



On B^{ic} -open set and B^{ic} -continuous function in topological spaces

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Abstract

In this paper, we have introduced a new class of sets in topological space called B^{ic} -open set and we have introduced and study the properties of B^{ic} -continuous function and topological properties.

Keywords: β -open set, B^{ic} -open set, B^{ic} -continuous function, β -irresolute function, B^{ic} -irresolute function.

1. Introduction and preliminaries

In general topology [15], the concept of β -open sets was introduced, The notion of β -open set was introduced by Abd El-Monsef et al. [2] introduced the notion of β -open sets and β -continuity in topological spaces and β -open sets have been referred to as semipreopen by Andrijevic [7, 3, 6, 4, 8, 10]. A Generalization of the concept of open sets is now well-known important notions in topology and its applications. Levine [6] introduced semi-open set and semi continuous function, [14, 5, 11, 13, 9, 1, 12]. introduced i -open set, i -continuous function and i -homeomorphism. This work is a generalization of topological concepts. We have studied an important type of open sets namely B^{ic} -open set and B^{ic} -continuous function.

Throughout this paper (X, τ) and (Y, τ) (simply, X and Y) represent topological spaces.

Definition 1.1. [5] Let X be a topological space. Then a subset A of X is said to be a β -open set if $A \subseteq cl(\int(cl(A)))$, and a β -closed set if $A \subseteq \int(cl(\int(A)))$. The family of all β -open (resp. β -closed) set sub sets of a space X will be as always denoted by $\beta O(X)$ (resp. $\beta c(X)$).

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Definition 1.2. [5] A subset A of a topological space X is said to be i -open set, if $A \subseteq Cl(A \cap O)$, where $\exists O \in \tau$ and $O \neq X, \emptyset$, The family of all i -open sets of a topological space is denoted by $iO(X)$. The complement of i -open sets of a topological space X is called i -closed sets.

Definition 1.3. A subset A of a topological space X is said to be $i\beta$ -open set if there exist, $O \in \beta$ -open and $O \neq X, \emptyset$, such that $A \subseteq Cl(A \cap O)$. The complement of $i\beta$ -open sets of a topological space X is called $i\beta$ -closed sets.

Definition 1.4. [5] Let X and Y be two a topological space, a function $f : X \rightarrow Y$ is said to be i -continuous functions [12] if the inverse image of every i -open subset of Y is an i -open set in X .

Definition 1.5. [5] Let X and Y be two a topological space, a function $f : X \rightarrow Y$ is said to be i -irresolute functions [12] if the inverse image of every i -semi-open subset of Y is an i -semi-open set in X .

2. B^{ic} -Open set and properties

Definition 2.1. Let X be a topological space and $A \subseteq X$. Then a β -open set A is called a B^{ic} -open set if $\forall x \in A \exists Fx$ i -closed set $\ni x \in Fx \subseteq A$. A is a B^{ic} -closed set if A^c is a B^{ic} -open set X . The family of all B^{ic} -open (resp. B^{ic} -closed) set subset of a space X will be as always denoted by $B^{ic}o(X)$ (resp. $B^{ic}c(X)$).

Example 2.2. Let $X = \{1, 2, 3\}$, $\tau = \{ \emptyset, X, \{1\}, \{2\}, \{1, 2\} \}$.
 $\beta o(X) = \{ \emptyset, X, \{1\}, \{2\}, \{1, 2\}, \{1, 3\}, \{2, 3\} \}$, $B^{ic}(X) = \{ \emptyset, X, \{1\}, \{2\}, \{3\}, \{1, 3\}, \{2, 3\} \}$.

Theorem 2.3. Let X be atop. S . and $A \subseteq X$. Then

- (i) If A is a B^{ic} -open set, then $cl(A)$ is a B^{ic} -open set also.
- (ii) If A is a B^{ic} -closed set, then $int(A)$ is a B^{ic} -closed set also.

Proof .

(i) Let A be B^{ic} -open, then A is a β -open, then $cl(A)$ is a β -open. Let $x \in cl(A)$. Since $cl(A)$ is closed set, $\exists Fx = cl(A)$ closed set in X such that $x \in A \subseteq cl(A)$, then $cl(A)$ is a B^{ic} -open, by Definition 2.1.

(ii) Let A be B^{ic} -closed, then A^c is a B^{ic} -open. So by (i), we get $cl(A^c)$ is B^{ic} -open, then is a B^{ic} -closed, therefore $\int(A)$ is a B^{ic} -closed set. \square

Definition 2.4. Let X be a topological space and $A \subseteq X$, we definition

$$cl(A^{B^{ic}}) = \cap \{ F : A \subseteq F, F \text{ is } B^{ic} \text{-closed in } X \}$$

$$\int(A^{B^{ic}}) = \cup \{ U : U \subseteq A, U \text{ is } B^{ic} \text{-open in } X \}$$

Remark 2.5. From Definition 2.1. Note that:

- (i) The open set and B^{ic} -open set are independent in general.
- (ii) Every B^{ic} -open set is a β -open set. The converse is not true in general.

Example 2.6. Let $X = \{a, b, c\}$, $\tau = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}\}$.

$$\begin{aligned} \beta o(X) &= \{\emptyset, X, \{a\}, \{b\}, \{a, b\}, \{a, c\}, \{b, c\}\} \\ io(X) &= \{\emptyset, X, \{a\}, \{b\}, \{a, b\}, \{a, c\}, \{b, c\}\}, \\ B^{ic}o(X) &= \{\emptyset, X, \{a\}, \{c\}, \{a, c\}, \{b, c\}\} \end{aligned}$$

- (i) Let $A = \{a, c\} \in B^{ic}o(X)$ but $A = \{a, c\} \notin \tau$, and $B = \{a, b\} \in \tau$ but $B = \{a, b\} \notin B^{ic}o(X)$.
- (ii) Let $A = \{b\} \in \beta o(X)$ but $A = \{b\} \notin B^{ic}o(X)$.

The following diagrams shows the relation among types of sets.

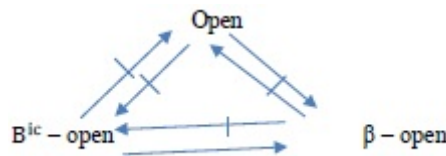


Figure 1:

3. B^{ic} -continuous function

In this section we recall some definition, examples, remarks theorems β -continuous, B^{ic} -continuous, β -open function, B^{ic} - open function, β - irresolute functions, B^{ic} - irresolute functions, β - homeomorphism, B^{ic} homeomorphism, β -contra continuous, B^{ic} - contra continuous.

Definition 3.1. Let $F : X \rightarrow Y$ be a function and $A \subseteq X$. Then:

- (i) F is called continuous function [8]. If for all open subset A of Y , then $F^{-1}(A)$ is open subset of X .
- (ii) F is called β - continuous function [6]. If for all open subset A of Y , then $(F^{-1}(A))$ is β - open subset of X .
- (iii) F is called B^{ic} -continuous function. If for all open subset A of Y , then $F^{-1}(A)$ is B^{ic} -open subset of X .

Theorem 3.2. Let X, Y be tow topological space and let $F : X \rightarrow Y$ be a function and $A \subseteq X$ Then:

- (i) Every continuous function is a β - continuous.
- (ii) Every B^{ic} continuous function is a β - continuous.

Proof . Let $F : X \rightarrow Y$ be a function

- (i) Let F be a continuous and let A be open in Y . Since F is continuous function, $F^{-1}(A)$ is open in X , then $F^{-1}(A)$ is a β - open in X . Hence F is a β - continuous.
- ii) Let F be a B^{ic} - continuous, and Let A be open in Y . Since F is B^{ic} continuous function, $F^{-1}(A)$ is B^{ic} -open in X , then $F^{-1}(A)$ is β -open in X . Hence F is a β -continuous. \square

The converse of the above theorem is not true in general.

Example 3.3. Let $F : X \rightarrow Y$ be a function and let

$$X = \{1, 2, 3\}, \quad \tau = \{\emptyset, X, \{1\}, \{2, 3\}\}, \quad \beta o(x) = \{\emptyset, X, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}\}.$$

$$B^{ic}o(x) = \{\emptyset, X, \{1\}, \{2, 3\}\}.$$

$$Y = \{a, b, c\}, \quad \acute{\tau} = \{\emptyset, Y, \{a\}, \{b\}, \{a, b\}\}, \quad \beta o(Y) = \{\emptyset, Y, \{a\}, \{b\}, \{a, b\}, \{a, c\}, \{b, c\}\}.$$

$$B^{ic}o(Y) = \{\emptyset, Y, \{a, c\}, \{b, c\}\}.$$

Define by $F(1) = a, F(2) = b, F(3) = c$. Note that F is β -continuous But

- (i) F not continuous. Since $A = \{b\}$ is an open in Y , but $F^{-1}(A)$ not open in X .
- (ii) F is not B^{ic} -continuous, since $A = \{b\}$ is an open in Y , but $F^{-1}(A)$ is not B^{ic} -open in X .

Remark 3.4. The continuous function and B^{ic} -continuous are independent in general.

Example 3.5. Let $F : X \rightarrow Y$ be a function. Let

$$X = \{1, 2, 3\}, \quad \tau = \{\emptyset, X, \{1\}, \{2\}, \{1, 2\}, \{1, 3\}\}, \quad \beta o(x) = \tau.$$

$$B^{ic}o(x) = \{\emptyset, X, \{2\}, \{1, 3\}\}.$$

$$Y = \{a, b, c\}, \quad \acute{\tau} = \{\emptyset, Y, \{a\}, \{b\}, \{a, b\}\}, \quad \beta o(Y) = \{\emptyset, Y, \{a\}, \{b\}, \{a, b\}, \{a, c\}, \{b, c\}\}.$$

$$B^{ic}o(Y) = \{\emptyset, Y, \{a, c\}, \{b, c\}\}.$$

Define by $F(1) = a, F(2) = b, F(3) = c$.
 Note that F is continuous function, but not B^{ic} -continuous function. Since $A = \{a\}$ is an open in Y , but $F^{-1}(A)$ is not B^{ic} -open in X .

Example 3.6. Let $F : X \rightarrow Y$ be a function and let

$$X = \{1, 2, 3\}, \quad \tau = \{\emptyset, X, \{1\}, \{3\}, \{1, 3\}\}, \quad \beta o(x) = \{\emptyset, X, \{1\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}\},$$

$$B^{ic}o(x) = \{\emptyset, X, \{1, 2\}, \{2, 3\}\}.$$

$$Y = \{a, b, c\}, \quad \acute{\tau} = \{\emptyset, Y, \{a, b\}\}.$$

Define $F(1) = a, F(2) = b, F(3) = c$.
 Note that F is B^{ic} -continuous. Since $A = \{a, b\}$ is an open in Y , but $F^{-1}(A)$ is not open in X .

The following diagram shows the relation among types of the continuous function.

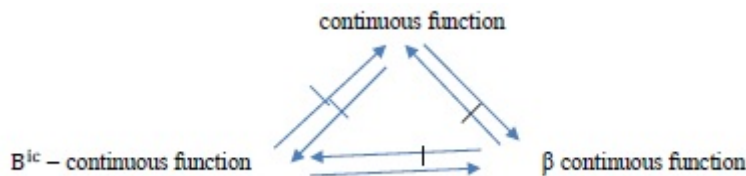


Figure 2:

Definition 3.7. Let $F : X \rightarrow Y$ be a function and $A \subseteq X$. Then:

- (i) F is called open (resp. closed) [8]. If A is an open (resp. closed) subset of X , then $F(A)$ is an open (resp. closed) subset of Y .

- (ii) F is called β -open (resp. β -closed). If A is open (resp. closed) subset of X , then $F(A)$ is a β -open (resp. β - closed) subset of Y .
- (iii) F is called B^{ic} -open (resp. B^{ic} -closed). If A open (resp. closed), subset of X , then $F(A)$ is a B^{ic} -open (resp. B^{ic} -closed) subset of Y .

Theorem 3.8. Let X, Y be a topological space and let $F : X \rightarrow Y$ be a function and AX . Then:

- (i) Every open function is a β -open function.
- (ii) Every closed function is a β -closed function.
- (iii) Every B^{ic} -open function is a β -open function.
- (iv) Every B^{ic} -closed function is a β -closed function.

Proof . Let $F : X \rightarrow Y$ be a function.

- (i) Suppose that F open function and let A open set in X . Since F open, then $F(A)$ open in Y , then $F(A)$ is a β - open in Y . Thus, F is β -open function.
- (ii) Similarly, part (i).
- (iii) Suppose that F is a B^{ic} - open function and let A be an open set in X . Since F is a B^{ic} - open, then $F(A)$ is a B^{ic} - open in Y , then $F(A)$ is a β - open in Y . Thus, F is a β - open function.
- (iv) Similarly, part (iii). \square

The Converse above theorem is not true in general.

Example 3.9. In Example 3.3, closed set in X are: $\emptyset, X, \{2, 3\}, \{1\}$. Closed set in Y are: $\emptyset, Y, X, \{b, c\}, \{a, c\}, \{c\}$. $B^{ic}(Y) = \{\emptyset, Y, \{b, c\}, \{a, c\}, \{c\}, \{b\}, \{a\}\}$.

$B^{ic}(Y) = \{\emptyset, Y, \{b\}, \{a\}\}$. Not that:

- (i) F is a β -open, but not open since $A = \{2, 3\}$ is open in X , but $F(A)$ is not open in Y .
- (ii) F is a β -closed, but not closed. Since $A = \{1\}$ is a closed set in X , but $F(A)$ not closed set in Y .
- (iii) F is a β - open, but not B^{ic} -open. Since $A = \{1\}$ is an open in X , but $F(A)$ not B^{ic} -open set in Y .
- (iv) F is a β -closed, but not B^{ic} -closed. Since $A = \{2, 3\}$ is a closed in X , but $F(A)$ is not B^{ic} -closed in Y .

Remark 3.10.

- (i) The open function and B^{ic} -open function is independent in general.
- (ii) The closed function and B^{ic} -closed function is independent in general

We can be showing that with two the following examples.

Example 3.11.

- (i) Let $F : X \rightarrow Y$ be function and let $X = \{a, b, c\}, \tau = \{\emptyset, X, \{b\}, \{b, c\}\}, \beta_o(x) = \{\emptyset, X, \{b\}, \{a, b\}, \{b, c\}\}, B^{ic}_o(x) = \{\emptyset, X\}$.
 $Y = \{1, 2, 3\}, \hat{\tau} = \{\emptyset, Y, \{2\}, \{3\}, \{2, 3\}, \{1, 2\}\}. \beta_o(Y) = \hat{\tau}, B^{ic}_o(Y) = \{\emptyset, Y, \{3\}, \{1, 2\}\}$. Define $F(a) = 1, F(b) = 2, F(c) = 3$.
- (ii) Let $F : X \rightarrow Y$ be a function and let $X = \{a, b, c\}, \tau = \{\emptyset, X, \{b, c\}\}, \beta_o(x) = \{\emptyset, X, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}\}, B^{ic}_o(x) = \{\emptyset, X\}$.
 $Y = \{1, 2, 3\}, \hat{\tau} = \{\emptyset, Y, \{1\}, \{3\}, \{1, 3\}\}. \beta_o(Y) = \{\emptyset, Y, \{1\}, \{3\}, \{1, 3\}, \{1, 2\}, \{2, 3\}\}, B^{ic}_o(Y) = \{\emptyset, Y, \{1, 2\}, \{2, 3\}\}$.

Define $F(a) = 1$, $F(b) = 2$, $F(c) = 3$.

In example (i). Note that:

- (1) F is open, but not B^{ic} -open. Since $A = \{b\}$ is open in X , but $F(A)$ is not B^{ic} -open set in Y .
- (2) F closed, but is not B^{ic} -closed. Since $A = \{a\}$ is a closed set in X , but $F(A)$ is not B^{ic} -closed in Y .

In Example (ii). Note that:

- (1) F is a B^{ic} -open, but not open. Since $A = \{b, c\}$ is an open set in X , but $F(A)$ is not open set in Y .
- (2) F is a B^{ic} -closed, but not closed. Since $A = \{a\}$ is a closed set in X , but $F(A)$ is not closed in Y .

The following diagram shows the relation among types of the open (closed) function.

Definition 3.12. Let $F : X \rightarrow Y$ be a function and $A \subseteq X$. Then:

- (i) F is said to be irresolute functions [8]. if the inverse image of every semi-open set in Y , is semi-open set in X .
- (ii) F is called β -irresolute functions [6]. If the inverse image of every β -open set in Y , is β -open set in X .
- (iii) F is called B^{ic} -irresolute functions. If the inverse image of every B^{ic} -open set in Y , is B^{ic} -open set in X .

Theorem 3.13. Let X, Y be a topological space and let $F : X \rightarrow Y$ be a function and $A \subseteq X$. Then:

- (i) Every irresolute function is a β -irresolute functions.
- (ii) Every B^{ic} -irresolute functions is a β -irresolute functions.

Proof . Let $F : X \rightarrow Y$ be a function.

- (i) Let F be an irresolute functions and let A be semi-open in Y . Since F is irresolute functions, $F^{-1}(A)$ is semi-open in X , then $F^{-1}(A)$ is a β -open in X . [because every semi-open set is β -open]. Hence F is a β -irresolute functions.
- (ii) Let F be a B^{ic} irresolute functions, and let A be open in Y . Since F is a B^{ic} irresolute function, $F^{-1}(A)$ is B^{ic} -open in X . Then $F^{-1}(A)$ is β -open in X , hence F is a β -continuous.

□

The converse of above theorem is not true in general.

Example 3.14. let $X = Y = \{a, b, c\}$, $\tau = \{\emptyset, X, \{b, c\}\}$. so $(X) = \{\emptyset, X, \{b, c\}\}$.

$$\beta o(X) = \{\emptyset, X, \{b\}, \{c\}, \{b, c\}, \{a, b\}, \{a, c\}\}. B^{ic}o(X) = \{\emptyset, X, \{c\}, \{a, b\}, \{a, c\}\}$$

$$\hat{\tau} = \{\emptyset, Y, \{a\}, \{c\}, \{a, c\}\}, \text{ so } (Y) = \{\emptyset, Y, \{a\}, \{c\}, \{a, b\}, \{b, c\}, \{a, c\}\}.$$

$$\beta o(Y) = \{\emptyset, Y, \{a\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}\}. B^{ic}o(Y) = \{\emptyset, Y, \{c\}, \{a, b\}, \{b, c\}\}$$

and Let $F : X \rightarrow Y$

- (i) F is a β -irresolute function but is not an irresolute function, because $\{a, c\}$ is semi-open set in Y . But $F^{-1}\{a, c\} = \{a, c\}$ is not semi-open set in X .
- (ii) F is not a B^{ic} -irresolute functions, since $A = \{b, c\}$ is an open in Y , but $F^{-1}(A)$ is not B^{ic} -open in X .

Remark 3.15. *The irresolute functions and B^{ic} -irresolute functions are independent in general.*

Example 3.16. *Let $F : X \rightarrow Y$ be a function*

Let $X = \{1, 2, 3\}$, $\tau = \{ \emptyset, X, \{1\}, \{2\}, \{1, 2\}, \{1, 3\} \}$, $\beta o(x) = Sox = \tau$.

$B^{ic} o(x) = \{ \emptyset, X, \{2\}, \{1, 3\} \}$. $Y = \{a, b, c\}$, $\tau' = \{ \emptyset, Y, \{a\}, \{b\}, \{a, b\} \}$.

$Sox = \{ \{a\}, \{b\}, \{a, b\}, \{a, c\}, \{b, c\} \}$, $\beta o(Y) = \{ \emptyset, Y, \{a\}, \{b\}, \{a, b\}, \{a, c\}, \{b, c\} \}$.

$B^{ic} o(Y) = \{ \emptyset, Y, \{a, c\}, \{b, c\} \}$. Define by $F(1) = a, F(2) = b, F(3) = c$.

Note that F is irresolute functions, but not B^{ic} is irresolute functions. Since $A = \{a\}$ is an open in Y , but $F^{-1}(A)$ not B^{ic} -open in X .

Example 3.17. *Let $F : X \rightarrow Y$ be a function and let $Y = X = \{1, 2, 3\}$, $\tau = \{ \emptyset, X, \{2, 3\} \}$.*

$So(x) = \{ \emptyset, X, \{2, 3\} \}$, $\beta o(x) = \{ \emptyset, X, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\} \}$

$B^{ic} o(x) = \{ \emptyset, X, \{2\}, \{3\}, \{1, 2\}, \{1, 3\} \}$, $\tau' = \{ \emptyset, Y, \{1\}, \{1, 3\} \}$.

$So(Y) = \{ \emptyset, Y, \{1\}, \{1, 2\}, \{1, 3\} \} = \beta o(Y)$, $B^{ic} o(Y) = \{ \emptyset, Y, \{1\}, \{1, 2\} \}$.

Note that F is a B^{ic} -irresolute function, but is not irresolute function. Since $A = \{1, 2\}$ is an β -open in Y , but $F^{-1}(A)$ is not semi-open in X .

The following diagram shows the relation among types of the continuous function.

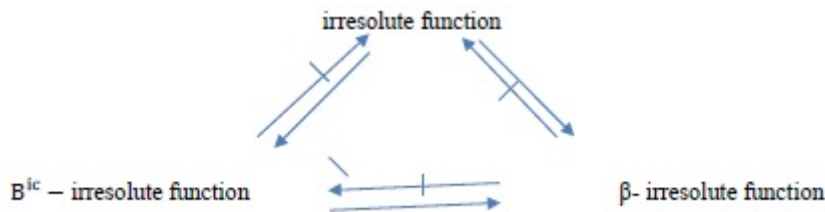


Figure 3:

4. Conclusion

In this paper, we introduce the concept of soft B^{ic} -interior and soft B^{ic} closure set in topological spaces and study some of their properties. We also introduce the concept of soft B^{ic} -open sets and soft B^{ic} -continuous functions, B^{ic} - irresolute function in topological spaces and some of their properties have been established. We hope that the findings in this paper are just the beginning of a new structure and not only will form the theoretical basis for further applications of topology on sets but also will lead to the development of information system and various fields in engineering.

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