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Blow up of solutions for a $r(x)-$Laplacian Lamé equation with variable-exponent nonlinearities and arbitrary initial energy level

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Abstract

In this paper, we consider the nonlinear $r(x)-$Laplacian Lamé equation

$$u_{tt} - \Delta_e u - div\big(|\nabla u|^{r(x)-2}\nabla u\big) + |u_t|^{m(x)-2}u_t = |u|^{p(x)-2}u$$

in a smoothly bounded domain $\Omega \subseteq R^n$, $n \geq 1$, where $r(.)$, $m(.)$ and $p(.)$ are continuous and measurable functions. Under suitable conditions on variable exponents and initial data, the blow-up of solutions is proved with negative initial energy as well as positive.

Keywords: blow-up, variable-exponent nonlinearities, elasticity operator, arbitrary initial energy
2010 MSC: Primary 35B44; Secondary 35L70, 74B20.

1. Introduction and preliminaries

In this paper, we consider the following initial boundary problem:

$$u_{tt} - \Delta_e u - div\big(|\nabla u|^{r(x)-2}\nabla u\big) + |u_t|^{m(x)-2}u_t = |u|^{p(x)-2}u, \ x \in \Omega \quad t > 0 \tag{1.1}$$

$$u(x,t) = \frac{\partial u}{\partial \nu}(x,t) = 0, \quad x \in \partial\Omega \quad t > 0 \tag{1.2}$$

$$u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \quad x \in \Omega, \tag{1.3}$$

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where $\Omega \subset R^n (n \geq 1)$ is a bounded domain. Δ_e denotes the elasticity operator, which is the differential operator defined by

$$\Delta_e u = \mu \Delta u + (\lambda + \mu) \nabla (div\ u),$$

λ and μ are the Lamé constants which satisfy the following conditions

$$\mu > 0, \quad \lambda + \mu \geq 0.$$

Here, $r(.)$, $m(.)$ and $p(.)$ are given measurable functions on $\overline{\Omega}$ such that:

$$\begin{gathered} 2 < r_1 \leq r(x) \leq r_2 \\ 2 < m_1 \leq m(x) \leq m_2 \\ 2 < p_1 \leq p(x) \leq p_2 \end{gathered} \tag{1.4}$$

with

$$\begin{gathered} r_1 := essinf_{x \in \overline{\Omega}} r(x), \ r_2 := esssup_{x \in \overline{\Omega}} r(x), \\ m_1 := essinf_{x \in \overline{\Omega}} m(x), \ m_2 := esssup_{x \in \overline{\Omega}} m(x), \\ p_1 := essinf_{x \in \overline{\Omega}} p(x), \ \ p_2 := esssup_{x \in \overline{\Omega}} p(x). \end{gathered}$$

Study of interaction between source terms and damping terms has attracted the attention of many mathematicians in the case of constant exponents. But less is know about the case of variable exponents. Modeling of some physical phenomena such as flows of electro-rheological fluids, filtration processes in porous media, and image processing produces equation with variable exponents.
In the case of constant exponent, the following equation

$$u_{tt} - \Delta u + g(u_t) = f(u) \tag{1.5}$$

in a bounded domain Ω, with smooth boundary, was considered at a large scale. When $g(u_t) = |u_t|^{m-2} u_t$ and $f(u) = |u|^{p-2} u$, firstly, Levine [13] discussed (5) for $m = 2$ and established blow-up result for solutions with negative initial energy. For more information about the equation (1.5), we refer the reader to [3, 12, 15, 19, 20, 22].
In case of variable exponent, Messaoudi and Talahmeh [16] investigated problem (1.1)-(1.3) without elasticity term i.e. $\lambda = \mu = 0$. They proved the blow-up of solutions under sufficient conditions on m, p, r and the initial data.
Shahrouzi [21] studied the solution behaviour of the following viscoelastic equation with variable-exponent nonlinearities

$$u_{tt} - \Delta u - div(|\nabla u|^{m(x)} \nabla u) + \int_0^t g(t-\tau) \Delta u(\tau) d\tau + h(x, t, u, \nabla u) + \beta u_t = |u|^{p(x)} u,$$

and proved general decay of solutions for appropriate initial data and when $h(x, t, u, \nabla u) \equiv 0$. Also, the blow up of solutions has been proved with positive initial energy and suitable conditions on datas when $\beta = 0$. For more results regarding this matter, we refer the reader to the review paper [17].
On the other hand, equations with elasticity operator has attracted considerable attention in recent years, where diverse types of dissipative mechanisms have been introduced and several stability and boundedness results have been obtained. Bchatnia and Guesmia [5] considered the elasticity system in 3-dimension bounded domain with infinite memories and proved that system is well-possed and

stable. Moreover, they established solutions converge to zero at infinity in terms of the growth of the infinite memories. Li and Bao [14] investigated a memory-type elasticity problem and obtained global existence and the general energy decay of solutions by using perturbed energy method. (see also [4, 6])
Recently, Antontsev et al. [2] considered the following nonlinear plate (or beam) Petrovsky equation with strong damping and source terms with variable exponents:

$$u_{tt} + \Delta^2 - \Delta u_t + |u_t|^{p(x)-2}u_t = |u|^{q(x)-2}u.$$

By using the Banach contraction mapping principle they obtained local weak solutions, under suitable assumptions on the variable exponents $p(.)$ and $q(.)$. Also, they proved that the solution is global if $p(.) \geq q(.)$ and if $p(.) < q(.)$ then there exists a solution with negative initial energy that blows up in finite time.(see also [18])

Motivated by the aforementioned works, in this paper, we try to extend the results of [16] to an elasticity equation. Indeed, by using different method we provide the sufficient conditions on variable exponents and initial data for the blow up of solutions of the problem (1.1)-(1.3) with negative initial energy as well as positive.

Throughout this paper we recall some notations and functionals. We denote by $\|.\|_q$ the L^q-norm over Ω . In particular, the L^2-norm is denoted $\|.\|$ in Ω and $\|.\|_{\Gamma_i}$ in Γ_i. Also $(.,.)$ denotes the usual L^2-inner product. In order to study problem (1.1)-(1.3), we need some theories about Lebesgue and Sobolev spaces with variable-exponents (for detailed, see [7, 8, 9, 10, 11]). Let $p(x) \geq 1$ and measurable, we assume that

$$C_+(\overline{\Omega}) = \{h | h \in C(\overline{\Omega}),\ h(x) > 1 \ for \ any \ x \in \overline{\Omega}\},$$

$$h^+ = \max_{\overline{\Omega}} h(x), \quad h^- = \min_{\overline{\Omega}} h(x) \quad for \ any \ h \in C(\overline{\Omega}),$$

$$L^{p(x)}(\Omega) = \Big\{ u |\ u \ is \ a \ measurable \ real-valued \ function, \ \int_\Omega |u(x)|^{p(x)} dx < \infty \Big\}.$$

We equip the Lebesgue space with a variable exponent, $L^{p(x)}(\Omega)$, with the following Luxembourg-type norm

$$\|u\|_{p(x)} := \inf \Big\{ \lambda > 0 \Big| \int_\Omega |\frac{u(x)}{\lambda}|^{p(x)} dx \leq 1 \Big\}.$$

The variable-exponent Lebesgue Sobolev space $W^{1,p(x)}(\Omega)$ is defined by

$$W^{1,p(x)}(\Omega) = \{u \in L^{p(x)}(\Omega) \ such \ that \ \nabla u \ exists \ and \ |\nabla u| \in L^{p(x)}(\Omega)\}.$$

This space is a Banach space with respect to the norm $\|u\|_{W^{1,p(x)}(\Omega)} = \|u\|_{p(x)} + \|\nabla u\|_{p(x)}$. Furthermore, let $W_0^{1,p(x)}(\Omega)$ be the closure of $C_0^\infty(\Omega)$ in $W^{1,p(x)}(\Omega)$. The dual of $W_0^{1,p(x)}(\Omega)$ is defined as $W^{-1,p'(x)}(\Omega)$, by the same way as the usual Sobolev spaces, where $\frac{1}{p(x)} + \frac{1}{p'(x)} = 1$.
If we define

$$p^*(x) = \begin{cases} essinf_{x \in \overline{\Omega}} \frac{Np(x)}{(N-p(x))}, & p^+ < N \\ \infty, & p^+ \geq N, \end{cases}$$

then we have

Lemma 1.1. *[7, 11] Let Ω be a bounded domain in R^n then for any measurable bounded exponent $p(x)$ we have*

(i) $W^{1,p(x)}(\Omega)$ *and* $W_0^{1,p(x)}(\Omega)$ *are separable Banach spaces;*
(ii) if $q \in C_+(\overline{\Omega})$ *and* $q(x) < p^*(x)$ *for any* $x \in \overline{\Omega}$, *then the imbedding* $W^{1,p(x)}(\Omega) \hookrightarrow L^{q(x)}(\Omega)$ *is compact and continuous;*
(iii) if $p(x)$ *is uniformly continuous in* Ω *then there exists a constant* $C > 0$, *such that*

$$\|u\|_{p(x)} \leq C\|\nabla u\|_{p(x)} \qquad \forall u \in W_0^{1,p(x)}(\Omega).$$

By (iii) of Lemma 1.1, we know that the space $W_0^{1,p(x)}(\Omega)$ has an equivalent norm given by $\|u\|_{W^{1,p(x)}(\Omega)} = \|\nabla u\|_{p(x)}$.
At this point, we state the local existence of solutions for the problem (1.1)-(1.3), that can be established employing the Galerkin method as in [1].

Theorem 1.2. *(Local existence) Let* $u_0 \in W_0^{1,r(.)}(\Omega)$, $u_1 \in L^2(\Omega)$ *and assume that the exponents* $r(.)$, $m(.)$ *and* $p(.)$ *satisfy conditions (1.4) then problem (1.1)-(1.3) has a unique weak solution such that*

$$u \in L^\infty\Big((0,T), W_0^{1,r(.)}(\Omega) \cap H^2(\Omega)\Big),\ u_1 \in L^\infty\Big((0,T), L^2(\Omega)\Big),$$

$$u_{tt} \in L^\infty\Big((0,T), W_0^{-1,r'(.)}(\Omega)\Big),$$

where $\frac{1}{r(.)} + \frac{1}{r'(.)} = 1$.

The energy function related with problem (1.1)-(1.3) is given by

$$E(t) = \frac{1}{2}(\|u_t\|^2 + \mu\|\nabla u\|^2 + (\lambda+\mu)\int_\Omega |div\ u|^2 dx) + \int_\Omega \frac{|\nabla u|^{r(x)}}{r(x)}dx - \int_\Omega \frac{|u|^{p(x)}}{p(x)}dx, \tag{1.6}$$

Lemma 1.3. *(Monotonicity of energy) Let* u *be a local solution of (1.1)-(1.3) and satisfy the conditions of Theorem 1.2. Then the energy functional along the solution satisfies*

$$E'(t) = -\int_\Omega |u_t|^{m(x)} dx \leq 0. \tag{1.7}$$

Proof . By multiplying equation (1.1) by u_t and integrating over Ω, using integration by parts, we obtain (1.7) for any regular solution. This equality remains valid for weak solutions by a simple density argument. □

2. Blow up result with negative initial energy level

In this section we prove the blow up of solutions with negative initial energy and suitable conditions on variable exponents. This result reads as follows:

Theorem 2.1. *Let the assumption of Theorem 1.2 be satisfied and assume that* μ *sufficiently large and*

$$r_2 \leq m_2 \leq \min\{p_1, \frac{K\varepsilon}{K\varepsilon + \sigma - 1}\}, \tag{2.1}$$

where σ, K *and* ε *are constants such that* $\frac{K\varepsilon}{2} < 1 - \sigma < 1$. *Then for* $E(0) < 0$, *the solution of problem (1.1)-(1.3) blows up in a finite time.*

Proof . To prove the blow-up result, let $H(t) = -E(t)$:

$$H(t) = \int_\Omega \frac{|u|^{p(x)}}{p(x)}dx - \frac{1}{2}(\|u_t\|^2 + \mu\|\nabla u\|^2 + (\lambda+\mu)\int_\Omega |div\ u|^2 dx) - \int_\Omega \frac{|\nabla u|^{r(x)}}{r(x)}dx, \tag{2.2}$$

therefore we have for negative initial energy, $H(t) \geq 0$ and

$$H(t) \leq \int_\Omega \frac{|u|^{p(x)}}{p(x)}dx \leq \frac{1}{p_1}\int_\Omega |u|^{p(x)}dx. \tag{2.3}$$

Now, define for sufficiently small $\varepsilon > 0$ and $\sigma < 1$

$$\psi(t) = H^{1-\sigma}(t) + \varepsilon \int_\Omega u u_t dx, \tag{2.4}$$

by differentiation we get

$$\begin{aligned}\psi'(t) &= (1-\sigma)H^{-\sigma}(t)H'(t) + \varepsilon\|u_t\|^2 + \varepsilon\int_\Omega u u_{tt} dt \\ &= (1-\sigma)H^{-\sigma}(t)H'(t) + \varepsilon\|u_t\|^2 - \varepsilon\mu\|\nabla u\|^2 - \varepsilon(\mu+\lambda)\int_\Omega |div\ u|^2 dx \\ &\quad - \varepsilon\int_\Omega |\nabla u|^{r(x)}dx - \varepsilon\int_\Omega u|u_t|^{m(x)-2}u_t dx + \varepsilon\int_\Omega |u|^{p(x)}dx. \end{aligned} \tag{2.5}$$

At this point, we derive from (2.2) that

$$\begin{aligned}\psi'(t) &= (1-\sigma)H^{-\sigma}(t)H'(t) + \varepsilon m_2 H(t) + \varepsilon\mu(\frac{m_2}{2}-1)\|\nabla u\|^2 + \varepsilon(1+\frac{m_2}{2})\|u_t\|^2 \\ &+\varepsilon(\mu+\lambda)(\frac{m_2}{2}-1)\int_\Omega |div\ u|^2 dx + \varepsilon m_2\int_\Omega \frac{|\nabla u|^{r(x)}}{r(x)}dx - \varepsilon m_2\int_\Omega \frac{|u|^{p(x)}}{p(x)}dx \\ &- \varepsilon\int_\Omega |\nabla u|^{r(x)}dx - \varepsilon\int_\Omega u|u_t|^{m(x)-2}u_t dx + \varepsilon\int_\Omega |u|^{p(x)}dx. \end{aligned} \tag{2.6}$$

Using the conditions of functions $r(.)$ and $p(.)$, deduce from (2.6)

$$\begin{aligned}\psi'(t) &\geq (1-\sigma)H^{-\sigma}(t)H'(t) + \varepsilon m_2 H(t) + \varepsilon\mu(\frac{m_2}{2}-1)\|\nabla u\|^2 + \varepsilon(1+\frac{m_2}{2})\|u_t\|^2 \\ &+\varepsilon(\mu+\lambda)(\frac{m_2}{2}-1)\int_\Omega |div\ u|^2 dx + \varepsilon(\frac{m_2}{r_2}-1)\int_\Omega |\nabla u|^{r(x)}dx \\ &+ \varepsilon(1-\frac{m_2}{p_1})\int_\Omega |u|^{p(x)}dx - \varepsilon\int_\Omega u|u_t|^{m(x)-2}u_t dx. \end{aligned} \tag{2.7}$$

On the other hand, by using the Young's inequality we have for any $\delta > 0$

$$\begin{aligned}\int_\Omega u|u_t|^{m(x)-1}dx &\leq \int_\Omega \frac{\delta^{m(x)}}{m(x)}|u|^{m(x)}dx + \int_\Omega \frac{m(x)-1}{m(x)}\delta^{-\frac{m(x)}{m(x)-1}}|u_t|^{m(x)}dx \\ &\leq \frac{\delta^{m_1}}{m_1}\int_\Omega |u|^{m(x)}dx + (\frac{m_2}{m_2-1})\delta^{-\frac{m_2}{m_2-1}}\int_\Omega |u_t|^{m(x)}dx. \end{aligned} \tag{2.8}$$

Let c_* be the best constant of embedding $H_0^1 \hookrightarrow L^{m(.)}(\Omega)$, then we have

$$\int_\Omega |u|^{m(x)}dx \leq \max\{\|u\|_{m(x)}^{m_1}, \|u\|_{m(x)}^{m_2}\} \leq \max\{c_*^{m_1}\|\nabla u\|^{m_1}, c_*^{m_2}\|\nabla u\|^{m_2}\}$$

$$\leq \max\{c_*^{m_1}\|\nabla u\|^{m_1-2}, c_*^{m_2}\|\nabla u\|^{m_2-2}\}\|\nabla u\|^2 \leq \overline{C}\|\nabla u\|^2. \tag{2.9}$$

Combining (2.8) with (2.9), we get ($\delta = 1$)

$$\int_\Omega u|u_t|^{m(x)-1}dx \leq \frac{\overline{C}}{m_1}\|\nabla u\|^2 + (\frac{m_2}{m_2-1})\int_\Omega |u_t|^{m(x)}dx. \tag{2.10}$$

By applying (2.10) into (2.7), we easily get

$$\psi'(t) \geq (1-\sigma)H^{-\sigma}(t)H'(t) + \varepsilon m_2 H(t) + \varepsilon \underbrace{\left[\mu(\frac{m_2}{2}-1) - \frac{\overline{C}}{m_1}\right]}_{I_1}\|\nabla u\|^2$$

$$+\varepsilon \underbrace{(1+\frac{m_2}{2})}_{I_2}\|u_t\|^2 + \varepsilon(\mu+\lambda)\underbrace{(\frac{m_2}{2}-1)}_{I_3}\int_\Omega |div\ u|^2dx + \varepsilon\underbrace{(\frac{m_2}{r_2}-1)}_{I_4}\int_\Omega |\nabla u|^{r(x)}dx$$

$$+\varepsilon\underbrace{(1-\frac{m_2}{p_1})}_{I_5}\int_\Omega |u|^{p(x)}dx - \varepsilon(\frac{m_2}{m_2-1})\int_\Omega |u_t|^{m(x)}dx. \tag{2.11}$$

Thanks to (1.7) and definition of $H(t)$, we have for appropriate constant K ($K < \frac{2}{\varepsilon}$)

$$-\int_\Omega |u_t|^{m(x)}dx = E'(t) = -H'(t) \geq -KH^{-\sigma}(t)H'(t),$$

substituting this inequality into (2.11) to obtain

$$\psi'(t) \geq \left[(1-\sigma) - \varepsilon K(\frac{m_2}{m_2-1})\right]H^{-\sigma}(t)H'(t) + \varepsilon\beta\big[H(t) + \|u_t\|^2 + \|\nabla u\|^2$$

$$+(\lambda+\mu)\int_\Omega |div\ u|^2dx + \int_\Omega |\nabla u|^{r(x)}dx + \int_\Omega |u|^{p(x)}dx\big]$$

$$\geq \varepsilon\beta\big[H(t) + \|u_t\|^2 + \|\nabla u\|^2 + (\lambda+\mu)\int_\Omega |div\ u|^2dx$$

$$+\int_\Omega |\nabla u|^{r(x)}dx + \int_\Omega |u|^{p(x)}dx\big], \tag{2.12}$$

where (2.1) has been used and

$$\beta = \min\{I_1,\ I_2,\ I_3,\ I_4,\ I_5\}.$$

Finally, inequality (2.12) implies that $\psi(t) \geq \psi(0) > 0\ \ \forall t \geq 0$.
Suppose that C is a generic constant, by using the Hölder and Young inequalities, we have

$$|\int_\Omega uu_tdx|^{\frac{1}{1-\sigma}} \leq C(\|u\|_{p_1}^{\frac{1}{1-\sigma}}\|u_t\|^{\frac{1}{1-\sigma}}) \leq C(\|u\|_{p_1}^{p_1} + \|u_t\|^2 + H(t)). \tag{2.13}$$

Combining (2.13) with (2.4), we obtain for some $\xi > 0$

$$\psi^{\frac{1}{1-\sigma}}(t) = \left[H^{1-\sigma}(t) + \varepsilon \int_\Omega uu_t dx\right]^{\frac{1}{1-\sigma}} \leq 2^{\frac{1}{1-\sigma}}\left(H(t) + \varepsilon^{\frac{1}{1-\sigma}} | \int_\Omega uu_t dx|^{\frac{1}{1-\sigma}}\right)$$

$$\leq C(\|u\|_{p_1}^{p_1} + \|u_t\|^2 + H(t)) \leq \xi^{-1}\psi'(t),$$

therefore

$$\psi'(t) \geq \xi\psi^{\frac{1}{1-\sigma}}(t). \tag{2.14}$$

Integrating (2.14) from 0 to t, we deduce

$$\psi^{\frac{\sigma}{1-\sigma}}(t) \geq \frac{1}{\psi^{-\frac{\sigma}{1-\sigma}}(0) - \frac{\xi\sigma t}{1-\sigma}}.$$

This shows that solutions blow up in finite time $t^* = \frac{1-\sigma}{\xi\sigma\psi^{\frac{\sigma}{1-\sigma}}(0)}$, and proof of Theorem 2.1 has been completed. □

3. Blow up result with positive initial energy level

In this section we shall prove that the solutions of (1.1)-(1.3) blow up in a finite time when variable exponents satisfy appropriate conditions and initial energy is positive. To prove this result, we assumed that:
(B1)

$$2(1 + \frac{\overline{C}}{\mu m_1}) < r_2 \leq p_1, \qquad \frac{m_2}{m_2 - 1} \leq 2\sqrt{\frac{r_2 + 2}{2r_2^2 C}[\mu(\frac{r_2 - 2}{2}) - \frac{\overline{C}}{m_1}]}.$$

Our main result in this section reads in the following theorem:

Theorem 3.1. *Suppose that the assumptions of Theorem 1.2 and (B1) hold. Moreover,* $E(0) > 0$ *(maybe large enough) is a given initial energy level. If we choose initial data* u_0, u_1 *satisfying*

$$\frac{m_2}{m_2 - 1} \int_\Omega u_0 u_1 dx > E(0), \tag{3.1}$$

then the solution of (1.1)-(1.3) blows up in finite time, i.e., there exists $T^* < +\infty$ *such that*

$$\lim_{t \to T^*} E(t) = +\infty.$$

Proof . Let define $A(t) = \int_\Omega uu_t dx$, then by using equation (1.3) we have

$$A'(t) = \|u_t\|^2 - \mu\|\nabla u\|^2 - (\lambda + \mu)\int_\Omega |div\ u|^2 dx - \int_\Omega |\nabla u|^{r(x)} dx + \int_\Omega |u|^{p(x)} dx - \int_\Omega u|u_t|^{m(x)-2} u_t dx.$$

By using the definition of $E(t)$, we obtain

$$\begin{aligned} A'(t) \geq\ & -r_2 E(t) + \frac{r_2 + 2}{2}\|u_t\|^2 + \frac{\mu}{2}(r_2 - 2)\|\nabla u\|^2 + \frac{(\lambda + \mu)}{2}(r_2 - 2)\int_\Omega |div\ u|^2 dx \\ & + \frac{p_1 - r_2}{p_1}\int_\Omega |u|^{p(x)} dx - \int_\Omega u|u_t|^{m(x)-2} u_t dx. \end{aligned} \tag{3.2}$$

Utilizing inequality (2.10) and (1.7), the last term of above inequality can be estimated as:

$$\int_\Omega u|u_t|^{m(x)-1}dx \leq \frac{\overline{C}}{m_1}\|\nabla u\|^2 - \frac{m_2}{m_2-1}E'(t). \tag{3.3}$$

Therefore by combining (3.2) with (3.3), we deduce

$$\begin{aligned} A'(t) \;\; &\geq \;\; -r_2E(t) + \frac{r_2+2}{2}\|u_t\|^2 + [\frac{\mu}{2}(r_2-2) - \frac{\overline{C}}{m_1}]\|\nabla u\|^2 \\ &\quad + \frac{(\lambda+\mu)}{2}(r_2-2)\int_\Omega |div\ u|^2dx + \frac{p_1-r_2}{p_1}\int_\Omega |u|^{p(x)}dx + \frac{m_2}{m_2-1}E'(t). \end{aligned}$$

Since $r_2 \geq 2 + \frac{2\overline{C}}{\mu m_1}$, we get

$$\begin{aligned} \frac{d}{dt}(A(t) - \frac{m_2}{m_2-1}E(t)) \;\; &\geq \;\; -r_2E(t) + \frac{r_2+2}{2}\|u_t\|^2 + \frac{1}{C}[\frac{\mu}{2}(r_2-2) - \frac{\overline{C}}{m_1}]\|u\|^2 \\ &\quad + \frac{(\lambda+\mu)}{2}(r_2-2)\int_\Omega |div\ u|^2dx + \frac{p_1-r_2}{p_1}\int_\Omega |u|^{p(x)}dx, \end{aligned} \tag{3.4}$$

where the part (iii) of Lemma 1.1 has been used.
Thanks to (B1), since $2 < r_2 \leq p_1$ we get from the last inequality

$$\frac{d}{dt}(A(t) - \frac{m_2}{m_2-1}E(t)) \geq -r_2E(t) + \frac{r_2+2}{2}\|u_t\|^2 + \frac{1}{C}[\frac{\mu}{2}(r_2-2) - \frac{\overline{C}}{m_1}]\|u\|^2. \tag{3.5}$$

Using Cauchy inequality, we have

$$\frac{r_2+2}{2}\|u_t\|^2 + \frac{1}{C}[\frac{\mu}{2}(r_2-2) - \frac{\overline{C}}{m_1}]\|u\|^2 \geq 2\sqrt{\frac{r_2+2}{2C}[\mu(\frac{r_2-2}{2}) - \frac{\overline{C}}{m_1}]A(t)}.$$

Therefore we get

$$\begin{aligned} \frac{d}{dt}(A(t) - \frac{m_2}{m_2-1}E(t)) \;\; &\geq \;\; -r_2E(t) + 2\sqrt{\frac{r_2+2}{2C}[\mu(\frac{r_2-2}{2}) - \frac{\overline{C}}{m_1}]A(t)} \\ &\geq \;\; \frac{r_2m_2}{m_2-1}(A(t) - \frac{m_2-1}{m_2}E(t)), \end{aligned} \tag{3.6}$$

where the condition (B1) has been used.
Now, define

$$H(t) = A(t) - \frac{m_2}{m_2-1}E(t), \tag{3.7}$$

where by hypotheses of Theorem 3.1 we have $H(0) \geq 0$. Thus inequality (3.6) yields

$$H'(t) \geq \frac{r_2m_2}{m_2-1}H(t),$$

and integration over t, we deduce

$$H(t) \geq e^{\frac{r_2m_2}{m_2-1}t}H(0), \qquad \forall t > 0. \tag{3.8}$$

It is easy to see that

$$e^{\frac{r_2 m_2}{m_2-1}t}H(0) \leq H(t) = A(t) - \frac{m_2}{m_2-1}E(t) \leq A(t) = \int_\Omega u u_t dx. \tag{3.9}$$

By using Hölder inequality, we have

$$\int_\Omega u u_t dx \leq \frac{C}{2}\|\nabla u\|^2 + \frac{1}{2}\|u_t\|^2. \tag{3.10}$$

Utilizing (3.10) into (3.9), we deduce

$$e^{\frac{r_2 m_2}{m_2-1}t}H(0) \leq H(t) \leq \frac{C}{2}\|\nabla u\|^2 + \frac{1}{2}\|u_t\|^2. \tag{3.11}$$

Also, by using Lemma 1.3 since u is global, we have for a positive constant C, we have $\|\nabla u\|^2 \leq C$. Thus inequality (3.11) shows that $\|u_t\|$ grows exponentially.
On the other hand, Monotonicity of energy yields

$$E(0) = E(t) - \int_0^t \int_\Omega |u_t(s)|^{m(x)} dx ds,$$

thus by assumption of Theorem 3.1 that $0 < E(t) \leq E(0)$, therefore we obtain

$$\int_0^t \int_\Omega |u_t(s)|^{m(x)} dx ds \leq E(0). \tag{3.12}$$

Finally, since $2 < m(x)$, we have

$$\int_0^t \|u_t(s)\|^2 ds \leq C \int_0^t \int_\Omega |u_t(s)|^{m(x)} dx ds \leq CE(0),$$

which contradicts the previous result that $\|u_t\|$ is exponentially growing. Therefore there exists a finite time T^* such that solutions of problem (1.1)-(1.3) blow up and proof of Theorem 3.1 is completed. □

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