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Solving Benny-Lin Equation by Adomain Decomposition Method with Genetic Algorithm

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Abstract

The nonlinear Benny-Lin equation has been solved In this paper using Adomian decomposition technique (ADM) with different initial conditions and the results shown in Figures (1-8), also by using modified Adomian decomposition technique with genetic algorithm to expound the optimal equation parameters. The proposed method (GA-ADM) guarantees that the optimal parameters will be achieved precisely regardless of the complexity and multiple values of the equation. The proposed method gives more accurate results than the ADM.

Keywords: Benny-Lin equation, Adomian-decomposition method, genetic algorithm

1. Introduction

The study of nonlinear partial differential equations (NPDEs), is one of the important applications in mathematics and physics including engineering sciences and biological sciences. One of the important partial differential equations is Benney–Lin equation, the (general form) low regularity of the initial value problem (IVP) of Benney–Lin equation:

$$\frac{\partial \omega}{\partial t}+\propto\omega\frac{\partial \omega}{\partial x}+\lambda\,\frac{\partial^3 \omega}{\partial x^3}+\beta\,\left(\frac{\partial^2 \omega}{\partial x^2}+\frac{\partial^4 \omega}{\partial x^4}\right)+\eta\,\frac{\partial^5 \omega}{\partial x^5}+\delta\,\frac{\partial \omega}{\partial x}=0,\quad x\in[0,L] \tag{1.1}$$

Benney [1] derived The Benney–Lin equation first and after by Lin [2]. It is a crucial general equation that can describe long waves evolution in fluid dynamics. the invariance properties of the

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time-fractional Benny–Lin equation investigated in[3], Travelling Wave Solutions to the Benney-Luke is presented in [4], Praveen [5] used reduced differential transform and the homotopy perturbation methods for solving fractional Benney–Lin equation, and He's homotopy perturbation and He's Variational iteration methods in [6].

The Adomian decomposition technique has recently become a recognized strategy in related scientific fields, and it does not need any differential assumptions or linearization to find the solutions to differential equations. Recently, many iteration methods were used for solving nonlinear partial differential equations [7, 8], such as the Variational iteration method [9, 10], Homotopy perturbation and analysis methods[11, 12], new iterative method [13]. Control strategies method for nonlinear ordinary differential equations[14, 15, 16]. The ADM method is improved via Bernstein polynomials in [17, 18, 19], with Legendre polynomials[20], and with Laguerre polynomials[21]. In recent years, researchers have been very interested in studying genetic algorithms, which have been applied for solutions to many complex applications. The firefly algorithm, developed by Yang in 2008, is one of the most important genetic algorithms and has managed to outperform many other algorithms in solving problems efficiently [22]. Optimal Parameters for Nonlinear Hirota-Satsuma Coupled KdV System by Using Hybrid Firefly Algorithm with Modified Adomian Decomposition has shown effectiveness and good performance in optimal solutions [23]. The idea for genetic algorithms was inspired by the flashing light behavior of fireflies. The flashes are used as a signal system for fireflies to attract other fireflies through the characteristics of their flashing [24].

In our work, the fifth-order non-linear Benny-Lin equation will be solved using the Adomain analytical method, and take some case studies as initial conditions which solved by M. Safari, D. Ganji and E. Sadeghi[6]. Then we take a special case study that identified the exact solution under a special term that achieves the equation and improve the results using the genetic algorithm to find the best coefficients for the non-linear equation of Benney–Lin. In Section 2 of this paper, Adomian decomposition techniques are presented. In Section 3 the Proposed method (GA_ADM) are described to solving a nonlinear Benny-Lin equation. Section 4 describes two main cases.

2. Adomian decomposition technique

Let's take the following equation:

$$L\omega + N\omega + R\omega = g\ (x) \tag{2.1}$$

since L represents an invertible linear term, R is the residual linear part and N be the non-linear term, from equation (2.1):

$$L\omega = g\ (x) - N\omega - R\omega, \tag{2.2}$$

By, applying the inverse L^{-1} to equation (2.2) then:

$$\omega = f(x) - L^{-1}\ N\omega - L^{-1}\ R\omega, \tag{2.3}$$

where $L^{-1} = \int_0^t (.)ds$, and $f(x)$ are the terms come by integrating the rest of the term $g(x)$ and from using the given initial condition. The ADM assumes that $N(\zeta)$ (non-linear term) can be computed by an infinite series of polynomials represents in form

$$N\ (\omega) = \sum_{n=0}^{\infty} A_n(\omega_o, \omega_1, \ldots, \omega_n)$$

where An are the ADM polynomials [25] defined as:

$$A_n = \frac{1}{n!}\,\frac{d^n}{d\lambda^n}\left[N\left(\sum_{i=0}^{\infty}\rho^i\omega_i\right)\right]_{\rho=0},\, n=0,1,2,\ldots \tag{2.4}$$

Now

$$\begin{aligned}\omega_0 &= L^{-1}\left(g\ \ (x)\right)\\ \omega_1 &= -L^{-1}\left(R\omega_0\right)-L^{-1}\left(N\omega_0\right),\\ \omega_2 &= -L^{-1}\left(R\omega_1\right)-L^{-1}\left(N\omega_1\right),\\ &\vdots\end{aligned} \tag{2.5}$$

and so on. These formulas can be computed using Maple 13 software. Above equation is governing equation of ADM. The obtained approximate solution:

$$\omega_V\left(x,t\right)=\sum_{i=0}^{V}\omega_i(x,t) \tag{2.6}$$

3. Proposed method (GA_ADM)

The partial differential equation is expressed within the form:

$$\Gamma\left(x,t,\omega\left(x,t\right),\frac{\partial}{\partial x}\omega\left(x,t\right)\frac{\partial}{\partial t}\omega\left(x,t\right),\frac{\partial^2}{\partial x^2}\omega\left(x,t\right),\frac{\partial^2}{\partial t^2}\omega\left(x,t\right),\ldots\ldots\right)=0 \tag{3.1}$$

where $x\in[x_0,x_1]$ and $t\in[t_0,t_1]$. The associated initial conditions are expressed as:

$$\omega\left(x,t_0\right)=\Gamma_0\left(x\right),$$

The fitness evaluation steps of the population will be as follows :

- For every chromosome i.
- Building an experimental solution $\omega_i(x,t)$ is expressed as follows.
- Calculate the quantity

$$E\left(H_i\right)=\sum_{j=1}^{N}\Gamma\left(x_j,t_j,\widehat{\omega}\left(x_i,t_j\right),\frac{\partial}{\partial x}\widehat{\omega}\left(x_i,t_j\right),\frac{\partial}{\partial t}\widehat{\omega}\left(x_i,t_j\right),\frac{\partial^2}{\partial x^2}\widehat{\omega}\left(x_i,t_j\right),\frac{\partial^2}{\partial t^2}\widehat{\omega}\left(x_i,t_j\right),\ldots\right)^2 \tag{3.2}$$

- Selection and reproduction (Chromosomes are selected for the next generation based on their fitness value using the mating process (crossover).
- Mutation (The mutation process introduces random changes in genes to preserve genetic diversity for a solution good.
- Stopping criteria (the algorithm stops if the number of generations exceeds or the required fitness is reached, That is, the condition in equation (3.2) is fulfilled so that the mean square error is as low as possible. Repeat steps.

The proposed method is based on determining the optimal parameters of the nonlinear Benney–Lin equation using GA for the solution (2.6). The result of the ADM solution series (2.6) is used to formulate the fitness function in the following equation:

$$F(\alpha,\lambda,\beta,\eta,\delta) = \sum_{i=1}^{n}\sum_{j=1}^{m}\left(\omega(x_i,t_j) - \widehat{\omega}(x_i,t_j)\right)^2 \tag{3.3}$$

where ω the solution of nonlinear Benney–Lin equation, ω the exact solutions for this equation, n and m represent the total numbers of steps used in the solution domain of x and t respectively, F in this equation represents the fitness function and it is solved by using the GA which involves three operators on its population (selection, crossover and mutation).

4. Applications

In this section, the solution of the equation (1.1) is divided into two main cases as follows:

4.1. First case

By applying the Adomian decomposition method with different initial conditions to Benney–Lin equation and finding the solutions that turns out excellent approximations . The initial conditions include[6]:

1. $\omega(x,0) = \alpha - 2\nu^2 \tan(\nu x)$ (4.1)
2. $\omega(x,0) = \alpha - 2\nu^2 \tan^2(\nu x)$ (4.2)
3. $\omega(x,0) = \alpha - 2\nu^2 \tanh(\nu x)$ (4.3)
4. $\omega(x,0) = \alpha - 2\nu^2 \tanh^2(\nu x)$ (4.4)
5. $\omega(x,0) = \alpha - 2\nu^2 \sec(\nu x)$ (4.5)
6. $\omega(x,0) = \alpha - 2\nu^2 \sec^2(\nu x)$ (4.6)
7. $\omega(x,0) = \alpha - 2\nu^2 \operatorname{sech}(\nu x)$ (4.7)
8. $\omega(x,0) = \alpha - 2\nu^2 \operatorname{sech}^2(\nu x)$ (4.8)

Using the initial (4.4) with $\alpha = \nu = 1$, the approximation solution for equation (1.1) by ADM is:

$$\begin{aligned}
\omega_0 &= 1 - 2\tanh^2(x) \\
\omega_1 &= 516 \tanh(x)\ t - 28\, t - 2388 \tanh^3(x)\ t + \cdots - 1440 \tanh^7(x)\ t \\
\omega_2 &= 10672880 \tanh^2(x)\ t^2 - 331298\, t^2 - 187023000 \tanh^6(x)\ t^2 + \cdots - 676896 \tanh(x)\ t^2 \\
&\ \ \vdots \\
\omega_{ADM} &= 1 - 2\tanh^2(x) + 516 \tanh(x)\ t - 28\, t - 2388 \tanh^3(x)\ t + 10672880 \tanh^2(x)\ t^2 \\
&\quad - 331298\, t^2 - 187023000 \tanh^6(x)\ t^2 + \cdots - 676896 \tanh(x)\ t^2
\end{aligned} \tag{4.9}$$

The following figures from 1 to 8 show the solution resulting from applying the method to each of the above-mentioned conditions:

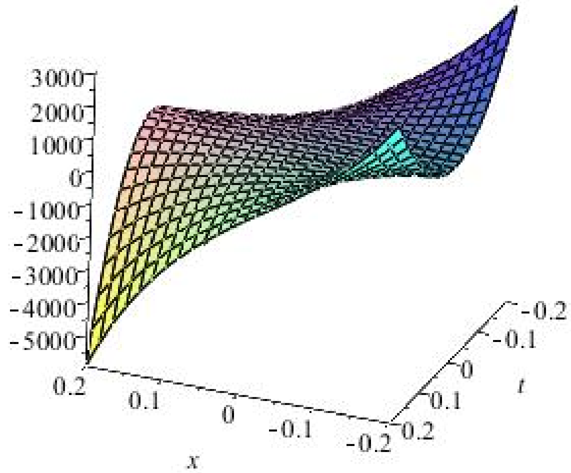

Figure 1: Appling eq. (4.1) with ADM

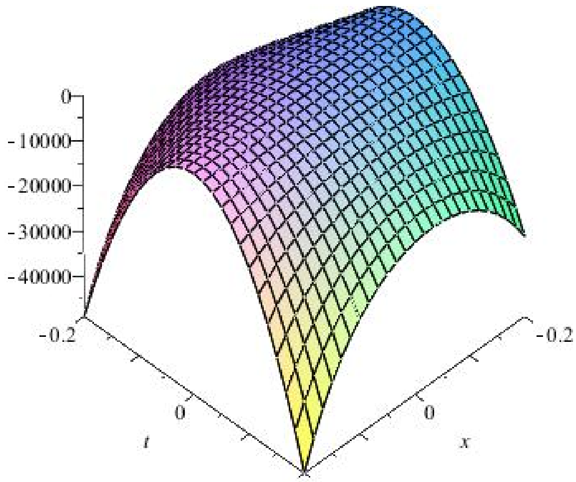

Figure 2: Appling eq. (4.2) with ADM

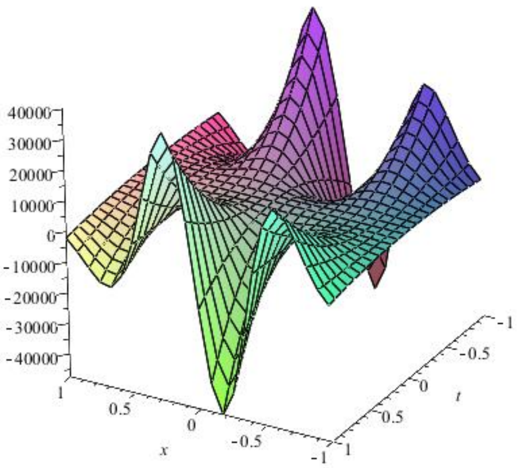

Figure 3: Appling eq. (4.3) with ADM

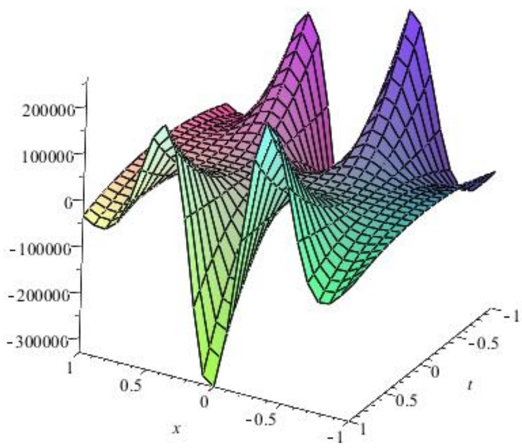

Figure 4: Appling eq. (4.4) with ADM

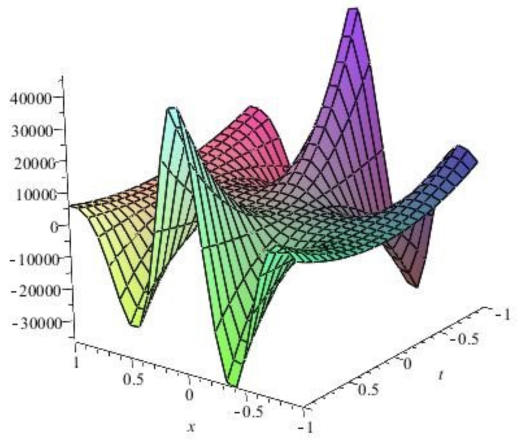

Figure 7: Appling eq. (4.7) with ADM

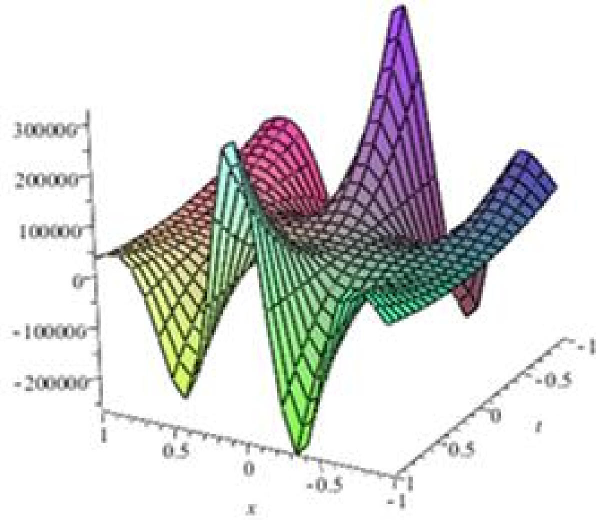

Figure 8: Appling eq. (4.8) with ADM

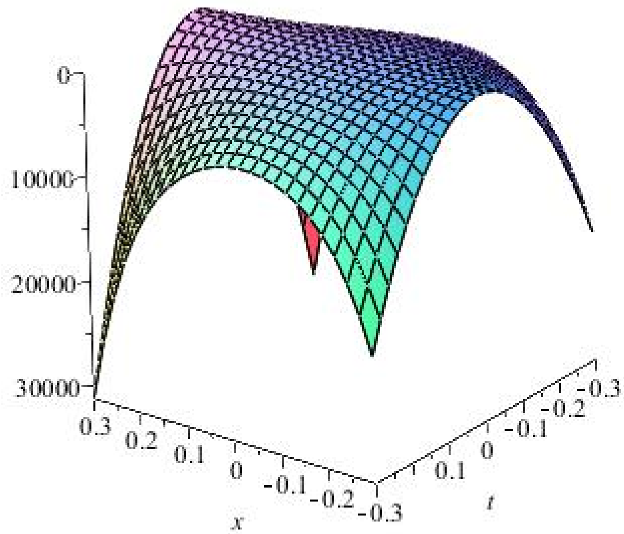

Figure 5: Appling eq. (4.5) with ADM

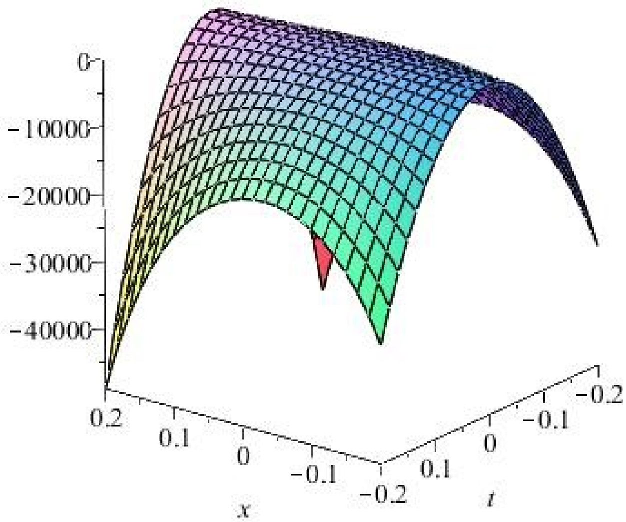

Figure 6: Appling eq. (4.6) with ADM

The solutions show good approximations to M. Safari etl [6] results, which takes the same values of equation parameters.

4.2. Second case study

Now we implementation suggested solution of eq. (1.1) and find test values of the constants in Benney–Lin equation by using GA techniques, these values achieved the equation as the semi-exact solution.

$$S_{exact} = \frac{e^x}{(1+t)} \tag{4.10}$$

with $\alpha = \frac{1}{e^x}$ and initial condition e^x , get the numerical solution of the Adomain polynomial as follows :

$$\begin{aligned}
\omega_0 =& e^x \\
\omega_1 =& -\alpha e^{2x}t + \lambda e^x t + 2\beta e^x t + \eta e^x t + \delta e^x t \\
\omega_2 =& -5\alpha\ \lambda\ e^{2x}t^2 - 12\alpha\ \beta e^{2x}t^2 - 17\alpha\ \eta e^{2x}t^2 - 2\alpha\ \delta\ e^{2x}t^2 + 2\lambda\ \beta e^x t^2 + \lambda\eta e^x t^2 + \lambda\ \delta e^x t^2 + 2\beta\ \eta e^x t^2 \\
&+ 2\beta\delta\ e^x t^2 + \eta\delta\ e^x t^2 + \frac{3}{2}\alpha^2\ e^{3x}t^2 + \frac{1}{2}\ \lambda^2 e^x t^2 + 2\beta^2\ e^x t^2 + \frac{1}{2}\eta^2\ e^x t^2 + \frac{1}{2}\delta^2 e^x t^2 \\
&\vdots
\end{aligned}$$

After simplifying get:

$$\begin{aligned}
\omega_{GA_ADM} =& e^x - \alpha e^{2x}t + \lambda e^x t + 2\beta e^x t + \eta e^x t + \delta e^x t - 5\alpha\beta e^{2x}t^2 - 17\alpha\eta e^{2x}t^2 - 2\alpha\delta e^{2x}t^2 + 2\lambda\beta e^x t^2 \\
&+ \lambda\eta e^x t^2 + \lambda\delta e^x t^2 + 2\beta\eta e^x t^2 + 2\beta\delta\ e^x t^2 + \eta\delta e^x t^2 + \frac{3}{2}\alpha^2\ e^{3x}t^2 \\
&+ \frac{1}{2}\ \lambda^2 e^x t^2 + 2\beta^2\ e^x t^2 + \frac{1}{2}\eta^2\ e^x t^2 + \frac{1}{2}\delta^2 e^x t^2 \ldots
\end{aligned} \tag{4.11}$$

Where ω_{GA_ADM} represent the numerical solution of the proposed method solving the nonlinear equation of Benney–Lin. The proposed method supposes a first standard model structure where some of the parameters are unknown. The objective of this method is to find the optimal parameters (λ,β,η,δ) of nonlinear Benney–Lin equation which minimizes differences between the model output vector and the real response vector.

The method of genetic algorithms is to generate new solutions from the possibilities encoded on the shape known as “chromosome”. Chromosomes combine or change to produce new individuals. It is useful for finding the optimal solution to multidimensional problems in which the values of the different variables can be encoded in the shape of the chromosome.

The idea of genetic algorithms lies in generating some solutions to the problem at random, then examining these solutions and comparing some criteria, and only the best solutions remain, while the least efficient solutions are neglected according to the biological rule of “survival of the fittest”. The programming Matlab® R2018a is used for applying the algorithm given in the following Table 1.

Table 1: Parameter values for GA

Parameter Name	**Values**
Population Size	50
Max Generations	100
Max Stall Generations	50
Function Tolerance	1e-30
Initial Population Range	[-100 100]
Crossover Fraction	0.8

The proposed method supposes an elementary standard model structure where some of the parameters are unknown. The objective of this method is to find the optimal parameters λ,β,η, *and* δ

for nonlinear Benney–Lin equation to minimize the differences between the model output vector and the real response vector. The best values for Eq. (1.1) are obtained from the following parameter values: $\lambda = 1, \beta = 1, \eta = -4, \delta = 1$

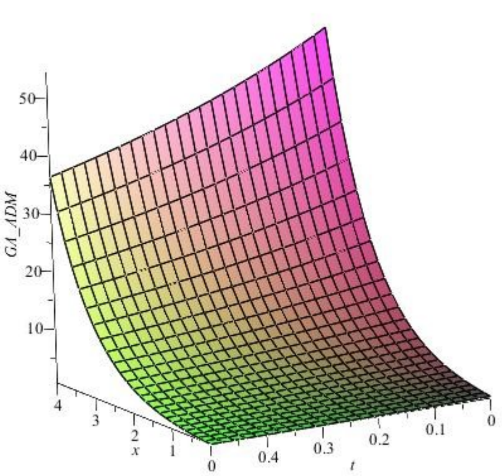

Figure 9: The numerical silution ω_{GA_ADM}

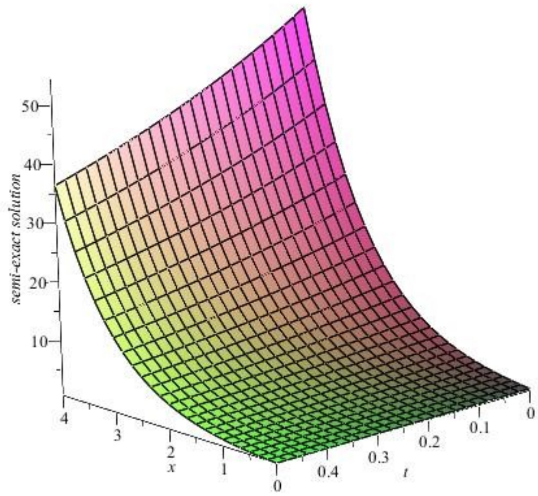

Figure 10: The semi-exact solution S_{exact}

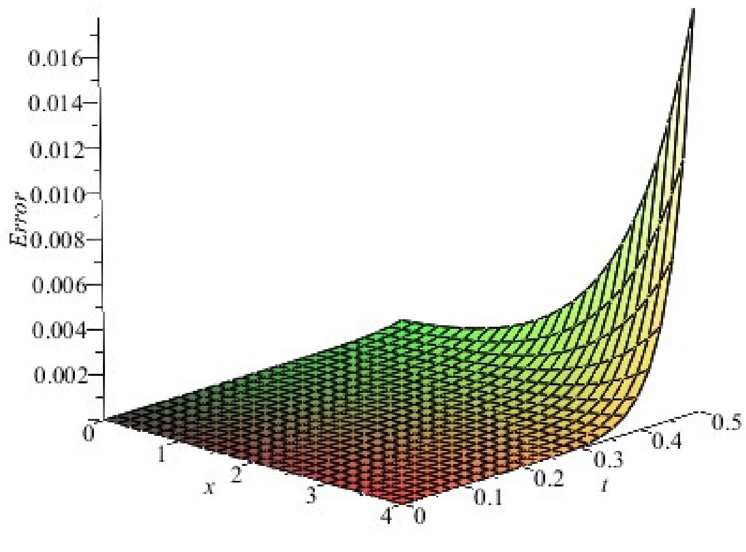

Figure 11: The global absolute error of x and t in Benney-Lin equation when $\alpha = \frac{1}{e^x}, \lambda = 1, \beta = 1, \eta = -4, \delta = 1$ and $n = 10$

We notice from the table 2 and figure 11, that the numerical solution approaches the exact solution based on the parameters found using the genetic algorithm, where the absolute error rate reaches

Table 2: Represent a comparison between the numerical solution ω_{GA_ADM} and S_{exact} to Benney–Lin equation with an absolute error when applying the values of equation parameters $\alpha=\frac{1}{e^x}$,$\lambda= 1$,$\beta= 1$,$\eta= -4$,$\delta= 1$ at $x \in [0, 2]$ and $t = 0.1, 0.3, 0.5$ with $n = 10$.

x	t	S_{exact}	ω_{GA_ADM}	Absolute Error
0		2.225540928492467	2.225540928492467	0.00000000000000
0.16		2.181902871071046	2.181902871071046	0.00000000000000
0.32		2.139943200473526	2.139943200473527	0. 88800000000000 E-15
0.48		2.099566913672139	2.099566913672215	0. 75939000000000 E-13
0.64		2.060686044900433	2.060686044902203	0. 17701400000000 E-11
0.80		2.023219025902243	2.023219025922476	0.202327040000000 E-10
0.96	0.1	1.987090114725418	1.987090114873060	0.147641899000000 E-9
1.12		1.95222888464251	1.9522288854330836	0.790567611000000 E-9
1.28		1.918569765941782	1.918569769316966	0.337518346600000 E-8
1.44		1.886051634315650	1.886051646437005	0.121213541430000 E-7
1.60		1.854617440410390	1.854617478392955	0.379825650930000 E-7
1.76		1.824213875813498	1.824213982405653	0.106592154880000 E-6
1.92		1.794791071364893	1.794791344474866	0.273109972859000 E-6
0		11.023176380641610	11.023176380641610	0.0
0.16		10.807035667295697	10.807035667295695	0.200000000000000 E-14
0.32		10.599208058309241	10.599208058309245	0.400000000000000 E-13
0.48		10.399223000605293	10.399223000605669	0.377000000000000 E-12
0.64		10.206644796890378	10.206644796899146	0.876800000000000 E-11
0.80		10.021069436946917	10.021069437047132	0.100215000000000 E-10
0.96	0.3	9.842121768430010	9.842121769161285	0.731275000000000 E-9
1.12		9.669452965475097	9.669452969390804	0.391570700000000 E-8
1.28		9.502738259173803	9.502738275891195	0.167173920000000 E-7
1.44		9.341674898848822	9.341674958886282	0.600374590000000 E-6
1.60		9.185980317201341	9.185980505330219	0.188128878000000 E-6
1.76		9.035390475935746	9.035391003890146	0.527954400000000 E-6
1.92		8.889658371485169	8.889659724207721	0.135272255300000 E-5
0		54.598150033144336	54.598150033144336	0.0
0.16		53.527598071710131	53.527598071710131	0.0
0.32		52.498221185715707	52.498221185715728	0.210000000000000 E-13
0.48		51.507688710513520	51.507688710515389	0.186900000000000 E-11
0.64		50.553842623281788	50.553842623325217	0.434280000000000 E-10
0.80		49.634681848313029	49.634681848809393	0.496364000000000 E-10
0.96		48.748348243878873	48.748348247500907	0.362203400000000 E-9
1.12	0.5	47.893114064161701	47.893114083556327	0.193946260000000 E-8
1.28		47.067370718227878	47.067370801029668	0.828017900000000 E-7
1.44		46.269618672156220	46.269618969523698	0.297367478000000 E-6
1.60		45.498458360953613	45.498459292762043	0.931808430000000 E-6
1.76		44.752581994380606	44.752584609355864	0.261497525900000 E-5
1.92		44.030766155761562	44.030772855840219	0.670007865700000 E-5

10^{-15} when x closed from the beginning of the interval and 10^{-6} when x closed from the end of the interval, Whereas the interval that is approved by the genetic algorithm for the parameters is $[-100,\ 100]$ with fitness function equation (3.3), and the population size of (the resulting solutions) is 50.

5. Conclusion

In this paper, we are interested to solve the nonlinear Benney–Lin equation by using a modified Adomian decomposition method with a genetic algorithm. The fundamental thought of GA_ADM is to calculate the best parameters of nonlinear Benney–Lin equation (α,λ,β,η,δ) as it can be widely applied to solve other nonlinear equations. In comparison with ADM, our proposed method GA-ADM guarantees that the optimal parameters will be achieved precisely regardless of the complexity and multiple values of the equations. The proposed method gives more accurate results were in great agreement with the exact solutions. Finally, the calculations in this work have been done using the MAPLE 13 software package and Matlab® R2018a Language for genetic programs.

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