Abstract

Using the fixed point method, we prove the Hyers-Ulam stability and the superstability of $n$-Jordan $*$-derivations in Fréchet locally $C^*$-algebras for the following generalized Jensen-type functional equation

$$f\left(\frac{a + b}{2}\right) + f\left(\frac{a - b}{2}\right) = f(a).$$

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1. Introduction and preliminaries

In this paper, assume that $n$ is an integer greater than 1.

Definition 1.1. Let $n \in \mathbb{N} - \{1\}$ and let $A$ be a ring and $B$ be an $A$-module. An additive map $D : A \to B$ is called $n$-Jordan derivation if

$$D(a^n) = D(a)a^{n-1} + aD(a)a^{n-2} + \ldots + a^{n-2}D(a)a + a^{n-1}D(a),$$

for all $a \in A$.

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The concept of $n$-jordan derivations was studied by Eshaghi Ghordji. (see also [7, 8, 13]).

**Definition 1.2.** Let $A$, $B$ be $C^*$-algebras. A $\mathbb{C}$-linear mapping $D : A \to B$ is called $n$-Jordan $\ast$-derivation if

$$D(a^n) = D(a)a^{n-1} + aD(a)a^{n-2} + \ldots + a^{n-2}D(a)a + a^{n-1}D(a),$$

$$D(a^\ast) = D(a)^\ast$$

for all $a \in A$.

We say functional equation $(\xi)$ is stable if any function $g$ satisfying the equation $(\xi)$ approximately is near to the true solution of $(\xi)$. We say that a functional equation is superstabile if every approximate solution is an exact solution of it.

The stability of functional equations was first introduced by Ulam [28] in 1940. More precisely, he proposed the following problem: Given a group $G_1$, a metric group $(G_2, d)$ and $\epsilon > 0$, does there exist a $\delta > 0$ such that if a mapping $f : G_1 \to G_2$ satisfies the inequality $d(f(ab), f(a)f(b)) < \delta$ for all $a, b \in G_1$, then there exists a homomorphism $T : G_1 \to G_2$ such that $d(f(a), T(a)) < \epsilon$ for all $a \in G_1$? As mentioned above, when this problem has a solution, we say that the homomorphisms from $G_1$ to $G_2$ are stable. In 1941, Hyers [16] gave a partial solution of Ulam’s problem for the case of approximate additive mappings under the assumption that $G_1$ and $G_2$ are Banach spaces. In 1950, Aoki [2] generalized the Hyers’ theorem for approximately additive mappings. In 1978, Rassias [27] generalized the theorem of Hyers by considering the stability problem with unbounded Cauchy differences.

**Theorem 1.3.** [27] Let $f : E \to E'$ be a mapping from a normed vector space $E$ into a Banach space $E'$ subject to the inequality

$$\|f(a + b) - f(a) - f(b)\| \leq \epsilon(\|a\|^p + \|b\|^p)$$

for all $a, b \in E$, where $\epsilon$ and $p$ are constants with $\epsilon > 0$ and $p < 1$. Then there exists a unique additive mapping $T : E \to E'$ such that

$$\|f(a) - T(a)\| \leq \frac{2\epsilon}{2 - 2^p}\|a\|^p$$

for all $a \in E$. If $p < 0$ then inequality (1.1) holds for all $a, b \neq 0$, and (1.2) holds for $a \neq 0$. Also, if the function $t \to f(ta)$ from $\mathbb{R}$ into $E'$ is continuous for each fixed $a \in X$, then $T$ is linear.

The result of the Rassias theorem was generalized by Forti [14] and Gavrută [15] who permitted the Cauchy difference to become arbitrary unbounded. Some results on the stability of functional equations in single variable and nonlinear iterative equations can be found in [11 29]. During the last decades several stability problems of functional equations have been investigated by many mathematicians (see [6, 9, 10, 11, 12, 17, 18, 20, 21, 22, 23, 24, 25]).

Let $X$ be a set. A function $d : X \times X \to [0, \infty]$ is called a generalized metric on $X$ if $d$ satisfies

1. $d(x, y) = 0$ if and only if $x = y$;
2. $d(x, y) = d(y, x)$ for all $x, y \in X$;
3. $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$.

We recall a fundamental result in fixed point theory.
Theorem 1.4. (3 5) Let \((X, d)\) be a complete generalized metric space and let \(J : X \to X\) be a strictly contractive mapping with Lipschitz constant \(L < 1\). Then for each given element \(x \in X\), either
\[
d(J^n x, J^{n+1} x) = \infty
\]
for all nonnegative integers \(n\) or there exists a positive integer \(n_0\) such that
\[
\begin{align*}
(1) & \quad d(J^n x, J^{n+1} x) < \infty, \quad \forall n \geq n_0; \\
(2) & \quad \text{the sequence} \ \{J^n x\} \ \text{converges to a fixed point} \ y^* \ \text{of} \ J; \\
(3) & \quad y^* \ \text{is the unique fixed point of} \ J \ \text{in the set} \ Y = \{y \in X \mid d(J^{n_0} x, y) < \infty\}; \\
(4) & \quad d(y, y^*) \leq \frac{1}{1-L} d(y, Jy) \ \text{for all} \ y \in Y.
\end{align*}
\]

Definition 1.5. A topological vector space \(X\) is a Fréchet space if it satisfies the following three properties:
\[
\begin{align*}
(1) & \quad \text{it is complete as a uniform space}; \\
(2) & \quad \text{it is locally convex}; \\
(3) & \quad \text{its topology can be induced by a translation invariant metric, i.e., a metric} \ d : X \times X \to \mathbb{R} \\
& \quad \text{such that} \ d(x, y) = d(x + a, y + a) \ \text{for all} \ a, x, y \in X.
\end{align*}
\]

For more detailed definitions of such terminologies, we can refer to [9]. Note that a ternary algebra is called a ternary Fréchet algebra if it is a Fréchet space with a metric \(d\).

Fréchet algebras, named after Maurice Fréchet, are special topological algebras as follows.

Note that the topology on \(A\) can be induced by a translation invariant metric, i.e., a metric \(d : X \times X \to \mathbb{R}\) such that \(d(x, y) = d(x + a, y + a)\) for all \(a, x, y \in X\).

Trivially, every Banach algebra is a Fréchet algebra as the norm induces a translation invariant metric and the space is complete with respect to this metric.

A locally \(C^*\)-algebra is a complete Hausdorff complex \(*\)-algebra \(A\) whose topology is determined by its continuous \(C^*\)-seminorms in the sense that a net \(\{a_i\}_{i \in I}\) converges to 0 if and if the net \(\{p(a_i)\}_{i \in I}\) converges to 0 for each continuous \(C^*\)-seminorm \(p\) on \(A\) (see [19 20]). The set of all continuous \(C^*\)-seminorms on \(A\) is denoted by \(S(A)\). A Fréchet locally \(C^*\)-algebra is a locally \(C^*\)-algebra whose topology is determined by a countable family of \(C^*\)-seminorms. Clearly, any \(C^*\)-algebra is a Fréchet locally \(C^*\)-algebra.

For given two locally \(C^*\)-algebras \(A\) and \(B\), a morphism of locally \(C^*\)-algebras from \(A\) to \(B\) is a continuous \(*\)-morphism \(\varphi\) from \(A\) to \(B\). An isomorphism of locally \(C^*\)-algebras from \(A\) to \(B\) is a bijective mapping \(\varphi : A \to B\) such that \(\varphi\) and \(\varphi^{-1}\) are morphisms of locally \(C^*\)-algebras.

Hilbert modules over locally \(C^*\)-algebras are generalization of Hilbert \(C^*\)-modules by allowing the inner product to take values in a locally \(C^*\)-algebra rather than in a \(C^*\)-algebra.

In this paper, using the fixed point method, we prove the Hyers-Ulam stability and the superstability of \(n\)-Jordan \(*\)-derivations in Fréchet locally \(C^*\)-algebras for the following generalized Jensen-type functional equation
\[
f \left( \frac{a + b}{2} \right) + f \left( \frac{a - b}{2} \right) = f(a).
\]

2. Stability of \(n\)-Jordan \(*\)-derivations

Lemma 2.1. (23) Let \(A, B\) be \(C^*\)-algebras, and let \(D : A \to B\) be a mapping such that
\[
\|D \left( \frac{a + b}{2} \right) + D \left( \frac{a - b}{2} \right) \|_B \leq \|D(a)\|_B,
\]
for all \(a, b \in A\). Then \(D\) is Cauchy additive.
Proof. By putting \( a = b = 0 \) in (2.1) we get \( \|2D(0)\| \leq \|D(0)\| \). So \( D(0) = 0 \), for all \( a, b \in A \).

Letting \( x = \frac{a+b}{2}, y = \frac{a-b}{2} \) in (2.1), we conclude that \( D \) is additive

\[
D(x) + D(y) = D(a) = D \left( \frac{a+b}{2} \right) = D(x+y).
\]

□

Now, we prove the Hyers-Ulam stability problem for \( n \)-Jordan \( \ast \)-derivations in Fréchet locally \( C^\ast \)-algebras.

**Theorem 2.2.** Let \( A, B \) be Fréchet locally \( C^\ast \)-algebras, and \( \theta \) be nonnegative real number. Let \( f : A \to B \) be a mapping such that

\[
\|\mu f(\frac{a+b}{2}) + \mu f(\frac{a-b}{2}) - f(\mu a) + f(c^n) - f(c)c^{n-1} + cf(c)c^{n-2} + \ldots + c^{n-2}f(c)c + c^{n-1}f(c) + f(d^\ast) - f(d)\ast \|_B \leq \theta
\]

for all \( \mu \in \mathbb{T}^1 := \{ \lambda \in \mathbb{C} : |\lambda| = 1 \} \) and all \( a, b, c, d \in A \). Then the mapping \( f : A \to B \) is an \( n \)-Jordan \( \ast \)-derivation.

Proof. Suppose that \( \mu = 1 \) and \( c, d = 0 \) in (2.2) by Lemma 2.1, the mapping \( f : A \to B \) is additive. By putting \( a = b \) and \( c = d = 0 \) in (2.2), we get

\[
\|\mu f(\frac{2a}{2}) + \mu f(0) - f(\mu a)\| \leq \theta,
\]

for all \( a \in A \) and \( \mu \in \mathbb{T}^1 \). So

\[
\mu f(a) = f(\mu a),
\]

for all \( a \in A \) and \( \mu \in \mathbb{T}^1 \).

By [4, Theorem 2.1], the mapping \( f : A \to B \) is \( \mathbb{C} \)-Linear. Letting \( a = b = d = 0 \) in (2.2), we get

\[
f(c^n) = f(c)c^{n-1} + cf(c)c^{n-2} + \ldots + c^{n-2}f(c)c + c^{n-1}f(c),
\]

for all \( c \in A \) and by letting \( a = b = c = 0 \) in (2.2), we have

\[
f(d^\ast) = f(d)^\ast,
\]

for all \( d \in A \). Hence the mapping \( f : A \to B \) is a \( n \)-Jordan \( \ast \)-derivation. □

**Theorem 2.3.** Let \( A, B \) be Frechet locally \( C^\ast \)-algebras and let \( \theta \) be nonnegative real number. Let \( f : A \to B \) be a mapping satisfying then the mapping \( f : A \to B \) is a \( n \)-Jordan \( \ast \)-derivation

Proof. The proof is similar to the proof of Theorem 2.2 □

Now we prove the Hyers-Ulam stability of \( n \)-Jordan derivations in \( C^\ast \)-algebras.

**Theorem 2.4.** Let \( A, B \) be Fréchet locally \( C^\ast \)-algebras. Let \( f : A \to B \) be a mapping for which there exists a function \( \varphi : A^4 \to \mathbb{R}^+ \) such that

\[
\psi(a,b,c,d) = \sum_{i=0}^{\infty} 2^{-i} \varphi(2^ia,2^ib,2^ic,2^id) < \infty,
\]

(2.3)
\[
\|\mu f(\frac{a+b}{2}) + \mu f(\frac{a-b}{2}) - f(\mu a) + f(c^n) - f(c)c^{n-1} + cf(c)c^{n-2} + \ldots + c^{n-2}f(c)c + c^{n-1}f(c) + f(d^*) - f(d^*)\|_B \leq \varphi(a,b,c,d)
\] (2.4)

for all \(a, b, c, d \in A\) and all \(\mu \in T^1\). Then there exists a unique \(n\)-Jordan \(*\)-derivation \(D : A \to B\) such that

\[
\|f(a) - D(a)\|_B \leq \psi(a,a,0,0)
\] (2.5)

for all \(a \in A\).

**Proof.** By putting \(\mu = 1\) and \(b = c = d = 0\) and replacing \(a\) by \(2a\) in (2.4), we get

\[
\|2f(\frac{2a}{2}) - f(2a)\|_B \leq \varphi(2a,0,0,0)
\] (2.6)

for all \(a \in A\). Using the induction method, we have

\[
\|f(a) - 2^{-n}f(2^n a)\|_B \leq \frac{1}{2^n} \sum_{i=1}^{n} \varphi(2^i a, 0, 0, 0)
\] (2.7)

for all \(a \in A\). Replace \(a\) by \(a^n\) in (2.6) and then divide by \(2^m\), we have

\[
\|f(a^m) - 2^{-n-m}f(2^{n+m} a)\|_B \leq \frac{1}{2^{n+m}} \sum_{i=m}^{m+n} \varphi(2^i a, 0, 0, 0)
\]

for all \(a \in A\). Hence, \(\{2^{-n}f(2^n a)\}\) is a Cauchy sequence. Since \(A\) is complete, then

\[
D(a) = \lim_{n} 2^{-n}f(2^n a)
\]

exists for all \(a \in A\). By (2.4) one can show that

\[
\|D(\frac{a+b}{2}) + D(\frac{a-b}{2}) - D(a)\|_B = \lim_{n} \frac{1}{2^n} \|f(2^{n-1}(a+b)) + f(2^{n-1}(a-b)) - f(2^n a)\|_B \leq \lim_{n} \frac{1}{2^n} \varphi(2^n a, 2^n b, 0, 0)
\] (2.8)

for all \(a, b \in A\). So

\[
D(\frac{a+b}{2}) + D(\frac{a-b}{2}) = D(a)
\]

for all \(a, b \in A\). Put \(x = \frac{a+b}{2}, y = \frac{a-b}{2}\) in above equation, we have

\[
D(x) + D(y) = D(a) = D(\frac{a+b}{2} + \frac{a-b}{2}) = D(x + y)
\]

for all \(x, y \in A\). Hence, \(D\) is Cauchy additive. On the other hand, we have

\[
D(\mu a) - \mu D(a) = \lim_{n} \frac{1}{2^n} \|f(\mu 2^n a) - \mu f(2^n a)\|_B \leq \lim_{n} \frac{1}{2^n} \varphi(2^n a, 2^n a, 0, 0) = 0
\]
for all \( \mu \in \mathbb{T}^1 \), and all \( a \in A \). So it is easy to show that \( D \) is linear. It follow from (2.4) that

\[
\|D(c^n) - D(c) e^{n-1} + cD(c)e^{n-2} + \ldots + e^{n-2}D(c)c + e^{n-1}D(c)\|_B \\
= \lim_m \left\| \frac{1}{2m} f(2^m c^n) - \frac{1}{2m} \left( f(2^m 2^{m(n-1)} c) + f(2^m 2^{m(n-2)} c) + \ldots \right) \right\|_B \\
+ f(2^{3m} 2^{m(n-3)} c) + \ldots + f(2^{m(n-1)} 2^m c) \leq \lim_m \frac{1}{2m} \varphi(0, 0, 0, 2^m c) \\
\leq \lim_m \frac{1}{2m} \varphi(0, 0, 0, 2^m) \\
= 0
\] (2.9)

for all \( c \in A \). and we have

\[
\|D(d^*) - D(d)^*\|_B = \lim_n \left\| \frac{1}{2n} f(2^n d^*) - \frac{1}{2n} (f(2^n d))^* \right\|_B \\
\leq \lim_n \frac{1}{2mn} \varphi(0, 0, 0, 2^n d) \\
= 0
\] (2.10)

for all \( d \in A \). Hence \( D : A \to B \) is a unique \( \ast \)-derivation. □

**Corollary 2.5.** Let \( A, B \) be Fréchet locally \( C^* \)-algebras, and let \( f : A \to B \) be a mapping with \( f(0) = 0 \) for which there exist constants \( \theta \geq 0 \) and \( p_1, p_2, p_3, p_4 \in (-\infty, 1) \) such that

\[
\|\mu f\left( \frac{a + b}{2} \right) + \mu f\left( \frac{a - b}{2} \right) - f(\mu a) + f(c^n) - f(c)e^{n-1} + cf(c)e^{n-2} + \ldots \]
\[
+ c^{n-2} f(c)c + e^{n-1} f(c) + f(d^*) - f(d)^* \|_B \\
\leq \theta(\|a\|_{A}^{p_1} + \|b\|_{A}^{p_2} + \|c\|_{A}^{p_3} + \|d\|_{A}^{p_4})
\] (2.11)

for all \( a, b, c, d \in A \) and all \( \mu \in \mathbb{T}^1 \). Then there exists a unique \( \ast \)-derivation \( D : A \to B \) such that

\[
\|f(a) - D(a)\|_B \leq \frac{2\theta\|a\|_{A}^{p_1}}{2 - 2^{p_1}}
\] (2.12)

for all \( a \in A \).

**Proof.** By putting \( \varphi(a, b, c, d) = \theta(\|a\|_{A}^{p_1} + \|b\|_{A}^{p_2} + \|c\|_{A}^{p_3} + \|d\|_{A}^{p_4}) \) in Theorem 2.3, we have

\[
\|f(a) - D(a)\|_B \leq \frac{2\theta\|a\|_{A}^{p_1}}{2 - 2^{p_1}}
\]

for all \( a \in A \), as desired. □

**Theorem 2.6.** Let \( A, B \) be Fréchet locally \( C^* \)-algebras. Let \( f : A \to B \) be a mapping for which there exists a function \( \varphi : A^4 \to \mathbb{R}^+ \) such that

\[
\psi(a, b, c, d) = \sum_{i=0}^{\infty} 2^i \varphi(2^{-i}a, 2^{-i}b, 2^{-i}c, 2^{-i}d) < \infty,
\] (2.13)

\[
\|\mu f\left( \frac{a + b}{2} \right) + \mu f\left( \frac{a - b}{2} \right) - f(\mu a) + f(c^n) - f(c)e^{n-1} + cf(c)e^{n-2} + \ldots \]
\[
+ c^{n-2} f(c)c + e^{n-1} f(c) + f(d^*) - f(d)^* \|_B \leq \varphi(a, b, c, d)
\] (2.14)
for all \( a, b, c, d \in A \) and all \( \mu \in \mathbb{T}^1 \). Then there exists a unique \( n \)-Jordan \(*\)-derivation \( D : A \to B \) such that
\[
\|f(a) - D(a)\|_B \leq \psi(a, a, 0, 0)
\]
for all \( a \in A \).

**Proof.** Suppose that \( \mu = 1 \) and \( b = c = d = 0 \) in (2.14), we get
\[
\|f(a) - 2f(2^{-1}a)\|_B \leq \varphi(a, 0, 0, 0)
\]
for all \( a \in A \). Using the induction method, we have
\[
\|f(a) - 2^nf(2^{-n}a)\|_B \leq \sum_{i=1}^{n} 2^i \varphi(2^{-i}a, 0, 0, 0)
\]
for all \( a \in A \). Replace \( a \) by \( a^n \) in (2.17) and then divide by \( 2^m \), we have
\[
\|f(a^n) - 2^{n+m}f(2^{-n-m}a)\|_B \leq \sum_{i=m}^{m+n} 2^i \varphi(2^{-i}a, 0, 0, 0)
\]
for all \( a \in A \). Hence, \( \{2^n f(2^{-n}a)\} \) is a Cauchy sequence. Since \( A \) is complete, then
\[
D(a) = \lim_n 2^n f(2^{-n}a)
\]
eexists for all \( a \in A \). By (2.14) one can show that
\[
\|D \left( \frac{a+b}{2} \right) + D \left( \frac{a-b}{2} \right) - D(a)\|_B
\]
\[
= \lim_n 2^n \|f(2^{-n-1}(a+b)) + f(2^{-n-1}(a-b)) - 2f(2^{-n}a)\|_B
\]
\[
\leq \lim_n 2^n \varphi(2^{-n-1}a, 2^{-n-1}b, 0, 0)
\]
for all \( a, b \in A \). So
\[
D \left( \frac{a+b}{2} \right) + D \left( \frac{a-b}{2} \right) = D(a)
\]
for all \( a, b \in A \). Put \( x = \frac{a+b}{2}, y = \frac{a-b}{2} \) in above equation, we have
\[
D(x) + D(y) = D(a) = D \left( \frac{a+b}{2} + \frac{a-b}{2} \right) = D(x+y)
\]
for all \( x, y \in A \). Hence, \( D \) is Cauchy additive.

The rest of proof is similar to the proof of Theorem 2.3. \( \square \)

**Corollary 2.7.** Let \( A, B \) be Fréchet locally \( C^* \)-algebras, and let \( f : A \to B \) be a mapping with \( f(0) = 0 \) for which there exist constants \( \theta \geq 0 \) and \( p_1, p_2, p_3, p_4 \in (-\infty, 1) \) such that
\[
\|\mu f \left( \frac{a+b}{2} \right) + \mu f \left( \frac{a-b}{2} \right) - f(\mu a) + f(c^n) - f(c)e^{n-1} + cf(c)e^{n-2} + \ldots \\
+ c^{n-2}f(c)c + c^{n-1}f(c) + f(d^n) - f(d^n)\|_B
\]
\[
\leq \theta (\|a\|^{p_1} + \|b\|^{p_2} + \|c\|^{p_3} + \|d\|^{p_4})
\]
for all \( a, b, c, d \in A \) and all \( \mu \in \mathbb{T}^1 \). Then there exists a unique \( n \)-Jordan \(*\)-derivation \( D : A \to B \) such that
\[
\|f(a) - D(a)\|_B \leq \frac{r \theta \|a\|_A^{p_1}}{2 - 2^n}
\]
for \( r < 1 \) and all \( a \in A \).
Proof. Letting $\varphi(a, b, c, d) = \theta(\|a\|^{p_1} + \|b\|^{p_2} + \|c\|^{p_3} + \|d\|^{p_4})$ in Theorem 2.5, we have
\[
\|f(a) - D(a)\|_B \leq r\theta\|a\|^{p_1}_A
\]
for $r < 1$ and all $a \in A$, as desired. □

References