\zeta^{m+1}\left([\eta h_1^p+(1-\eta)h_2^p]^{\frac{1}{p}}\right)d\eta,\end{aligned} \tag{2.8}$$

where $G_\eta=[\eta h_1^p+(1-\eta)h_2^p]$ *and*

(ii) *for* $p<0$*, we have*

$$\begin{aligned}&\left(\frac{\zeta^m(h_1)+\zeta^m(h_2)}{2}\right)-\frac{\Gamma(m-\alpha+1)}{2(h_1^p-h_2^p)^{m-\alpha}}\left[(-1)^m({}^C\mathcal{D}^{\alpha}_{h_1^p-}\zeta)(\mu(h_2^p))+({}^C\mathcal{D}^{\alpha}_{h_2^p+}\zeta)(\mu(h_1^p))\right]\\&=\frac{(h_1^p-h_2^p)}{2p}\int_0^1\left[\eta^{m-\alpha}-(1-\eta)^{m-\alpha}\right]H_\eta^{\frac{1}{p}-1}\zeta^{m+1}\left([\eta h_2^p+(1-\eta)h_1^p]^{\frac{1}{p}}\right)d\eta,\end{aligned} \tag{2.9}$$

where $H_\eta=[\eta h_2^p+(1-\eta)h_1^p]$.

Proof . (i) Consider,

$$\begin{aligned} J &= \frac{(h_2^p - h_1^p)}{2p} \int_0^1 \left[(1-\eta)^{m-\alpha} - \eta^{m-\alpha}\right] G_\eta^{\frac{1}{p}-1} \zeta^{m+1}\left([\eta h_1^p + (1-\eta)h_2^p]^{\frac{1}{p}}\right) d\eta \\ &= \frac{(h_2^p - h_1^p)}{2p} \int_0^1 (1-\eta)^{m-\alpha} G_\eta^{\frac{1}{p}-1} \zeta^{m+1}\left([\eta h_1^p + (1-\eta)h_2^p]^{\frac{1}{p}}\right) d\eta \\ &\quad - \frac{(h_2^p - h_1^p)}{2p} \int_0^1 \eta^{m-\alpha} G_\eta^{\frac{1}{p}-1} \zeta^{m+1}\left([\eta h_1^p + (1-\eta)h_2^p]^{\frac{1}{p}}\right) d\eta \\ &= J_1 - J_2. \end{aligned} \tag{2.10}$$

Then applying by parts integration, we achieve

$$\begin{aligned} J_1 &= \frac{(h_2^p - h_1^p)}{2p} \int_0^1 (1-\eta)^{m-\alpha} G_\eta^{\frac{1}{p}-1} \zeta^{m+1}\left([\eta h_1^p + (1-\eta)h_2^p]^{\frac{1}{p}}\right) d\eta \\ &= \frac{(h_2^p - h_1^p)}{2p} \left[\frac{p(1-\eta)^{m-\alpha}}{h_1^p - h_2^p} \zeta^m\left([\eta h_1^p + (1-\eta)h_2^p]^{\frac{1}{p}}\right)\Big|_0^1 \right. \\ &\quad \left. + \frac{p(m-\alpha)}{h_2^p - h_1^p} \int_0^1 (1-\eta)^{m-\alpha-1} \zeta^m\left([\eta h_1^p + (1-\eta)h_2^p]^{\frac{1}{p}}\right) d\eta \right] \\ &= \frac{(h_2^p - h_1^p)}{2p} \left[\frac{p}{(h_2^p - h_1^p)} \zeta^m(h_2) - \frac{p(m-\alpha)}{h_2^p - h_1^p} \int_{h_2^p}^{h_1^p} \left(\frac{h_1^p - s}{h_1^p - h_2^p}\right)^{m-\alpha-1} \frac{\zeta^m(\mu(s))}{h_1^p - h_2^p} ds \right] \\ &= \frac{\zeta^m(h_2)}{2} - \frac{\Gamma(m-\alpha+1)}{2(h_2^p - h_1^p)^{m-\alpha}} (-1)^m ({}^C\mathcal{D}^\alpha_{h_2^p -}\zeta)(\mu(h_1^p)). \end{aligned} \tag{2.11}$$

Similarly

$$\begin{aligned} J_2 &= \frac{(h_2^p - h_1^p)}{2p} \int_0^1 \eta^{m-\alpha} G_\eta^{\frac{1}{p}-1} \zeta^{m+1}\left([\eta h_1^p + (1-\eta)h_2^p]^{\frac{1}{p}}\right) d\eta \\ &= \frac{(h_2^p - h_1^p)}{2p} \left[\frac{p\eta^{m-\alpha}}{h_1^p - h_2^p} \zeta^m\left([\eta h_1^p + (1-\eta)h_2^p]^{\frac{1}{p}}\right)\Big|_0^1 \right. \\ &\quad \left. - \frac{p(m-\alpha)}{h_2^p - h_1^p} \int_0^1 \eta^{m-\alpha-1} \zeta^m\left([\eta h_1^p + (1-\eta)h_2^p]^{\frac{1}{p}}\right) d\eta \right] \\ &= \frac{(h_2^p - h_1^p)}{2p} \left[- \frac{p}{(h_2^p - h_1^p)} \zeta^m(h_1) + \frac{p(m-\alpha)}{h_2^p - h_1^p} \int_{h_2^p}^{h_1^p} \left(\frac{h_2^p - s}{h_2^p - h_1^p}\right)^{m-\alpha-1} \frac{\zeta^m(\mu(s))}{h_1^p - h_2^p} ds \right] \\ &= -\frac{\zeta^m(h_1)}{2} + \frac{\Gamma(m-\alpha+1)}{2(h_2^p - h_1^p)^{m-\alpha}} ({}^C\mathcal{D}^\alpha_{h_1^p +}\zeta)(\mu(h_2^p)). \end{aligned} \tag{2.12}$$

Hence from (2.11) and (2.11) we get (2.8).

(ii) The proof is similar to (i). □

Remark 2.5. *In Lemma* 2.4 (i), *if one takes* $p = 1$ *in identity* 2.8 *then one gets identity* 2.1 *of Lemma* 2.2 *in* [8].

Theorem 2.6. *Let* $\zeta : [h_1, h_2] \subset (0, \infty) \to \mathbb{R}$ *be a function such that* $\zeta \in C^m[h_1, h_2]$ *with* $h_1 < h_2$. *If* $|\zeta^{m+1}|^q$ *is p-convex with* $q \geq 1$, *then* *(i) for $p>1$, we have*

$$\left|\left(\frac{\zeta^m(h_1)+\zeta^m(h_2)}{2}\right)-\frac{\Gamma(m-\alpha+1)}{2(h_2^p-h_1^p)^{m-\alpha}}\left[\left({}^C\mathcal{D}^{\alpha}_{h_1^p+}\zeta\right)(\mu(h_2^p))+(-1)^m\left({}^C\mathcal{D}^{\alpha}_{h_2^p-}\zeta\right)(\mu(h_1^p))\right]\right| \\ \leq \frac{(h_2^p-h_1^p)}{2p}\nu_1^{1-\frac{1}{q}}\left(1-\frac{1}{2^{m-\alpha}}\right)^{\frac{1}{q}}\left(\frac{|\zeta^{m+1}(h_1)|^q+|\zeta^{m+1}(h_2)|^q}{m-\alpha+1}\right)^{\frac{1}{q}}. \tag{2.13}$$

Where $\nu_1=\frac{h_2^{1-p}}{2}\ {}_2F_1\left(1-\frac{1}{p},1;2;1-\frac{h_1^p}{h_2^p}\right)$.

(ii) for $p<1$, we have

$$\left|\left(\frac{\zeta^m(h_1)+\zeta^m(h_2)}{2}\right)-\frac{\Gamma(m-\alpha+1)}{2(h_1^p-h_2^p)^{m-\alpha}}\left[(-1)^m\left({}^C\mathcal{D}^{\alpha}_{h_1^p-}\zeta\right)(\mu(h_2^p))+\left({}^C\mathcal{D}^{\alpha}_{h_2^p+}\zeta\right)(\mu(h_1^p))\right]\right| \\ \leq \frac{(h_2^p-h_1^p)}{2p}\nu_2^{1-\frac{1}{q}}\left(1-\frac{1}{2^{m-\alpha}}\right)^{\frac{1}{q}}\left(\frac{|\zeta^{m+1}(h_1)|^q+|\zeta^{m+1}(h_2)|^q}{m-\alpha+1}\right)^{\frac{1}{q}}. \tag{2.14}$$

Where $\nu_2=\frac{h_2^{p-1}}{2}\ {}_2F_1\left(1-\frac{1}{p},1;2;1-\frac{h_2^p}{h_1^p}\right)$.

Proof . Applying Power mean inequality on Lemma 2.4 and p-convexity of $|\zeta^{m+1}|^q$, we achieve

$$\begin{aligned}
&\left|\left(\frac{\zeta^m(h_1)+\zeta^m(h_2)}{2}\right)-\frac{\Gamma(m-\alpha+1)}{2(h_2^p-h_1^p)^{m-\alpha}}\left[\left({}^C\mathcal{D}^{\alpha}_{h_1^p+}\zeta\right)(\mu(h_2^p))+(-1)^m\left({}^C\mathcal{D}^{\alpha}_{h_2^p-}\zeta\right)(\mu(h_1^p))\right]\right| \\
&=\frac{(h_2^p-h_1^p)}{2p}\left|\int_0^1\left[(1-\eta)^{m-\alpha}-\eta^{m-\alpha}\right]G_\eta^{\frac{1}{p}-1}\zeta^{m+1}\left([\eta h_1^p+(1-\eta)h_2^p]^{\frac{1}{p}}\right)d\eta\right| \\
&\leq\frac{(h_2^p-h_1^p)}{2p}\left(\int_0^1 G_\eta^{\frac{1}{p}-1}d\eta\right)^{1-\frac{1}{q}} \\
&\quad\times\left(\int_0^1\left|(1-\eta)^{m-\alpha}-\eta^{m-\alpha}\right||\zeta^{m+1}\left([\eta h_1^p+(1-\eta)h_2^p]^{\frac{1}{p}}\right)|^q d\eta\right)^{1/q} \\
&\leq\frac{(h_2^p-h_1^p)}{2p}\left(\int_0^1 G_\eta^{\frac{1}{p}-1}d\eta\right)^{1-\frac{1}{q}} \\
&\quad\times\left(\int_0^1\left|(1-\eta)^{m-\alpha}-\eta^{m-\alpha}\right|\left(\eta|\zeta^{m+1}(h_1)|^q+(1-\eta)|\zeta^{m+1}(h_2)|^q\right)d\eta\right)^{1/q} \\
&=\frac{(h_2^p-h_1^p)}{2p}\left(\int_0^1 G_\eta^{\frac{1}{p}-1}d\eta\right)^{1-\frac{1}{q}} \\
&\quad\times\left(\int_0^{1/2}\left[(1-\eta)^{m-\alpha}-\eta^{m-\alpha}\right]\left(\eta|\zeta^{m+1}(h_1)|^q+(1-\eta)|\zeta^{m+1}(h_2)|^q\right)d\eta\right. \\
&\quad\left.+\int_{1/2}^1\left[\eta^{m-\alpha}-(1-\eta)^{m-\alpha}\right]\left(\eta|\zeta^{m+1}(h_1)|^q+(1-\eta)|\zeta^{m+1}(h_2)|^q\right)d\eta\right)^{1/q}.
\end{aligned} \tag{2.15}$$

Where

$$\nu_1=\int_0^1 G_\eta^{\frac{1}{p}-1}d\eta=\frac{h_2^{1-p}}{2}\ {}_2F_1\left(1-\frac{1}{p},1;2;1-\frac{h_1^p}{h_2^p}\right), \tag{2.16}$$

Also note that

$$\int_0^{1/2} \eta\left[(1-\eta)^{m-\alpha}-\eta^{m-\alpha}\right] = \frac{1}{(m-\alpha+1)(m-\alpha+2)} - \frac{\frac{1}{2^{m-\alpha+1}}}{m-\alpha+1}, \tag{2.17}$$

$$\int_0^{1/2} (1-\eta)\left[(1-\eta)^{m-\alpha}-\eta^{m-\alpha}\right] = \frac{1}{m-\alpha+2} - \frac{\frac{1}{2^{m-\alpha+1}}}{m-\alpha+1}, \tag{2.18}$$

$$\int_{1/2}^{1} \eta\left[\eta^{m-\alpha}-(1-\eta)^{m-\alpha}\right] = \frac{1}{m-\alpha+2} - \frac{\frac{1}{2^{m-\alpha+1}}}{m-\alpha+1}, \tag{2.19}$$

$$\int_{1/2}^{1} (1-\eta)\left[\eta^{m-\alpha}-(1-\eta)^{m-\alpha}\right] = \frac{1}{(m-\alpha+1)(m-\alpha+2)} - \frac{\frac{1}{2^{m-\alpha+1}}}{m-\alpha+1}. \tag{2.20}$$

By substituting values from (2.16) to (2.20) in (2.15) and after simple calculations, we get the inequality (2.13).

(ii) Proof is similar to (i). □

Remark 2.7. *In Theorem* 2.6 (i)*, if we take* $p=1$ *and* $q=1$ *then we get inequality* 2.8 *of Thoerem* 2.4 *in* [8].

Theorem 2.8. *Let* $\zeta : [h_1, h_2] \subset (0,\infty) \to \mathbb{R}$ *be a function such that* $\zeta \in C^m[h_1,h_2]$ *with* $h_1 < h_2$*. If* $|\zeta^{m+1}|^q$ *is p-convex with* $q \geq 1$*, then*

(i) *for* $p > 1$*, we have*

$$\begin{aligned} &\left|\left(\frac{\zeta^m(h_1)+\zeta^m(h_2)}{2}\right) - \frac{\Gamma(m-\alpha+1)}{2(h_2^p-h_1^p)^{m-\alpha}}\left[({}^{C}\mathcal{D}^{\alpha}_{h_1^p+}\zeta)(\mu(h_2^p)) + (-1)^m({}^{C}\mathcal{D}^{\alpha}_{h_2^p-}\zeta)(\mu(h_1^p))\right]\right| \\ &\leq \frac{(h_2^p-h_1^p)}{2p}\left(\frac{2}{m-\alpha+1}\right)^{1-\frac{1}{q}}\left(\nu_3|\zeta^{m+1}(h_1)|^q + \nu_4|\zeta^{m+1}(h_2)|^q\right)^{\frac{1}{q}}. \end{aligned} \tag{2.21}$$

Where

$$\nu_3 = \frac{h_2^{1-p}}{2}\ {}_2F_1\left(1-\frac{1}{p}, 2; 3; 1-\frac{h_1^p}{h_2^p}\right) \quad \text{and} \quad \nu_4 = \frac{h_2^{1-p}}{2}\ {}_2F_1\left(1-\frac{1}{p}, 1; 3; 1-\frac{h_1^p}{h_2^p}\right).$$

(ii) *for* $p < 1$*, we have*

$$\begin{aligned} &\left|\left(\frac{\zeta^m(h_1)+\zeta^m(h_2)}{2}\right) - \frac{\Gamma(m-\alpha+1)}{2(h_1^p-h_2^p)^{m-\alpha}}\left[(-1)^m({}^{C}\mathcal{D}^{\alpha}_{h_1^p-}\zeta)(\mu(h_2^p)) + ({}^{C}\mathcal{D}^{\alpha}_{h_2^p+}\zeta)(\mu(h_1^p))\right]\right| \\ &\leq \frac{(h_2^p-h_1^p)}{2p}\left(\frac{2}{m-\alpha+1}\right)^{1-\frac{1}{q}}\left(\nu_5|\zeta^{m+1}(h_1)|^q + \nu_6|\zeta^{m+1}(h_2)|^q\right)^{\frac{1}{q}}. \end{aligned} \tag{2.22}$$

Where

$$\nu_5 = \frac{h_2^{p-1}}{2}\ {}_2F_1\left(1-\frac{1}{p}, 1; 3; 1-\frac{h_2^p}{h_1^p}\right) \quad \text{and} \quad \nu_6 = \frac{h_2^{p-1}}{2}\ {}_2F_1\left(1-\frac{1}{p}, 2; 3; 1-\frac{h_2^p}{h_1^p}\right).$$

Proof . Applying Power mean inequality on Lemma 2.4 and p-convexity of $|\zeta^{m+1}|^q$, we achieve

$$\begin{aligned}
&\left|\left(\frac{\zeta^m(h_1)+\zeta^m(h_2)}{2}\right)-\frac{\Gamma(m-\alpha+1)}{2(h_2^p-h_1^p)^{m-\alpha}}\left[\left({}^C\mathcal{D}^{\alpha}_{h_1^p+}\zeta\right)(\mu(h_2^p))+(-1)^m\left({}^C\mathcal{D}^{\alpha}_{h_2^p-}\zeta\right)(\mu(h_1^p))\right]\right|\\
&=\frac{(h_2^p-h_1^p)}{2p}\left|\int_0^1\left[(1-\eta)^{m-\alpha}-\eta^{m-\alpha}\right]G_\eta^{\frac{1}{p}-1}\zeta^{m+1}\left([\eta h_1^p+(1-\eta)h_2^p]^{\frac{1}{p}}\right)d\eta\right|\\
&\le\frac{(h_2^p-h_1^p)}{2p}\left(\int_0^1\left|(1-\eta)^{m-\alpha}-\eta^{m-\alpha}\right|d\eta\right)^{1-\frac{1}{q}}\\
&\quad\times\left(\int_0^1 G_\eta^{\frac{1}{p}-1}|\zeta^{m+1}\left([\eta h_1^p+(1-\eta)h_2^p]^{\frac{1}{p}}\right)|^q d\eta\right)^{1/q}\\
&\le\frac{(h_2^p-h_1^p)}{2p}\left(\int_0^1\left|(1-\eta)^{m-\alpha}-\eta^{m-\alpha}\right|d\eta\right)^{1-\frac{1}{q}}\\
&\quad\times\left(\int_0^1 G_\eta^{\frac{1}{p}-1}\left(\eta|\zeta^{m+1}(h_1)|^q+(1-\eta)|\zeta^{m+1}(h_2)|^q\right)d\eta\right)^{1/q}\\
&=\frac{(h_2^p-h_1^p)}{2p}\left(\int_0^1\left[(1-\eta)^{m-\alpha}+\eta^{m-\alpha}\right]d\eta\right)^{1-\frac{1}{q}}\\
&\quad\times\left(\int_0^1 G_\eta^{\frac{1}{p}-1}\left(\eta|\zeta^{m+1}(h_1)|^q+(1-\eta)|\zeta^{m+1}(h_2)|^q\right)d\eta\right)^{1/q}.
\end{aligned}\tag{2.23}$$

Note that

$$\int_0^1\left[(1-\eta)^{m-\alpha}+\eta^{m-\alpha}\right]d\eta=\frac{2}{m-\alpha+1},$$

$$\int_0^1\eta G_\eta^{\frac{1}{p}-1}d\eta=\frac{h_2^{1-p}}{2}\,{}_2F_1\left(1-\frac{1}{p},2;3;1-\frac{h_1^p}{h_2^p}\right),$$

and

$$\int_0^1(1-\eta)G_\eta^{\frac{1}{p}-1}d\eta=\frac{h_2^{1-p}}{2}\,{}_2F_1\left(1-\frac{1}{p},1;3;1-\frac{h_1^p}{h_2^p}\right).$$

By substituting above values in (2.23) and after simple calculations, we get the inequality (2.21).

(ii) Proof is similar to (i). □

Corollary 2.9. *Consider the similar assumptions of Theorem 2.8. Let $p=1$, then following holds for convex functions:*

$$\begin{aligned}
&\left|\left(\frac{\zeta^m(h_1)+\zeta^m(h_2)}{2}\right)-\frac{\Gamma(m-\alpha+1)}{2(h_2-h_1)^{m-\alpha}}\left[\left({}^C\mathcal{D}^{\alpha}_{h_1+}\zeta\right)(\mu(h_2))+(-1)^m\left({}^C\mathcal{D}^{\alpha}_{h_2-}\zeta\right)(\mu(h_1))\right]\right|\\
&\le\frac{(h_2-h_1)}{2}\left(\frac{2}{m-\alpha+1}\right)^{1-\frac{1}{q}}\left(\frac{|\zeta^{m+1}(h_1)|^q+|\zeta^{m+1}(h_2)|^q}{2}\right)^{\frac{1}{q}}.
\end{aligned}\tag{2.24}$$

Remark 2.10. *In Lemma 2.4 (ii), Theorem 2.6 (ii) and 2.8 (ii), if one takes $p=-1$ then one gets similar results for harmonically convex functions.*

3. Hadamard-Fejér type inequalities

Kunt and Iscan definine p-symmetric functions as:

Definition 3.1 ([15]). *Let $p \in \mathbb{R}\backslash\{0\}$. A function $\Upsilon : [h_1, h_2] \subseteq (0, \infty) \to \mathbb{R}$ is called p-symmetric along $\left[\frac{h_1^p+h_2^p}{2}\right]^{1/p}$, if*

$$\Upsilon(s) = \Upsilon\left([h_1^p + h_2^p - s^p]^{\frac{1}{p}}\right)$$

holds for all $s \in [h_1, h_2]$.

To prove Hadamard-Fejér inequality we need following lemma.

Lemma 3.2. *Let $p \in \mathbb{R}\backslash\{0\}$. Let $\Upsilon : [h_1, h_2] \subseteq (0, \infty) \to \mathbb{R}$be a function such that $\Upsilon \in C^m[h_1, h_2]$. If Υ^m is p-symmetric over $\left[\frac{h_1^p+h_2^p}{2}\right]^{1/p}$, then*
(i) for $p > 0$

$$\begin{aligned} &({}^C\mathcal{D}^{\alpha}_{h_1^p+}\Upsilon)(\mu(h_2^p)) \\ &= \frac{1}{2}\left[({}^C\mathcal{D}^{\alpha}_{h_1^p+}\Upsilon)(\mu(h_2^p)) + (-1)^m({}^C\mathcal{D}^{\alpha}_{h_2^p-}\Upsilon)(\mu(h_1^p))\right] \\ &= (-1)^m({}^C\mathcal{D}^{\alpha}_{h_2^p-}\Upsilon)(\mu(h_1^p)), \end{aligned} \tag{3.1}$$

with $\alpha > 0$ and where $\mu(s) = s^{\frac{1}{p}}$, for all $s \in [h_1^p, h_2^p]$.
(ii) for $p < 0$

$$\begin{aligned} &(-1)^m({}^C\mathcal{D}^{\alpha}_{h_1^p-}\Upsilon)(\mu(h_2^p)) \\ &= \frac{1}{2}\left[(-1)^m({}^C\mathcal{D}^{\alpha}_{h_1^p-}\Upsilon)(\mu(h_2^p)) + ({}^C\mathcal{D}^{\alpha}_{h_2^p+}\Upsilon)(\mu(h_1^p))\right] \\ &= ({}^C\mathcal{D}^{\alpha}_{h_2^p+}\Upsilon)(\mu(h_1^p)), \end{aligned} \tag{3.2}$$

with $\alpha > 0$ and $\mu(s) = s^{\frac{1}{p}}$, for all $s \in [h_2^p, h_1^p]$.

Proof . Since Υ^m is p-symmetric along $\left[\frac{h_1^p+h_2^p}{2}\right]^{1/p}$, then by definition we have $\Upsilon^m(s^{\frac{1}{p}}) = \Upsilon^m\left([h_1^p + h_2^p - s]^{\frac{1}{p}}\right)$, for all $s \in [h_1^p, h_2^p]$. Thus we have

$$\begin{aligned} ({}^C\mathcal{D}^{\alpha}_{h_1^p+}\Upsilon)(\mu(h_2^p)) &= \frac{1}{\Gamma(m-\alpha)}\int_{h_1^p}^{h_2^p} \frac{\Upsilon^m\left(x^{\frac{1}{p}}\right)}{(h_2^p - x)^{m-\alpha+1}}dx \\ &= \frac{1}{\Gamma(m-\alpha)}\int_{h_1^p}^{h_2^p} \frac{\Upsilon^m\left([h_1^p + h_2^p - s]^{\frac{1}{p}}\right)}{(s - h_1^p)^{m-\alpha+1}}ds \\ &= \frac{1}{\Gamma(m-\alpha)}\int_{h_1^p}^{h_2^p} \frac{\Upsilon^m\left(s^{\frac{1}{p}}\right)}{(s - h_1^p)^{m-\alpha+1}}ds \\ &= (-1)^m({}^C\mathcal{D}^{\alpha}_{h_2-}\Upsilon)(\mu(h_1^p)). \end{aligned} \tag{3.3}$$

(*ii*) The proof is similar to (*i*). □

Remark 3.3. *In Lemma* 3.2 *(i), if one takes* $p = 1$ *in the identity* 3.1 *then one gets the identity of Lemma* 2.1 *in* [8].

Theorem 3.4. *Let* $p \in \mathbb{R}\backslash\{0\}$. *Let* $\Upsilon, \zeta : [h_1, h_2] \subset (0, \infty) \to \mathbb{R}$, $h_1 < h_2$, *be the functions such that* $\Upsilon, \zeta \in C^m[h_1, h_2]$. *Also consider* ζ^m *is positve and p-convex and* Υ^m *is nonnegative and p-symmetric over* $\left[\frac{h_1^p+h_2^p}{2}\right]^{1/p}$, *then*
(i) for $p > 0$

$$\begin{aligned}
&\zeta\left(\left[\frac{h_1^p + h_2^p}{2}\right]^{1/p}\right)\left[({}^C\mathcal{D}^{\alpha}_{h_1^p+}\Upsilon)(\mu(h_2^p)) + (-1)^m({}^C\mathcal{D}^{\alpha}_{h_2^p-}\Upsilon)(\mu(h_1^p))\right] \\
&\le \left[({}^C\mathcal{D}^{\alpha}_{h_1^p+}(\zeta * \Upsilon))(\mu(h_2^p)) + (-1)^m({}^C\mathcal{D}^{\alpha}_{h_2^p-}(\zeta * \Upsilon))(\mu(h_1^p))\right] \\
&\le \frac{\zeta^m(h_1) + \zeta^m(h_2)}{2}\left[({}^C\mathcal{D}^{\alpha}_{h_1^p+}\Upsilon)(\mu(h_2^p)) + (-1)^m({}^C\mathcal{D}^{\alpha}_{h_2^p-}\Upsilon)(\mu(h_1^p))\right],
\end{aligned} \tag{3.4}$$

where $\mu(s) = s^{\frac{1}{p}}$, *for all* $s \in [h_1^p, h_2^p]$.
(ii) for $p < 0$

$$\begin{aligned}
&\zeta\left(\left[\frac{h_1^p + h_2^p}{2}\right]^{1/p}\right)\left[(-1)^m({}^C\mathcal{D}^{\alpha}_{h_1^p-}\Upsilon)(\mu(h_2^p)) + ({}^C\mathcal{D}^{\alpha}_{h_2^p+}\Upsilon)(\mu(h_1^p))\right] \\
&\le \left[(-1)^m({}^C\mathcal{D}^{\alpha}_{h_1^p-}(\zeta * \Upsilon))(\mu(h_2^p)) + ({}^C\mathcal{D}^{\alpha}_{h_2^p+}(\zeta * \Upsilon))(\mu(h_1^p))\right] \\
&\le \frac{\zeta^m(h_1) + \zeta^m(h_2)}{2}\left[(-1)^m({}^C\mathcal{D}^{\alpha}_{h_1^p-}\Upsilon)(\mu(h_2^p)) + ({}^C\mathcal{D}^{\alpha}_{h_2^p+}\Upsilon)(\mu(h_1^p))\right],
\end{aligned} \tag{3.5}$$

where $\mu(s) = s^{\frac{1}{p}}$, $s \in [h_2^p, h_1^p]$.

Proof . From p-convexity of ζ^m, we get

$$\zeta^m\left(\left[\frac{a^p + b^p}{2}\right]^{\frac{1}{p}}\right) \le \frac{\zeta^m(a) + \zeta^m(b)}{2}.$$

Taking $a^p = \eta h_1^p + (1-\eta)h_2^p$ and $b^p = (1-\eta)h_1^p + \eta h_2^p$ with $\eta \in [0, 1]$, we get

$$\zeta^m\left(\left[\frac{h_1^p + h_2^p}{2}\right]^{\frac{1}{p}}\right) \le \frac{\zeta^m\left([\eta h_1^p + (1-\eta)h_2^p]^{\frac{1}{p}}\right) + \zeta^m\left([(1-\eta)h_1^p + \eta h_2^p]^{\frac{1}{p}}\right)}{2}. \tag{3.6}$$

Multiplying (3.6) by $\eta^{m-\alpha-1}\,\Upsilon\left([\eta h_1^p + (1-\eta)h_2^p]^{\frac{1}{p}}\right)$ on both sides with $\eta \in (0, 1)$, $\alpha > 0$ and then integrating over $\eta \in [0, 1]$, we obtain

$$\begin{aligned}
&2\zeta^m\left(\left[\frac{h_1^p + h_2^p}{2}\right]^{\frac{1}{p}}\right)\int_0^1 \eta^{m-\alpha-1}\,\Upsilon^m\left([\eta h_1^p + (1-\eta)h_2^p]^{\frac{1}{p}}\right)d\eta \\
&\le \int_0^1 \eta^{m-\alpha-1}\zeta^m\left([\eta h_1^p + (1-\eta)h_2^p]^{\frac{1}{p}}\right)\Upsilon^m\left([\eta h_1^p + (1-\eta)h_2^p]^{\frac{1}{p}}\right)d\eta \\
&\quad + \int_0^1 \eta^{m-\alpha-1}\zeta^m\left([\eta h_2^p + (1-\eta)h_1^p]^{\frac{1}{p}}\right)\Upsilon^m\left([\eta h_1^p + (1-\eta)h_2^p]^{\frac{1}{p}}\right)d\eta.
\end{aligned}$$

Then by change of variable and symmetric property of Υ^m, we get

$$\begin{aligned}
&\frac{2}{(h_2^p-h_1^p)^{m-\alpha}}\zeta^m\left(\left[\frac{h_1^p+h_2^p}{2}\right]^{\frac{1}{p}}\right)\int_{h_1^p}^{h_2^p}(h_2^p-s)^{m-\alpha-1}\,\Upsilon^m\left(s^{\frac{1}{p}}\right)ds\\
&\le\frac{1}{(h_2^p-h_1^p)^{m-\alpha}}\int_{h_1^p}^{h_2^p}(h_2^p-s)^{m-\alpha-1}\,\zeta^m\left(s^{\frac{1}{p}}\right)\Upsilon^m\left(s^{\frac{1}{p}}\right)ds\\
&\quad+\frac{1}{(h_2^p-h_1^p)^{m-\alpha}}\int_{h_1^p}^{h_2^p}(h_2^p-s)^{m-\alpha-1}\,\zeta^m\left([h_1^p+h_2^p-s]^{\frac{1}{p}}\right)\Upsilon^m\left(s^{\frac{1}{p}}\right)ds\\
&=\frac{1}{(h_2^p-h_1^p)^{m-\alpha}}\int_{h_1^p}^{h_2^p}(h_2^p-s)^{m-\alpha-1}\,\zeta^m\left(s^{\frac{1}{p}}\right)\Upsilon^m\left(s^{\frac{1}{p}}\right)ds\\
&\quad+\frac{1}{(h_2^p-h_1^p)^{m-\alpha}}\int_{h_1^p}^{h_2^p}(s-h_1^p)^{m-\alpha-1}\,\zeta^m\left(s^{\frac{1}{p}}\right)\Upsilon^m\left([h_1^p+h_2^p-s]^{\frac{1}{p}}\right)ds.
\end{aligned}$$

Hence by Lemma 3.2, we find

$$\begin{aligned}
&\frac{1}{(h_2^p-h_1^p)^{m-\alpha}}\zeta\left(\left[\frac{h_1^p+h_2^p}{2}\right]^{1/p}\right)\left[({}^{C}\mathcal{D}^{\alpha}_{h_1^p+}\Upsilon)(\mu(h_2^p))+(-1)^m({}^{C}\mathcal{D}^{\alpha}_{h_2^p-}\Upsilon)(\mu(h_1^p))\right]\\
&\le\frac{1}{(h_2^p-h_1^p)^{m-\alpha}}\left[({}^{C}\mathcal{D}^{\alpha}_{h_1^p+}(\zeta*\Upsilon))(\mu(h_2^p))+(-1)^m({}^{C}\mathcal{D}^{\alpha}_{h_2^p-}(\zeta*\Upsilon)(\mu(h_1^p))\right]
\end{aligned}\tag{3.7}$$

Now consider,

$$\zeta^m\left([\eta h_1^p+(1-\eta)h_2^p]^{\frac{1}{p}}\right)+\zeta^m\left([\eta h_2^p+(1-\eta)h_1^p]^{\frac{1}{p}}\right)\le[\zeta^m(h_1)+\zeta^m(h_2)]\,.\tag{3.8}$$

Multiplying (3.8) by $\eta^{m-\alpha}\,\Upsilon\left([\eta h_1^p+(1-\eta)h_2^p]^{\frac{1}{p}}\right)$ on both sides with $\eta\in(0,1)$, $\alpha>0$, and then integrating over $\eta\in[0,1]$, we obtain

$$\begin{aligned}
&\int_0^1\zeta^m\left([\eta h_1^p+(1-\eta)h_2^p]^{\frac{1}{p}}\right)\eta^{m-\alpha}\,\Upsilon\left([\eta h_1^p+(1-\eta)h_2^p]^{\frac{1}{p}}\right)d\eta\\
&\quad+\int_0^1\zeta^m\left([\eta h_2^p+(1-\eta)h_1^p]^{\frac{1}{p}}\right)\eta^{m-\alpha}\,\Upsilon\left([\eta h_1^p+(1-\eta)h_2^p]^{\frac{1}{p}}\right)d\eta\\
&\le[\zeta^m(h_1)+\zeta^m(h_2)]\int_0^1\eta^{m-\alpha}\,\Upsilon\left([\eta h_1^p+(1-\eta)h_2^p]^{\frac{1}{p}}\right)d\eta
\end{aligned}\tag{3.9}$$

that is, we get

$$\begin{aligned}
&\frac{1}{(h_2^p-h_1^p)^{m-\alpha}}\left[({}^{C}\mathcal{D}^{\alpha}_{h_1^p+}(\zeta*\Upsilon))(\mu(h_2^p))+(-1)^m({}^{C}\mathcal{D}^{\alpha}_{h_2^p-}(\zeta*\Upsilon)(\mu(h_1^p))\right]\\
&\le\frac{\zeta^m(h_1)+\zeta^m(h_2)}{2(h_2^p-h_1^p)^{m-\alpha}}\left[({}^{C}\mathcal{D}^{\alpha}_{h_1^p+}\Upsilon)(\mu(h_2^p))+(-1)^m({}^{C}\mathcal{D}^{\alpha}_{h_2^p-}\Upsilon)(\mu(h_1^p))\right].
\end{aligned}\tag{3.10}$$

Hence from inequalities (3.7) and (3.10), we get (3.4).

(*ii*) The proof is analogous to (*i*). □

Remark 3.5. *In Theorem* 3.4 (*i*)*, if one takes* $p=1$ *in the inequaliity* 3.4 *then one gets inequality* 3.3 *of Thoerem* 3.2 *in* [8].

Lemma 3.6. *Let $p \in \mathbb{R}\backslash\{0\}$. Let $\Upsilon, \zeta : [h_1, h_2] \subset (0, \infty) \to \mathbb{R}$, $h_1 < h_2$, be the functions such that $\Upsilon, \zeta \in C^m[h_1, h_2]$. Also consider ζ^m is positve and Υ^m is nonnegative and p-symmetric over $\left[\frac{h_1^p+h_2^p}{2}\right]^{1/p}$. Then*
(i) for $p > 0$, the following inequality holds:

$$\begin{aligned}
&\frac{\zeta^m(h_1)+\zeta^m(h_2)}{2}\Big[({}^C\mathcal{D}^{\alpha}_{h_1^p+}\Upsilon)(\mu(h_2^p))+(-1)^m({}^C\mathcal{D}^{\alpha}_{h_2^p-}\Upsilon)(\mu(h_1^p))\Big]\\
&-\Big[({}^C\mathcal{D}^{\alpha}_{h_1^p+}(\zeta*\Upsilon))(\mu(h_2^p))+(-1)^m({}^C\mathcal{D}^{\alpha}_{h_2^p-}(\zeta*\Upsilon))(\mu(h_1^p))\Big]\\
&\le \frac{1}{\Gamma(m-\alpha)}\int_{h_1^p}^{h_2^p}\Big[\int_{h_1^p}^{t}(h_2^p-s)^{m-\alpha-1}\,\Upsilon^m\,(\mu(s))ds\\
&-\int_{t}^{h_2^p}(s-h_1^p)^{m-\alpha-1}\,\Upsilon^m\,(\mu(s))ds\Big]\zeta^{m+1}(\mu(t))dt.
\end{aligned} \tag{3.11}$$

Where $\mu(s) = s^{1/p}$, for all $s \in [h_1^p, h_2^p]$.
(ii) for $p < 0$, the following inequality holds:

$$\begin{aligned}
&\frac{\zeta^m(h_1)+\zeta^m(h_2)}{2}\Big[(-1)^m({}^C\mathcal{D}^{\alpha}_{h_1^p-}\Upsilon)(\mu(h_2^p))+({}^C\mathcal{D}^{\alpha}_{h_2^p+}\Upsilon)(\mu(h_1^p))\Big]\\
&-\Big[(-1)^m({}^C\mathcal{D}^{\alpha}_{h_1^p-}(\zeta*\Upsilon))(\mu(h_2^p))+({}^C\mathcal{D}^{\alpha}_{h_2^p+}(\zeta*\Upsilon))(\mu(h_1^p))\Big]\\
&\le \frac{1}{\Gamma(m-\alpha)}\int_{h_2^p}^{h_1^p}\Big[\int_{h_2^p}^{t}(h_1^p-s)^{m-\alpha-1}\,\Upsilon^m\,(\mu(s))ds\\
&-\int_{t}^{h_1^p}(s-h_2^p)^{m-\alpha-1}\,\Upsilon^m\,(\mu(s))ds\Big]\zeta^{m+1}(\mu(t))dt.
\end{aligned} \tag{3.12}$$

Where $\mu(s) = s^{1/p}$, for all $s \in [h_2^p, h_1^p]$.

Proof . (i) Note that

$$\begin{aligned}
F &= \int_{h_1^p}^{h_2^p}\left(\int_{h_1^p}^{t}(h_2^p-s)^{m-\alpha-1}\,\Upsilon^m\,(\mu(s))ds\right)\zeta^{m+1}(\mu(t))dt\\
&\quad-\int_{h_1^p}^{h_2^p}\left(\int_{t}^{h_2^p}(s-h_1^p)^{m-\alpha-1}\,\Upsilon^m\,(\mu(s))ds\right)\zeta^{m+1}(\mu(t))dt\\
&= F_1 - F_2.
\end{aligned} \tag{3.13}$$

By integrating by parts and using change of variable and Lemma 3.2, we get

$$\begin{aligned} F_1 &= \left(\int_{h_1^p}^{t}(h_2^p-s)^{m-\alpha-1}\,\Upsilon^m\,(\mu(s))ds\right)\zeta^m(\mu(t))\Big|_{h_1^p}^{h_2^p} \\ &\quad -\int_{h_1^p}^{h_2^p}(h_2^p-t)^{m-\alpha-1}\,\Upsilon^m\,(\mu(t))\zeta^m(\mu(t))dt \\ &= \Gamma(m-\alpha)\left[\zeta^m(\mu(h_2^p))(\,{}^C\mathcal{D}^{\alpha}_{h_1^p+}\Upsilon)(\mu(h_2^p)) - (\,{}^C\mathcal{D}^{\alpha}_{h_1^p+}(\zeta*\Upsilon))(\mu(h_2^p))\right] \\ &= \Gamma(m-\alpha)\left[\frac{\zeta^m(\mu(h_2^p))}{2}\left[(\,{}^C\mathcal{D}^{\alpha}_{h_1^p+}\Upsilon)(\mu(h_2^p)) + (-1)^m(\,{}^C\mathcal{D}^{\alpha}_{h_2^p-}\Upsilon)(\mu(h_1^p))\right]\right. \\ &\quad \left. -(\,{}^C\mathcal{D}^{\alpha}_{h_1^p+}(\zeta*\Upsilon))(\mu(h_2^p))\right]. \end{aligned} \tag{3.14}$$

Similarly,

$$\begin{aligned} F_2 &= \left(\int_{t}^{h_2^p}(s-h_1^p)^{m-\alpha-1}\,\Upsilon^m\,(\mu(s))ds\right)\zeta^m(\mu(t))\Big|_{h_1^p}^{h_2^p} \\ &\quad +\int_{h_1^p}^{h_2^p}(t-h_1^p)^{m-\alpha-1}\,\Upsilon^m\,(\mu(t))\zeta^m(\mu(t))dt \\ &= \Gamma(m-\alpha)\left[\frac{\zeta^m(\mu(h_1^p))}{2}\left[(\,{}^C\mathcal{D}^{\alpha}_{h_1^p+}\Upsilon)(\mu(h_2^p)) + (-1)^m(\,{}^C\mathcal{D}^{\alpha}_{h_2^p-}\Upsilon)(\mu(h_1^p))\right]\right. \\ &\quad \left. -(-1)^m(\,{}^C\mathcal{D}^{\alpha}_{h_2^p-}(\zeta*\Upsilon))(\mu(h_1^p))\right]. \end{aligned} \tag{3.15}$$

Then from (3.14) and (3.15), we find

$$\begin{aligned} F &= F_1 - F_2 \\ &= \Gamma(m-\alpha)\left[\frac{\zeta^m(h_1)+\zeta^m(h_2)}{2}\left[(\,{}^C\mathcal{D}^{\alpha}_{h_1^p+}\Upsilon)(\mu(h_2^p)) + (-1)^m(\,{}^C\mathcal{D}^{\alpha}_{h_2^p-}\Upsilon)(\mu(h_1^p))\right]\right. \\ &\quad \left. -\left[(\,{}^C\mathcal{D}^{\alpha}_{h_1^p+}(\zeta*\Upsilon))(\mu(h_2^p)) + (-1)^m(\,{}^C\mathcal{D}^{\alpha}_{h_2^p-}(\zeta*\Upsilon))(\mu(h_1^p))\right]\right]. \end{aligned} \tag{3.16}$$

Multiplying (3.16) by $\frac{1}{\Gamma(m-\alpha)}$ we obtain (3.11).
(ii) Proof is similar to (i). □

Remark 3.7. *In Lemma* 3.6 (i)*, if one takes* $p = 1$ *in the identity* 3.11 *then one gets the identity of Lemma* 3.3 *in* [8]*.*

Note that in Lemma 3.2 (ii), Theorem 3.4 (ii) and Lemma 3.6 (ii), if one takes $p = -1$ then one gets similar results for harmonically convex functions. Also from Lemma 3.6 we can have many other Hadamard-Fejér type inequalities for p-convex functions and hence for convex as well as harmonically convex functions.

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