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Some trapezoid type inequalities for generalized fractional integral

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Abstract

In this paper, we have established some trapezoid type inequalities for generalized fractional integral.The results presented here would provide some fractional inequalities and Riemann-Liouville type fractional operators.

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2010 MSC: Primary 26D07, 26D10 ; Secondary 26D15, 26A33.

1. Introduction

Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex mapping defined on the interval I of real numbers and $a, b \in I$, with $a < b$. The following double inequality is well known in the literature as the Hermite-Hadamard inequality [2]:

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)\,dx \leq \frac{f(a)+f(b)}{2}. \tag{1.1}$$

The most well-known inequalities related to the integral mean of a convex function are the Hermite Hadamard inequalities. Over the years several papers have focused on the inequalities (1.1). For some of them, see (

In this section we summarize the generalized fractional integrals defined by Sarikaya and Ertuğral in [8].

Let's define a function $\varphi : [0, \infty) \to [0, \infty)$ satisfiying the following conditions :

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$$\int_0^1 \frac{\varphi(t)}{t} dt < \infty.$$

We define the following left-sided and right-sided generalized fractional integral operators, respectively, as follows:

$$_{a^+}I_\varphi f(x) = \int_a^x \frac{\varphi(x-t)}{x-t} f(t)dt, \quad x > a, \tag{1.2}$$

$$_{b^-}I_\varphi f(x) = \int_x^b \frac{\varphi(t-x)}{t-x} f(t)dt, \quad x < b. \tag{1.3}$$

The most important feature of generalized fractional integrals is that they generalize some types of fractional integrals such as Riemann-Liouville fractional integral, k-Riemann-Liouville fractional integral, Katugampola fractional integrals, conformable fractional integral, Hadamard fractional integrals, etc. These important special cases of the integral operators (1.2) and (1.3) are mentioned below.

i) If we take $\varphi(t) = t$, the operator (1.2) and (1.3) reduce to the Riemann integral as follows:

$$I_{a^+} f(x) = \int_a^x f(t)dt, \quad x > a,$$

$$I_{b^-} f(x) = \int_x^b f(t)dt, \quad x < b.$$

ii) If we take $\varphi(t) = \frac{t^\alpha}{\Gamma(\alpha)}$, the operator (1.2) and (1.3) reduce to the Riemann-Liouville fractional integral as follows:

$$I_{a^+}^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)dt, \quad x > a,$$

$$I_{b^-}^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} f(t)dt, \quad x < b.$$

iii) If we take $\varphi(t) = \frac{1}{k\Gamma_k(\alpha)} t^{\frac{\alpha}{k}}$, the operator (1.2) and (1.3) reduce to the k-Riemann-Liouville fractional integral as follows:

$$I_{a^+,k}^\alpha f(x) = \frac{1}{k\Gamma_k(\alpha)} \int_a^x (x-t)^{\frac{\alpha}{k}-1} f(t)dt, \quad x > a,$$

$$I_{b^-,k}^\alpha f(x) = \frac{1}{k\Gamma_k(\alpha)} \int_x^b (t-x)^{\frac{\alpha}{k}-1} f(t)dt, \quad x < b$$

where

$$\Gamma_k(\alpha) = \int_0^\infty t^{\alpha-1} e^{-\frac{t^k}{k}} dt, \quad \mathcal{R}(\alpha) > 0$$

and

$$\Gamma_k(\alpha) = k^{\frac{\alpha}{k}-1} \Gamma\left(\frac{\alpha}{k}\right), \quad \mathcal{R}(\alpha) > 0; k > 0$$

are given by Mubeen and Habibullah in [5].

2. Main Results

Throughout this study, for brevity, we define

$$\Delta(t)=\int_t^1 \frac{\varphi\left(\left(x-a\right)u\right)}{u}du<\infty,\ \nabla(t)=\int_t^1 \frac{\varphi\left(\left(b-x\right)u\right)}{u}du<\infty.$$

In this section, using generalized fractional integral operators, we begin by the following theorem:

Lemma 2.1. *Let $f: I \to \mathbb{R}$ be an absolutely continuous mapping on I° such that $f' \in L_1([a,b])$, where $a,b \in I^{\circ}$ with $a<b$. Then the following equality holds:*

$$\begin{aligned} &\frac{\nabla(0) f(b)+\Delta(0) f(a)}{b-a}-\frac{1}{b-a}\left[{}_{x^+}I_{\varphi}f(b)+{}_{x^-}I_{\varphi}f(a)\right] \qquad (2.1)\\ =\ &\frac{b-x}{b-a}\int_0^1 \nabla(t) f'(tx+(1-t)b)\,dt-\frac{x-a}{b-a}\int_0^1 \Delta(t) f'(tx+(1-t)a)\,dt. \end{aligned}$$

Proof . It suffices to note that

$$I = \frac{b-x}{b-a}\int_0^1 \nabla(t) f'(tx+(1-t)b)\,dt-\frac{x-a}{b-a}\int_0^1 \Delta(t) f'(tx+(1-t)a)\,dt.$$

Integrating by parts, we obtain

$$\begin{aligned} I_1 &= \frac{b-x}{b-a}\left[\nabla(t) f(tx+(1-t)b)\frac{1}{x-b}\Big|_0^1 \right.\\ &\left.+\frac{1}{x-b}\int_0^1 \frac{\varphi((b-x)t)}{t} f(tx+(1-t)b)\right]\\ &= \frac{b-x}{b-a}\left[\frac{\nabla(0) f(b)}{b-x}-\frac{1}{b-x}\int_x^b \frac{\varphi(b-s)}{b-s} f(s)\,ds\right]\\ &= \frac{\nabla(0) f(b)-{}_{x^+}I_{\varphi}f(b)}{b-a} \qquad (2.2) \end{aligned}$$

and similarly we get,

$$\begin{aligned} I_2 &= \frac{x-a}{b-a}\int_0^1 \Delta(t) f'(tx+(1-t)a)\,dt \qquad (2.3)\\ &= \frac{x-a}{b-a}\left[\frac{\Delta(0) f(a)}{x-a}-\frac{1}{x-a}\int_a^x \frac{\varphi(s-a)}{s-a} f(s)\,ds\right]\\ &= \frac{\Delta(0) f(a)-{}_{x^-}I_{\varphi}f(a)}{b-a}. \end{aligned}$$

By substracting equation (2.2) and (2.3), we have

$$I_1-I_2=\frac{\nabla(0) f(b)+\Delta(0) f(a)}{b-a}-\frac{1}{b-a}\left[{}_{x^+}I_{\varphi}f(b)+{}_{x^-}I_{\varphi}f(a)\right]$$

that is desired result. □

Remark 2.2. *In Lemma 2.1 if we take $\varphi(t)=t$, then Theorem 2.1 reduce to Lemma 1 in [4].*

Remark 2.3. *In Lemma 2.1, if we take $\varphi(t)=\frac{t^{\alpha}}{\Gamma(\alpha)}$, then Theorem 2.1 Lemma 1 in [3].*

Corollary 2.4. *In Lemma 2.1, if we take $\varphi(t)=\frac{t^{\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}$, then Theorem 2.1 we have the following inequality*

$$\begin{aligned} & \frac{(b-x)^{\frac{\alpha}{k}+1}}{(b-a)\,\Gamma_k(\alpha+k)}\int_0^1\left(1-t^{\frac{\alpha}{k}}\right)f'(tx+(1-t)\,b)\,dt \\ & -\frac{(x-a)^{\frac{\alpha}{k}+1}}{(b-a)\,\Gamma_k(\alpha+k)}\int_0^1\left(1-t^{\frac{\alpha}{k}}\right)f'(tx+(1-t)\,a)\,dt \\ = & \frac{(b-x)^{\frac{\alpha}{k}}f(b)+(x-a)^{\frac{\alpha}{k}}f(a)}{(b-a)\,\Gamma_k(\alpha+k)}-\frac{1}{b-a}\left[I^{\alpha}_{x^-,k}f(a)+I^{\alpha}_{x^+,k}f(b)\right]. \end{aligned}$$

Theorem 2.5. *Let $f:I=[a,b]\subset R\rightarrow R$ be an absolutely continuous mapping on I° such that $f'\in L_1([a,b])$, where $a,b\in I^{\circ}$ with $a<b$. If the mapping $|f'|$ is convex on $[a,b]$, then we have the following inequality*

$$\begin{aligned} & \left|\frac{\bigtriangledown(0)\,f(b)+\Delta(0)\,f(a)}{b-a}-\frac{1}{b-a}\left[{}_{x^+}I_{\varphi}f(b)+{}_{x^-}I_{\varphi}f(a)\right]\right| \qquad (2.4) \\ \leq & \frac{b-x}{b-a}\left|f'(x)\right|\int_0^1\left|\bigtriangledown(t)\right|tdt+\frac{x-a}{b-a}\left|f'(x)\right|\int_0^1\left|\Delta(t)\right|tdt \\ & +\frac{b-x}{b-a}\left|f'(b)\right|\int_0^1\left|\bigtriangledown(t)\right|(1-t)\,dt+\frac{x-a}{b-a}\left|f'(a)\right|\int_0^1\left|\Delta(t)\right|(1-t)\,dt. \end{aligned}$$

Proof . From Lemma 2.1 and $|f'|$ is convex on $[a,b]$,we get

$$\begin{aligned} & \left|\frac{\bigtriangledown(0)\,f(b)+\Delta(0)\,f(a)}{b-a}-\frac{1}{b-a}\left[{}_{x^+}I_{\varphi}f(b)+{}_{x^-}I_{\varphi}f(a)\right]\right| \\ \leq & \left|\frac{b-x}{b-a}\int_0^1\bigtriangledown(t)\,f'(tx+(1-t)\,b)\,dt-\frac{x-a}{b-a}\int_0^1\Delta(t)\,f'(tx+(1-t)\,a)\,dt\right| \\ \leq & \frac{b-x}{b-a}\int_0^1\left|\bigtriangledown(t)\right|\left|f'(tx+(1-t)\,b)\right|dt+\frac{x-a}{b-a}\int_0^1\left|\Delta(t)\right|\left|f'(tx+(1-t)\,a)\right|dt \\ \leq & \frac{b-x}{b-a}\int_0^1\left|\bigtriangledown(t)\right|\left(t\left|f'(x)\right|+(1-t)\left|f'(b)\right|\right)dt \\ & +\frac{x-a}{b-a}\int_0^1\left|\Delta(t)\right|\left(t\left|f'(x)\right|+(1-t)\left|f'(a)\right|\right)dt \end{aligned}$$

$$\leq \frac{b-x}{b-a}\left|f'(x)\right|\int\limits_0^1\left|\triangledown(t)\right|tdt+\frac{x-a}{b-a}\left|f'(x)\right|\int\limits_0^1\left|\Delta(t)\right|tdt$$
$$+\frac{b-x}{b-a}\left|f'(b)\right|\int\limits_0^1\left|\triangledown(t)\right|(1-t)\,dt+\frac{x-a}{b-a}\left|f'(a)\right|\int\limits_0^1\left|\Delta(t)\right|(1-t)\,dt.$$

This completes the proof. □

Remark 2.6. *Under assumption of Theorem 2.5 with $\varphi(t)=t$, then Theorem 2.5 reduce to Theorem 4 in [4].*

Corollary 2.7. *Under assumption of Theorem 2.5 with $\varphi(t)=\frac{t^\alpha}{\Gamma(\alpha)}$, we have the following inequality*

$$\left|\frac{f(b)(b-x)^\alpha+f(a)(x-a)^\alpha}{b-a}-\frac{\Gamma(\alpha+1)}{b-a}\left[J_{x^-}f(a)+J_{x^+}f(b)\right]\right|$$
$$\leq \frac{\alpha}{2(\alpha+2)}\left[\frac{(b-x)^{\alpha+1}+(x-a)^{\alpha+1}}{b-a}\right]\left|f'(x)\right|$$
$$+\frac{\alpha^2+3\alpha}{2(\alpha+2)}\left[\frac{(b-x)^{\alpha+1}\left|f'(b)\right|+(x-a)^{\alpha+1}\left|f'(a)\right|}{b-a}\right].$$

Corollary 2.8. *Under assumption of Theorem 2.5 with $\varphi(t)=\frac{t^{\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}$, we have the following inequality*

$$\frac{(b-x)^{\frac{\alpha}{k}+1}}{(b-a)\Gamma_k(\alpha+k)}\int\limits_0^1\left(1-t^{\frac{\alpha}{k}}\right)f'(tx+(1-t)b)\,dt$$
$$-\frac{(x-a)^{\frac{\alpha}{k}+1}}{(b-a)\Gamma_k(\alpha+k)}\int\limits_0^1\left(1-t^{\frac{\alpha}{k}}\right)f'(tx+(1-t)a)\,dt$$
$$\leq \frac{(b-x)^{\frac{\alpha}{k}+1}+(x-a)^{\frac{\alpha}{k}+1}}{(b-a)\Gamma_k(\alpha+k)}\left(\frac{\alpha}{\alpha+2k}\right)\left|f'(x)\right|$$
$$+\frac{1}{(b-a)\Gamma_k(\alpha+k)}\left(\frac{1}{2}-\frac{k}{\alpha+k}+\frac{k}{\alpha+2k}\right)$$
$$\times\left[(b-x)^{\frac{\alpha}{k}+1}\left|f'(b)\right|+(x-a)^{\frac{\alpha}{k}+1}\left|f'(a)\right|\right].$$

Theorem 2.9. *Let $f:I=[a,b]\subset R\to R$ be an absolutely continuous mapping on I° such that $f'\in L_1([a,b])$, where $a,b\in I^\circ$ with $a<b$. If the mapping $|f'|^q$, $q>1$, is convex on $[a,b]$, then we have the following inequality*

$$\left|\frac{\triangledown(0)f(b)+\Delta(0)f(a)}{b-a}-\frac{1}{b-a}\left[{}_{x^+}I_\varphi f(b)+{}_{x^-}I_\varphi f(a)\right]\right|$$
$$\leq \frac{b-x}{b-a}\left(\int_0^1\left|\triangledown(t)\right|^p dt\right)^{\frac{1}{p}}\left(\frac{\left|f'(x)\right|^q+\left|f'(b)\right|^q}{2}\right)^{\frac{1}{q}}$$
$$+\frac{x-a}{b-a}\left(\int_0^1\left|\Delta(t)\right|^p dt\right)^{\frac{1}{p}}\left(\frac{\left|f'(x)\right|^q+\left|f'(a)\right|^q}{2}\right)^{\frac{1}{q}}$$

where $\frac{1}{p}+\frac{1}{q}=1$.

Proof . From Lemma 2.1 and by Hölder's inequality, we get

$$\begin{aligned}
&\left|\frac{\nabla(0) f(b)+\Delta(0) f(a)}{b-a}-\frac{1}{b-a}\left[{}_{x^{+}}I_{\varphi} f(b)+\ {}_{x^{-}}I_{\varphi} f(a)\right]\right| \\
\leq\ & \frac{b-x}{b-a}\left(\int_0^1|\nabla(t)|^p dt\right)^{\frac{1}{p}}\left(\int_0^1\left|f'(tx+(1-t)b)\right|^q dt\right)^{\frac{1}{q}} \\
&+\frac{x-a}{b-a}\left(\int_0^1|\Delta(t)|^p dt\right)^{\frac{1}{q}}\left(\int_0^1\left|f'(tx+(1-t)a)\right|^q dt\right)^{\frac{1}{q}} \\
\leq\ & \frac{b-x}{b-a}\left(\int_0^1|\nabla(t)|^p dt\right)^{\frac{1}{p}}\left(\int_0^1\left(t\left|f'(x)\right|^q+(1-t)\left|f'(b)\right|^q\right) dt\right)^{\frac{1}{q}} \\
&+\frac{x-a}{b-a}\left(\int_0^1|\Delta(t)|^p\right)^{\frac{1}{p}}\left(\int_0^1\left(t\left|f'(x)\right|^q+(1-t)\left|f'(a)\right|^q\right) dt\right)^{\frac{1}{q}} \\
\leq\ & \frac{b-x}{b-a}\left(\int_0^1|\nabla(t)|^p dt\right)^{\frac{1}{p}}\left(\frac{\left|f'(x)\right|^q+\left|f'(b)\right|^q}{2}\right)^{\frac{1}{q}} \\
&+\frac{x-a}{b-a}\left(\int_0^1|\Delta(t)|^p dt\right)^{\frac{1}{p}}\left(\frac{\left|f'(x)\right|^q+\left|f'(a)\right|^q}{2}\right)^{\frac{1}{q}}.
\end{aligned}$$

□

Remark 2.10. *Under assumption of Theorem 2.9 with* $\varphi(t)=t$, *then Theorem 2.9 reduce to Theorem 5 in [4].*

Corollary 2.11. *Under assumption of Theorem 2.9 with* $\varphi(t)=\frac{t^{\alpha}}{\Gamma(\alpha)}$, *we have the following inequality*

$$\begin{aligned}
&\left|\frac{f(b)(b-x)^{\alpha}+f(a)(x-a)^{\alpha}}{b-a}-\frac{\Gamma(\alpha+1)}{b-a}\left[J_{x^{-}} f(a)+J_{x^{+}} f(b)\right]\right| \\
\leq\ & \left(\frac{\alpha+2}{\alpha+1}\right)\frac{(b-x)^{\alpha+1}}{(b-a)}\left(\frac{\left|f'(x)\right|^q+\left|f'(b)\right|^q}{2}\right)^{\frac{1}{q}} \\
&+\left(\frac{\alpha+2}{\alpha+1}\right)\frac{(x-a)^{\alpha+1}}{(b-a)}\left(\frac{\left|f'(x)\right|^q+\left|f'(a)\right|^q}{2}\right)^{\frac{1}{q}}.
\end{aligned}$$

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