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A self-adaptive hybrid inertial algorithm for split feasibility problems in Banach spaces

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Abstract

In this paper, we introduce a new self-adaptive hybrid algorithm of inertial form for solving Split Feasibility Problem (SFP) which also solve a Monotone Inclusion Problem (MIP) and a Fixed Point Problem (FPP) in p-uniformly convex and uniformly smooth Banach spaces. Motivated by the self-adaptive technique, we incorporate the inertial technique to accelerate the convergence of the proposed method. Under standard and mild assumption of monotonicity of the SFP associated mapping, we establish the strong convergence of the sequence generated by our algorithm which does not require a prior knowledge of the norm of the bounded linear operator. Some numerical examples are presented to illustrate the performance of our method as well as comparing it with some related methods in the literature.

Keywords: split feasibility problem, Bregman distance, uniformly smooth Banach spaces, uniformly convex Banach spaces, variational inequalities, Bregman and metric projection.
2010 MSC: 47H06, 47H09, 47H10, 46N10, 47J25.

1. Introduction

Let E_1 and E_2 be real Banach spaces, C and Q be non-empty, closed convex subsets of E_1 and E_2 respectively, E_1^* and E_2^* be the duals of E_1 and E_2 respectively, $A : E_1 \to E_2$ be a bounded linear operator and $A^* : E_2^* \to E_1^*$ be the adjoint of A. We shall denote the value of the functional $x^* \in E^*$

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at $x \in E$ by $\langle x^*, x\rangle$. The Split Feasibility Problem (shortly SFP) introduced by Censor and Elfving [10] in 1994 can be defined as follows:

$$\text{Find } \ x^* \in C \quad \text{such that} \quad Ax^* \in Q. \tag{1.1}$$

We denote the set of solution of SFP (1.1) by $\Omega = C \cap A^{-1}(Q) = \{x^* \in C : Ax^* \in Q\}$.
The SFP has recently attracted much attention from many researchers due to its application in modelling real-world problems such as inverse problem in signal processing, radiotherapy, data compression (see for example [7, 11, 12, 16, 29, 37, 48]). Furthermore, many algorithms have been introduced by several authors for solving the SFP and related optimization problems (for example, see [1, 2, 9, 17, 20, 21, 22, 24, 23, 25, 19, 28, 36, 41, 52, 54, 58, 59, 60]). A very popular algorithm which is often called CQ algorithm defined below, was proposed by Byrne [8] to solve the SFP in real Hilbert spaces:

$$x_{n+1} = P_C\left(x_n - \mu A^*(I - P_Q)Ax_n\right), \quad \forall n \geq 1, \tag{1.2}$$

where

$$\mu \in \left(0, \frac{2}{\|A\|^2}\right), \tag{1.3}$$

P_C and P_Q denote the metric projections of E_1 onto C and E_2 onto Q, respectively. It was proved that the sequence $\{x_n\}$ generated by (1.2) converges weakly to a solution of the SFP provided the step size μ satisfies the condition (1.3). As a result of this CQ algorithm, several iterative algorithms have been invented for solving SFP in Hilbert spaces and Banach spaces (see for example, [56, 18]). Let H be a real Hilbert space, F be a strictly convex, reflexive smooth Banach space, J_F denotes the duality mapping on F, C and Q be non-empty closed convex subsets of H and F, respectively. The following algorithm was proposed by Alsulami and Takahashi [6] in 2015: for any $x_1 \in H$,

$$x_{n+1} = \alpha_n x_n + (1-\alpha_n)P_C(x_n - rA^*J_F(I - P_Q)Ax_n), \quad n \geq 1. \tag{1.4}$$

It was proved that for some $a, b \in \mathbb{R}$ if, $0 < a \leq \alpha_n \leq b < 1$ and $0 < r\|A\|^2 < 2$, where $0 < r < \infty$ and $\{\alpha_n\} \subset [0,1]$, then $\{x_n\}$ weakly converges to $\omega_0 = \lim_{n\to\infty} P_{C\cap A^{-1}Q}x_n$, where $w_0 \in C \cap A^{-1}Q$. Furthermore, they introduced the following Halpern's type iteration in order to obtain strong convergence result. Let $\{t_n\}$ be a sequence in H such that $t_n \to t \in H$ and $x_1, t_1 \in H$,

$$\begin{cases} \nu_n = \lambda_n t_n + (1-\lambda_n)P_C(x_n - rA^*J_F(I - P_Q)Ax_n), \\ x_{n+1} = \alpha_n x_n + (1-\alpha_n)\nu_n, \quad n \geq 1, \end{cases} \tag{1.5}$$

where $0 < r < \infty$ and $\{\alpha_n\} \subset (0,1)$. It was proved that the sequence $\{x_n\}$ defined by (1.5) converges strongly to a point $\omega_0 \in C \cap A^{-1}Q$, for some $\omega_0 = P_{C\cap A^{-1}Q}t_1$, $\forall\ a, b \in \mathbb{R}$ if, $0 < r\|A\|^2 < 2$, $\lim_{n\to\infty} \lambda_n = 0$, $\sum_{n=1}^{\infty} \lambda_n = \infty$, and $0 < a \leq \alpha_n \leq b < 1$.
Recently, Suantai et al. 2019 [49] considered the following modified SFP:

$$\text{Find } \ x \in F(T) \cap C \ \text{ such that } \ Ax \in Q. \tag{1.6}$$

Clearly, when $F(T) = C$, then (1.6) reduces to (1.1). Suantai et al. [49] proved the following weak and strong convergence theorems using Mann's iteration and Halpern's type iteration process, respectively for solving SFP and fixed point problem for nonexpansive mappings.

Theorem 1.1. *Let H be a Hilbert space, F be a strictly convex, reflexive and smooth Banach space, C and Q be non-empty, closed and convex subsets of H and F, J_F be the duality mapping on F, P_Q and P_C denote the metric projections of F on Q and H on C, respectively. Let $T : C \to C$ be nonexpansive mapping. Suppose $\Gamma \neq \emptyset$, where $\Gamma = F(T) \cap C \cap A^{-1}Q$, for $x_1 \in C$, define $\{x_n\}$ by*

$$x_{n+1} = \alpha_n x_n + (1-\alpha_n) T P_C \left(x_n - \gamma_n \frac{f(x_n)}{\|g(x_n)\|^2 + \|x_n - Tx_n\|^2} g(x_n) \right), \tag{1.7}$$

where $g(x_n) = A^* J_F (I - P_Q) A x_n$, $f(x_n) = \frac{1}{2}\|(I-P_Q)Ax_n\|^2$, $\{\gamma_n\} \subset (0,4)$, $\forall n \in \mathbb{N}$ which satisfies the following conditions:

1. $\liminf_{n\to\infty} \gamma_n(4-\gamma_n) > 0$,
2. $0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty} \alpha_n < 1$.

Then $\{x_n\}$ weakly converges to $\omega_0 \in \Gamma$, where $\omega_0 = \lim\limits_{n\to\infty} P_\Gamma x_n$.

Theorem 1.2. *Let H be a Hilbert space, F be a strictly convex, reflexive and smooth Banach space, C and Q be non-empty, closed and convex subsets of H and F, J_F be the duality mapping on F, P_Q and P_C denote the metric projections of F on Q and H onto C, respectively. Let $T : C \to C$ be nonexpansive mapping. Suppose $\Gamma \neq \emptyset$, where $\Gamma = F(T) \cap C \cap A^{-1}Q$. Let $x_1 \in C$, $\{t_n\}$ be a sequence in C such that $t_n \to t$, and let $\{x_n\}$ be a sequence defined by*

$$x_{n+1} = \alpha_n x_n + (1-\alpha_n) \left(\lambda_n t_n + (1-\lambda_n) T P_C \left(x_n - \gamma_n \frac{f(x_n)}{\|g(x_n)\|^2 + \|x_n - Tx_n\|^2} g(x_n) \right) \right), \tag{1.8}$$

where $g(x_n) = A^* J_F (I - P_Q) A x_n$, $f(x_n) = \frac{1}{2}\|(I-P_Q)Ax_n\|^2$, $\{\gamma_n\} \subset (0,4)$, $\lambda_n \subset (0,1)$, $\{\alpha_n\} \subset (0,1)$, $\forall n \in \mathbb{N}$ which satisfy the following conditions:

1. $\liminf_{n\to\infty} \gamma_n(4-\gamma_n) > 0$,
2. $0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty} \alpha_n < 1$,
3. $\lim_{n\to\infty} \lambda_n = 0$ *and* $\sum_{n=1}^{\infty} \lambda_n = \infty$.

Then $\{x_n\}$ strongly converges to $w_0 \in \Gamma$.

Also, Polyak [40] proposed an inertial extrapolation to speed up convergence rate of smooth convex minimization problem. The main idea of this method is to make use of two previous iterates in order to update the next iterate. Due to the fact that the presence of inertial term in an algorithm speed up the convergence rate, inertial type algorithms have been widely studied by authors (see [1, 3, 5, 22, 30, 34, 35, 38] and the references therein).
Let $A : E \to 2^{E^*}$ be a set-valued mapping, $D(A)$ denotes the effective domain of A defined by $D(A) = \{x \in E : Ax \neq \emptyset\}$, $R(A)$ denotes the range of A which can be defined by $R(A) = \cup_{x\in D(A)} Ax$, $G(A)$ denotes the graph of A where $G(A) = \{(x, x^*) \in E \times E^* : x^* \in Ax\}$. A set-valued mapping A is said to be monotone if for all $x, y \in D(A)$, $v^* \in Ax$ and $w^* \in Ay$, we have $\langle v^* - w^*, x - y\rangle \geq 0$. Furthermore, A is said to be maximal monotone if its graph is not contained in the graph of any other monotone operator on E. It is known that if A is maximal monotone, then $A^{-1}(0) = \{z \in E : 0^* \in Az\}$ is a closed convex set (see [2, 15, 31, 32]).

Motivated by the above results, in this paper, we study the following modified SFP:

$$\text{Find } \ x \in F(T) \cap C \quad \text{such that } \ Ax \in B^{-1}(0), \tag{1.9}$$

where $B : E_2 \to 2^{E_2^*}$ is a maximal monotone operator. Obviously, the SFP (1.9) is more general than (1.6) and (1.1). When $B = \partial i_Q$, a maximal monotone operator and the subdifferential of the indicator function on Q, then (1.9) reduces to (1.6). We introduced a self-adaptive hybrid iterative algorithm for approximating solution of Problem (1.9) in p-uniformly convex and uniformly smooth Banach spaces. Our algorithm is designed such that its implementation does not require a prior knowledge of the norm of the bounded linear operator.

2. Preliminaries

In this section, we recall some basic definitions and preliminaries results which will be useful for our results in this paper. We denote the strong and weak convergence of the sequence $\{x_n\}$ to a point x by $x_n \to x$ and $x_n \rightharpoonup x$ respectively.
Let E be a real Banach space and $1 < q \leq 2 \leq p < \infty$ where $\frac{1}{p} + \frac{1}{q} = 1$. The modulus of smoothness of E denoted by $\rho_E(\tau)$ is the function $\rho_E : [0, \infty) \to [0, \infty)$ defined by

$$\begin{aligned}\rho_E(\tau) &= \sup\left\{\frac{\|x-y\| + \|x+y\|}{2} - 1 : \|x\| = 1, \|y\| = \tau\right\}\\ &= \sup\left\{\frac{\|x-\tau y\| + \|x+\tau y\|}{2} - 1 : \|x\| = 1 = \|y\|\right\}.\end{aligned}$$

E is uniformly smooth if and only if $\lim_{\tau\to 0^+} \frac{\rho_E(\tau)}{\tau} = 0$ and E is said to be q-uniformly smooth if there exists a constant $D_q > 0$ such that $\rho_E(\tau) \leq D_q\tau^q$. The modulus of convexity of E denoted by $\delta_E(\epsilon)$ is the function $\delta_E : (0, 2] \to [0, 1]$ defined by

$$\delta_E(\epsilon) = \inf\left\{1 - \frac{\|x+y\|}{2} : \|x\| = \|y\| = 1; \epsilon = \|x-y\|\right\}.$$

E is unifomly convex if and only if $\delta_E(\epsilon) > 0$, for all $\epsilon \in (0, 2]$ and E is p-uniformly convex if there is a constant $C_p > 0$ such that $\delta_E(\epsilon) \geq C_p\epsilon^p$ for all $\epsilon \in (0, 2]$. Every uniformly convex Banach space is strictly convex and reflexive. It is known that if E is p-uniformly convex and uniformly smooth, then its dual E^* is q-uniformly smooth and uniformly convex.

Definition 2.1. *[4] Let $p > 1$, the generalised duality mapping $J_E^p : E \to 2^{E^*}$ is defined by*

$$J_E^p = \left\{x^* \in E^* : \langle x^*, x\rangle = \|x\|^p, \|x^*\| = \|x\|^{p-1}\right\}.$$

It is known that when E is uniformly smooth, then J_p^E is norm to norm uniformly continuous on bounded subsets of E and E is smooth if and only if J_p^E is single valued. Also, when E is reflexive and strictly convex then $J_E^p = (J_{E^*}^q)^{-1}$ is one-to-one and surjective, where $J_{E^*}^q$ is the duality mapping of E^*(see [13]). Furthermore, J_E^p is said to be weak-to-weak continuous if

$$x_n \rightharpoonup x \Rightarrow \langle J_E^P x_n, y\rangle \to \langle J_E^p x, y\rangle, \quad \textit{for any } \ y \in E.$$

It is known that $l_p(p > 1)$ has such a property, but $L_p(p > 2)$ does not share this property. The following inequality was proved by Xu [57].

Lemma 2.2. *[53, 57]*
Let $x, y \in E$. If E is a q-uniformly smooth Banach space, then there exists a $D_q > 0$ such that

$$\|x-y\|^q \leq \|x\|^q - q\langle J_E^q(x), y\rangle + D_q\|y\|^q. \tag{2.1}$$

Definition 2.3. *[51] A function $f : E \to \mathbb{R} \cup \{+\infty\}$ is said to be*

(i) proper if its effective domain $domf = \{x \in E : f(x) < +\infty\}$ is non-empty,
(ii) convex if $f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y),\ \forall \lambda \in (0,1), x, y \in D(f)$,
(iii) lower semi-continuous at $x_0 \in D(f)$ if $f(x_0) \leq \lim_{x\to x_0} \inf f(x)$.

Let $x \in int(domf)$, for any $y \in E$, the directional derivative of f at x denoted by $f^0(x,y)$ is defined by

$$f^0(x,y) := \lim_{t\to 0^+} \frac{f(x+ty) - f(x)}{t}. \tag{2.2}$$

If the limit at $t \to 0^+$ in (2.2) exists for each y, then the function f is said to be Gâteaux differentiable at x. In this case $f^0(x,y) = \langle \nabla f(x), y\rangle$ (or $f'(x)$), where $\nabla f(x)$ is the value of the gradient of f at x.

Definition 2.4. *Let $f : E \to \mathbb{R}$ be a Gâteaux differentiable and convex function. The Bregman distance denoted as $\Delta_f : domf \times domf \to [0, +\infty)$ is defined as*

$$\Delta_f(x,y) = f(y) - f(x) - \langle f^{'}(x), y - x\rangle, x, y \in E. \tag{2.3}$$

Note that $\Delta_f(x,y) \geq 0$ (see [26, 55]). It is worthy to note that the duality mapping J_E^p is actually the derivative of the function $f_p(x) = \frac{1}{p}\|x\|^p$ for $2 \leq p < \infty$. Hence, if $f = f_p$ in (2.3), the Bregman distance with respect to f_p now becomes

$$\begin{aligned}\Delta_p(x,y) &= \frac{1}{q}\|x\|^p - \langle J_E^p x, y\rangle + \frac{1}{p}\|y\|^p\\ &= \frac{1}{p}\left(\|y\|^p - \|x\|^p\right) + \langle J_E^p x, x - y\rangle\\ &= \frac{1}{q}\left(\|x\|^p - \|y\|^p\right) - \langle J_E^p x - J_E^p y, x\rangle.\end{aligned}$$

It is generally known that the Bregman distance is not a metric as a result of absence of symmetry, but it possesses some distance-like properties which are stated below:

$$\Delta_p(x,y) = \Delta_p(x,z) + \Delta_p(z,y) + \langle J_E^p x - J_E^p z, z - y\rangle, \tag{2.4}$$

and

$$\Delta_p(x,y) + \Delta_p(y,x) = \langle J_E^p x - J_E^p y, x - y\rangle.$$

The relationship between the metric and Bregman distance in p-uniformly convex space is as follow:

$$\tau\|x - y\|^p \leq \Delta_p(x,y) \leq \langle J_E^p x - J_E^p y, x - y\rangle, \tag{2.5}$$

where $\tau > 0$ is a fixed number.
Let C be a non-empty closed convex subset of E. The Bregman projection is defined as

$$\Pi_C x = \arg\min_{y\in C} \Delta_p(y,x),\ x \in E,$$

and the metric projection can be defined similarly as

$$P_C x = \arg\min_{y\in C} \|y - x\|,\ x \in E.$$

The Bregman projection is the unique minimizer of the Bregman distance and can be characterized by a variational inequality (see [44, 45]):

$$\langle J_E^p(x) - J_E^p(\Pi_C x), z - \Pi_C x\rangle \leq 0, \ \forall z \in C, \tag{2.6}$$

from which we have

$$\Delta_p(\Pi_C x, z) \leq \Delta_p(x, z) - \Delta_p(x, \Pi_C x), \ \forall z \in C. \tag{2.7}$$

The metric projection which is also the unique minimizer of the norm distance can be characterized by the following variational inequality:

$$\langle J_E^p(x - P_C x), z - P_C x\rangle \leq 0, \ \forall z \in C. \tag{2.8}$$

We define the functional $V_p : E \times E \to [0, \infty]$ associated with $f_p(x) = \frac{1}{p}\|x\|^p$ by

$$V_p(x, \bar{x}) = \frac{1}{p}\|x\|^p - \langle \bar{x}, x\rangle + \frac{1}{q}\|\bar{x}\|^q, \ x \in E, \ \bar{x} \in E^*, \tag{2.9}$$

where $V_p(x, \bar{x}) \geq 0$. It then follows that

$$V_p(x, \bar{x}) = \Delta_p(x, J_{E^*}^q(\bar{x})), \ \forall x \in E, \ \bar{x} \in E^*.$$

Chuasuk et al [14] proved the following inequality

$$V_p(x, \bar{x}) + \langle \bar{y}, J_{E^*}^q(\bar{x}) - x\rangle \leq V_p(x, \bar{x} + \bar{y}), \ \forall x \in E, \bar{x}, \ \bar{y} \in E^*.$$

Furthermore, V_p is convex in the second variable, and thus, for all $z \in E$, $\{x_i\}_{i=1}^N$, and $\{t_i\}_{i=1}^N \subset (0, 1)$, $\sum_{i=1}^N t_i = 1$ we have (see [47])

$$\Delta_p\left(z, J_{E^*}^q\left(\sum_{i=1}^N t_i J_E^p(x_i)\right)\right) = V_p\left(z, \left(\sum_{i=1}^N t_i J_E^p(x_i)\right)\right) \leq \sum_{i=1}^N t_i \Delta_p(z, x_i). \tag{2.10}$$

Let C be a non-empty, closed and convex subset of a smooth Banach space E and let $T : C \to C$. A point $x^* \in C$ is called an asymptotic fixed point of T if a sequence $\{x_n\}_{n\in\mathbb{N}}$ exists in C and converges weakly to x^* such that $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$. We denote the set of all asymptotic fixed points of T by $\hat{F}(T)$. Moreover, a point $x^* \in C$ is said to be a strong asymptotic fixed point of T if there exists a sequence $\{x_n\}_{n\in\mathbb{N}}$ in C which converges strongly to x^* such that $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$. We denote the set of all strong asymptotic fixed points of T by $\tilde{F}(T)$. It follows from the definitions that $F(T) \subset \tilde{F}(T) \subset \hat{F}(T)$, see [39].

Definition 2.5. *[42] Let T be a mapping such that $T : C \to E$. T is said to be*

(i) nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$ for each $x, y \in C$,
(ii) quasi-nonexpansive if $\|Tx - y^*\| \leq \|x - y^*\|$ such that $F(T) \neq \emptyset$, $\forall x \in C$ and $y^* \in F(T)$.

Definition 2.6. *[43] Let $T : C \to E$ be a mapping. T is said to be*

1. *Bregman nonexpansive if*

$$\Delta_p(Tx, Ty) \leq \Delta_p(x, y), \quad \forall x, y \in C,$$

2. *Bregman quasi-nonexpansive if* $F(T) \neq \emptyset$ *and*

$$\Delta_p(y^*, Tx) \leq \Delta_p(y^*, x), \quad \forall x \in C, y^* \in F(T),$$

3. *Bregman weak relatively nonexpansive if* $\tilde{F}(T) \neq \emptyset$, $\tilde{F}(T) = F(T)$ *and*

$$\Delta_p(y^*, Tx) \leq \Delta_p(y^*, x) \quad \forall x \in C, y^* \in F(T),$$

4. *Bregman relatively nonexpansive if* $F(T) \neq \emptyset$, $\hat{F}(T) = F(T)$ *and*

$$\Delta_p(y^*, Tx) \leq \Delta_p(y^*, x) \quad \forall x \in C, y^* \in F(T).$$

From the definitions, it is evident that the class of Bregman quasi-nonexpansive maps contains the class of Bregman weak relatively nonexpansive maps. The class of Bregman weak relatively nonexpansive maps contains the class of Bregman relatively nonexpansive maps.
Let E be a smooth, strictly convex and reflexive Banach space, $A : E \to 2^{E^*}$ be a maximal monotone operator. We define a mapping $Q_r^A : E \to D(A)$ by (see [50])

$$Q_r^A(x) = (I + r(J_E^p)^{-1}A)^{-1}(x), \text{ for all } x \in E \text{ and } r > 0.$$

This mapping is known as metric resolvent of A. Obviously, for all $r > 0$, we have

$$0 \in J_E^p(Q_r^A(x) - x) + rAQ_r^A(x), \tag{2.11}$$

and $F(Q_r^A) = A^{-1}(0)$. Furthermore, for all $x, y \in E$ and by the monotonicity of A, we can show that

$$\langle J_E^p(x - Q_r^A(x)) - J_E^p(y - Q_r^A(y)), Q_r^A(x) - Q_r^A(y)\rangle \geq 0. \tag{2.12}$$

From (2.11), we have for all $x, y \in E$

$$\frac{J_E^p(x - Q_r^A(x))}{r} \in AQ_r^A(x), \tag{2.13}$$

and

$$\frac{J_E^p(y - Q_r^A(y))}{r} \in AQ_r^A(y). \tag{2.14}$$

Since A is monotone, we can obtain (2.12) from (2.13) and (2.14). This implies that for all $x \in E$, $t \in A^{-1}(0)$, and whenever $A^{-1}(0) \neq \emptyset$, we have

$$\langle J_E^P(x - Q_r^A(x)), Q_r^A(x) - t\rangle \geq 0. \tag{2.15}$$

Lemma 2.7. *[27] Let C be a non-empty, closed and convex subset of a reflexive, strictly convex and smooth Banach space E, $x_0 \in C$ and $x \in E$. Then the following assertions are equivalent:*

1. $x_0 = \Pi_C(x)$;
2. $\langle J_E^p(x_0) - J_E^p(x), z - x_0 \geq 0\rangle, \quad \forall z \in C.$

Furthermore, $\forall y \in C$, we have,

$$\Delta_p(\Pi_C(x), y) + \Delta_p(x, \Pi_C(x)) \leq \Delta_p(x, y).$$

Lemma 2.8. *[53] Let E be a smooth and uniformly convex real Banach space. Let $\{x_n\}$ and $\{y_n\}$ be two bounded sequences in E. Then $\lim_{n\to\infty} \Delta_p(x_n, y_n) = 0$ if and only if*

$$\lim_{n\to\infty} \|x_n - y_n\| = 0.$$

Lemma 2.9. *[57] Let $q \geq 1$ and $r > 0$ be two fixed real numbers, then a Banach space E is uniformly convex if and only if there exists a continuous, strictly, increasing and convex function $g : \mathbb{R}^+ \to \mathbb{R}^+, g(0) = 0$ such that for all $x, y \in B_r$ and $0 \leq \alpha \leq 1$,*

$$\|\alpha x + (1-\alpha)y\|^q \leq \alpha\|x\|^q + (1-\alpha)\|y\|^q - W_q(\alpha)g(\|x - y\|), \tag{2.16}$$

where $W_q := \alpha^q(1-\alpha) + \alpha(1-\alpha)^q$ and $B_r := \{x \in E : \|x\| \leq r\}$.

3. Main results

In this section, we present our inertial technique for solving the modified SFP (1.9) in Banach spaces. We also prove a strong convergence result for the sequence generated by our algorithm.

Algorithm 3.1. *Let E_1, E_2 be p-uniformly convex and uniformly smooth real Banach spaces, C and Q be non-empty closed convex subsets of E_1 and E_2 respectively, and $A : E_1 \to E_2$ be a bounded linear operator with $A^* : E_2^* \to E_1^*$. Let $T : C \to C$ be a Bregman weak relatively nonexpansive mapping, and $B : E_2 \to 2^{E_2^*}$ be a maximal monotone operator. Suppose $\Gamma = F(T) \cap C \cap A^{-1}(B^{-1}(0)) \neq \emptyset$. Let $\{\alpha_n\}$ be a sequence in [0,1], with $0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty} \alpha_n < 1$, $x_0, x_1 \in C = C_1 = H_1$, $\{\theta_n\} \subset (0,1)$ be a real sequence and $r_n > 0$. Assuming the $(n-1)$th and nth iterates have been constructed, we calculate the next iterate $(n+1)$th via the formula*

$$\begin{cases} w_n = J^q_{E_1^*}\left[J^p_{E_1}(x_n) + \theta_n(J^p_{E_1}(x_n) - J^p_{E_1}(x_{n-1}))\right], \\ v_n = \Pi_C J^q_{E_1^*}\left[J^p_{E_1}(w_n) - \mu_n A^* J^p_{E_2}(I - Q^B_{r_n})Aw_n\right], \\ u_n = J^q_{E_1^*}\left[\alpha_n J^p_{E_1}(v_n) + (1-\alpha_n)J^p_{E_1}(Tv_n)\right], \\ C_n = \{u \in E_1 : \Delta_p(u, u_n) \leq \Delta_p(u, w_n)\}, \\ H_n = \{u \in E_1 : \langle x_n - u, J^p_{E_1}(x_1) - J^p_{E_1}(x_n)\rangle \geq 0\}, \\ x_{n+1} = \Pi_{C_n\cap H_n} x_1, \quad \forall n \in \mathbb{N}, \end{cases} \tag{3.1}$$

where μ_n is a positive number satisfying

$$\mu_n^{q-1} = \begin{cases} \frac{q\|(I-Q^B_{r_n})Aw_n\|^p}{D_q\|A^*J^p_{E_2}(I-Q^B_{r_n})Aw_n\|^q}, & \text{if } Aw_n \neq Q^B_{r_n}Aw_n, \\ \epsilon, & \text{if } Aw_n = Q^B_{r_n}Aw_n, \end{cases} \tag{3.2}$$

for any $\epsilon > 0$.

Note that the step size defined in (3.2) does not require a prior knowledge or estimate of the operator norm $\|A\|$. This is very important because in practice, it is very difficult to estimate the norm of bounded linear operators (for simple estimate, see [46]).
Next, we prove some necessary results which will be used to establish our main theorem.
First, we show that the sequence $\{x_n\}$ generated by Algorithm (3.1) is well-defined.

Lemma 3.2. *Let $\{x_n\}$ be generated by* (3.1)*, then $\{x_n\}$ is well-defined.*

Proof . We need to show that $C_n \cap H_n$ is a non-empty closed and convex set $\forall n \geq 1$. It is obvious that H_n is closed and convex while C_n is closed. So we show that C_n is also convex. Observe that

$$\Delta_p(u, u_n) \leq \Delta_p(u, w_n)$$

is equivalent to

$$\langle J^p_{E_1}(w_n) - J^p_{E_1}(u_n), u\rangle \leq \frac{1}{q}\left(\|w_n\|^p - \|u_n\|^p\right).$$

Hence C_n is a half space and so convex. This implies that $C_n \cap H_n$ is closed and convex for $n \in \mathbb{N}$. Furthermore, we need to show that $C_n \cap H_n$ is non-empty. It is sufficient to show that $\Gamma \subset C_n \cap H_n$. Let $x^* \in \Gamma$, then

$$\begin{aligned}\Delta_p(x^*, u_n) &= \Delta_p\left(x^*, J^q_{E_1^*}\left[\alpha_n J^p_{E_1} v_n + (1-\alpha_n)J^p_{E_1} T v_n\right]\right)\\ &\leq \alpha_n \Delta_p(x^*, v_n) + (1-\alpha_n)\Delta_p(x^*, T v_n)\\ &\leq \alpha_n \Delta_p(x^*, v_n) + (1-\alpha_n)\Delta_p(x^*, v_n)\\ &= \Delta_p(x^*, v_n).\end{aligned} \tag{3.3}$$

Also from Lemma 2.2 and (2.9), we have

$$\begin{aligned}\Delta_p(x^*, v_n) &= \Delta_p\left(x^*, \Pi_C J^q_{E_1^*}\left[J^p_{E_1}(w_n) - \mu_n A^* J^p_{E_2}(I - Q^B_{r_n})Aw_n\right]\right)\\ &\leq \Delta_p\left(x^*, J^q_{E_1^*}\left[J^p_{E_1}(w_n) - \mu_n A^* J^p_{E_2}(I - Q^B_{r_n})Aw_n\right]\right)\\ &= V_p(x^*, [J^p_{E_1}(w_n) - \mu_n A^* J^p_{E_2}(I - Q^B_{r_n})Aw_n])\\ &= \frac{\|x^*\|^p}{p} - \langle J^p_{E_1} w_n, x^*\rangle + \langle \mu_n A^* J^p_{E_2}(I - Q^B_{r_n})Aw_n, x^*\rangle\\ &+ \frac{1}{q}\|J^p_{E_1} w_n - \mu_n A^* J^p_{E_2}(I - Q^B_{r_n})Aw_n\|^q\\ &\leq \frac{\|x^*\|^p}{p} - \langle J^p_{E_1} w_n, x^*\rangle + \mu_n\langle J^p_{E_2}(I - Q^B_{r_n})Aw_n, Ax^*\rangle + \frac{1}{q}\|J^p_{E_1} w_n\|^q\\ &- \mu_n\langle J^p_{E_2}(I - Q^B_{r_n})Aw_n, Aw_n\rangle + \frac{D_q\mu_n^q}{q}\|A^* J^p_{E_2}(I - Q^B_{r_n})Aw_n\|^q\\ &= \frac{\|x^*\|^p}{p} - \langle J^p_{E_1} w_n, x^*\rangle + \frac{1}{q}\|J^p_{E_1} w_n\|^q + \mu_n\langle J^p_{E_2}(I - Q^B_{r_n})Aw_n, Ax^* - Aw_n\rangle\\ &+ \frac{D_q\mu_n^q}{q}\|A^* J^p_{E_2}(I - Q^B_{r_n})Aw_n\|^q\\ &= \Delta_p(x^*, w_n) + \mu_n\langle J^p_{E_2}(I - Q^B_{r_n})Aw_n, Ax^* - Q^B_{r_n}Aw_n + Q^B_{r_n}Aw_n - Aw_n\rangle\\ &+ \frac{D_q\mu_n^q}{q}\|A^* J^p_{E_2}(I - Q^B_{r_n})Aw_n\|^q\\ &= \Delta_p(x^*, w_n) + \mu_n\langle J^p_{E_2}(I - Q^B_{r_n})Aw_n, Ax^* - Q^B_{r_n}Aw_n\rangle\\ &- \mu_n\langle J^p_{E_2}(I - Q^B_{r_n})Aw_n, Aw_n - Q^B_{r_n}Aw_n\rangle + \frac{D_q\mu_n^q}{q}\|A^* J^p_{E_2}(I - Q^B_{r_n})Aw_n\|^q.\end{aligned}$$

From (2.15), we have

$$\begin{aligned}\Delta_p(x^*, v_n) &\leq \Delta_p(x^*, w_n) - \mu_n\langle J_{E_2}^p(I - Q_{r_n}^B)Aw_n, (I - Q_{r_n}^B)Aw_n\rangle \\ &\quad + \frac{D_q\mu_n^q}{q}\|A^*J_{E_2}^p(I - Q_{r_n}^B)Aw_n\|^q \\ &= \Delta_p(x^*, w_n) - \mu_n\left\{\|(I - Q_{r_n}^B)Aw_n\|^p - \frac{D_q\mu_n^{q-1}}{q}\|J_{E_1}^p(I - Q_{r_n}^B)Aw_n\|^q\right\}. \qquad (3.4)\end{aligned}$$

Hence, from (3.2), we have

$$\Delta_p(x^*, v_n) \leq \Delta_p(x^*, w_n).$$

This implies that

$$\Delta_p(x^*, u_n) \leq \Delta_p(x^*, w_n).$$

So $\Gamma \subset C_n$, for all $n \in \mathbb{N}$. Since $x_{n+1} = \Pi_{C_n \cap H_n} x_1$, then $\langle J_{E_1}^p x_1 - J_{E_1}^p x_{n+1}, v - x_{n+1}\rangle \leq 0$, $\forall v \in C_n \cap H_n \subset C$. In particular, for $x^* \in \Gamma$, we have $\langle J_{E_1}^p x_1 - J_{E_1}^p x_{n+1}, x^* - x_{n+1}\rangle \leq 0$. This implies that $\Gamma \subset H_n$ for all $n \in \mathbb{N}$. So we obtain that $\Gamma \subset C_n \cap H_n$ for all $n \in \mathbb{N}$. Therefore, $C_n \cap H_n$ is non-empty and thus $x_{n+1} = \Pi_{C_n \cap H_n} x_1$ is well-defined. □

Lemma 3.3. *Let $\{x_n\}$ be a sequence generated by Algorithm 3.1. Then*

(i) $\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0$,

(ii) $\lim_{n\to\infty} \|x_n - w_n\| = 0$,

(iii) $\lim_{n\to\infty} \|Tv_n - v_n\| = 0$,

(iv) $\lim_{n\to\infty} \|x_n - v_n\| = 0$,

(v) $\lim_{n\to\infty} \|A^*J_{E_2}^p(I - Q_{r_n}^B)Aw_n\| = 0$.

Proof . (i) Let $w \in \Gamma$. Since $\Gamma \subset C_n \cap H_n$, $\forall n \geq 1$ and $x_{n+1} = \Pi_{C_n \cap H_n} x_1$, it follows that

$$\Delta_p(x_{n+1}, x_1) \leq \Delta_p(w, x_1), \forall\ n \geq 1.$$

Thus $\{\Delta_p(x_{n+1}, x_1)\}$ is bounded.
We observe that $x_{n+1} \in H_n$ and by (2.6) we have

$$\langle J_{E_1}^p(x_n) - J_{E_1}^p(x_1), x_n - x_{n+1}\rangle \leq 0,$$

also by (2.7) we have

$$\Delta_p(x_{n+1}, x_n) \leq \Delta_p(x_{n+1}, x_1) - \Delta_p(x_n, x_1),\ \forall n \geq 1, \qquad (3.5)$$

which implies that

$$\Delta_p(x_n, x_1) \leq \Delta_p(x_{n+1}, x_1) - \Delta_p(x_{n+1}, x_n).$$

Thus

$$\Delta_p(x_n, x_1) \leq \Delta_p(x_{n+1}, x_1),$$

therefore, $\{\Delta_p(x_n, x_1)\}$ is a bounded monotone nondecreasing sequence. Hence, $\lim_{n\to\infty} \{\Delta_p(x_n, x_1)\}$ exists.

From (3.5), we have $\lim_{n\to\infty} \Delta_p(x_{n+1}, x_n) = 0$. Thus, using Lemma 2.8

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0. \tag{3.6}$$

(ii) Since $J^p_{E_1}$ is uniformly continuous on bounded subsets of E_1, we have from (3.6) that

$$\lim_{n\to\infty} \|J^p_{E_1}(x_{n+1}) - J^p_{E_1}(x_n)\| = \lim_{n\to\infty} \|J^p_{E_1}(x_n) - J^p_{E_1}(x_{n-1})\| = 0.$$

From (3.1), we have

$$w_n = J^q_{E_1^*}\left(J^p_{E_1}(x_n) + \theta_n(J^p_{E_1}(x_n) - J^p_{E_1}(x_{n-1}))\right)$$

then

$$J^p_{E_1} w_n = J^p_{E_1} x_n + \theta_n(J^p_{E_1}(x_n) - J^p_{E_1}(x_{n-1}),$$

which gives

$$\|J^p_{E_1}(w_n) - J^p_{E_1}(x_n)\| = |\theta_n|\|J^p_{E_1}(x_n) - J^p_{E_1}(x_{n-1})\|.$$

Therefore

$$\lim_{n\to\infty} \|J^p_{E_1}(w_n) - J^p_{E_1}(x_n)\| = 0.$$

Since $J^q_{E_1^*}$ is also uniformly continuous on bounded subsets of E_1^*, we have

$$\lim_{n\to\infty} \|x_n - w_n\| = 0. \tag{3.7}$$

(iii) From (3.6) and (3.7), we obtain

$$\begin{aligned}\|x_{n+1} - w_n\| &= \|x_{n+1} - x_n + x_n - w_n\| \\ &\le \|x_{n+1} - x_n\| + \|x_n - w_n\| \to 0 \text{ as } n \to \infty.\end{aligned}$$

Note that from the construction of C_n, we have that

$$\Delta_p(x_{n+1}, u_n) \le \Delta_p(x_{n+1}, w_n) \to 0 \text{ as } n \to \infty,$$

therefore by Lemma 2.8, we have

$$\lim_{n\to\infty} \|x_{n+1} - u_n\| = 0.$$

Again, since $\|x_n - u_n\| \le \|x_n - x_{n+1}\| + \|x_{n+1} - u_n\|$, it then follows that

$$\lim_{n\to\infty} \|x_n - u_n\| = 0. \tag{3.8}$$

It follows from (3.7) and (3.8) that

$$\lim_{n\to\infty} ||w_n - u_n\| = 0. \tag{3.9}$$

Using Lemma 2.9, we have

$$\begin{aligned}
\Delta_p(x^*, u_n) &= \Delta_p\left(x^*, J^q_{E_1^*}\left[\alpha_n J^p_{E_1} v_n + (1-\alpha_n) J^p_{E_1} T v_n\right]\right) \\
&= V_p\left(x^*, \alpha_n J^p_{E_1} v_n + (1-\alpha_n) J^p_{E_1} T v_n\right) \\
&= \frac{1}{p}\|x^*\|^p - \langle \alpha_n J^p_{E_1} v_n, x^*\rangle - \langle (1-\alpha_n) J^p_{E_1} T v_n, x^*\rangle \\
&\quad + \frac{1}{q}\|\alpha_n J^p_{E_1} v_n + (1-\alpha_n) J^p_{E_1} T v_n\|^q \\
&\leq \frac{1}{p}\|x^*\|^p - \alpha_n \langle J^p_{E_1} v_n, x^*\rangle - (1-\alpha_n)\langle J^p_{E_1} T v_n, x^*\rangle \\
&\quad + \frac{1}{q}\alpha_n \|v_n\|^p + \frac{(1-\alpha_n)}{q}\|T v_n\|^p - \frac{W_q(\alpha_n)}{q} g\left(\|J^p_{E_1} v_n - J^p_{E_1} T v_n\|\right) \qquad (3.10) \\
&= \alpha_n \frac{1}{p}\|x^*\|^p + (1-\alpha_n)\frac{1}{p}\|x^*\|^p - \alpha_n\langle J^p_{E_1} v_n, x^*\rangle - (1-\alpha_n)\langle J^p_{E_1} T v_n, x^*\rangle \\
&\quad + \frac{1}{q}\alpha_n \|v_n\|^p + \frac{(1-\alpha_n)}{q}\|T v_n\|^p - \frac{W_q(\alpha_n)}{q} g\left(\|J^p_{E_1} v_n - J^p_{E_1} T v_n\|\right) \\
&= \alpha_n \left\{\frac{1}{p}\|x^*\|^p - \langle J^p_{E_1} v_n, x^*\rangle + \frac{1}{q}\|v_n\|^p\right\} \\
&\quad + (1-\alpha_n)\left\{\frac{1}{p}\|x^*\|^p - \langle J^p_{E_1} T v_n, x^*\rangle + \frac{1}{q}\|T v_n\|^p\right\} \\
&\quad - \frac{W_q(\alpha_n)}{q} g\left(\|J^p_{E_1} v_n - J^p_{E_1} T v_n\|\right) \\
&= \alpha_n \Delta_p(x^*, v_n) + (1-\alpha_n)\Delta_p(x^*, T v_n) - \frac{W_q(\alpha_n)}{q} g\left(\|J^p_{E_1} v_n - J^p_{E_1} T v_n\|\right) \\
&\leq \alpha_n \Delta_p(x^*, v_n) + (1-\alpha_n)\Delta_p(x^*, v_n) - \frac{W_q(\alpha_n)}{q} g\left(\|J^p_{E_1} v_n - J^p_{E_1} T v_n\|\right) \\
&= \Delta_p(x^*, v_n) - \frac{W_q(\alpha_n)}{q} g\left(\|J^p_{E_1} v_n - J^p_{E_1} T v_n\|\right) \\
&\leq \Delta_p(x^*, w_n) - \frac{W_q(\alpha_n)}{q} g\left(\|J^p_{E_1} v_n - J^p_{E_1} T v_n\|\right). \qquad (3.11)
\end{aligned}$$

Hence, from (2.4) and (2.5), we get

$$\begin{aligned}
\frac{W_q(\alpha_n)}{q} g\left(\|J^p_{E_1} v_n - J^p_{E_1} T v_n\|\right) &\leq \Delta_p(x^*, w_n) - \Delta_p(x^*, u_n) \\
&= \Delta_p(u_n, w_n) + \langle J^p_{E_1} x^* - J^p_{E_1} u_n, u_n - w_n\rangle \\
&\leq \langle J^p_{E_1} u_n - J^p_{E_1} w_n, u_n - w_n\rangle + \langle J^p_{E_1} x^* - J^p_{E_1} u_n, u_n - w_n\rangle \\
&= \langle J^p_{E_1} x^* - J^p_{E_1} w_n, u_n - w_n\rangle. \qquad (3.12)
\end{aligned}$$

From (3.9), we have

$$\lim_{n\to\infty} \frac{W_q(\alpha_n)}{q} g\left(\|J^p_{E_1} v_n - J^p_{E_1} T v_n\|\right) = 0,$$

which implies

$$\lim_{n\to\infty} g\left(\|J^p_{E_1} v_n - J^p_{E_1} T v_n\|\right) = 0.$$

By the property of mapping g, we obtain

$$\lim_{n\to\infty} \|J_{E_1}^p v_n - J_{E_1}^p T v_n\| = 0.$$

Since $J_{E_1^*}^q$ is uniformly continuous on bounded subsets of E_1^*, we have

$$\lim_{n\to\infty} \|Tv_n - v_n\| = 0. \tag{3.13}$$

(iv) From Algorithm 3.1, we have that,

$$J_{E_1}^P u_n - J_{E_1}^P v_n = (1-\alpha_n)(J_{E_1}^P T v_n - J_{E_1}^P v_n).$$

Since $0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty} \alpha_n < 1$ and from (3.13), we have

$$\lim_{n\to\infty} \|J_{E_1}^P u_n - J_{E_1}^P v_n\| = 0,$$

Hence

$$\lim_{n\to\infty} \|u_n - v_n\| = 0.$$

Since $\|w_n - v_n\| \leq \|w_n - u_n\| + \|u_n - v_n\|$, then from (3.9) we have

$$\lim_{n\to\infty} \|w_n - v_n\| = 0. \tag{3.14}$$

Therefore, from (3.7), we have

$$\lim_{n\to\infty} \|v_n - x_n\| = 0.$$

(v) From (3.4), we have

$$\begin{aligned}
\mu_n \left\{ \|(I - \mathrm{Q}_{r_n}^B)Aw_n\|^p - \frac{D_q \mu_n^{q-1}}{q} \|A^* J_{E_2}^p (I - \mathrm{Q}_{r_n}^B)Aw_n\|^q \right\} &\leq \Delta_p(x^*, w_n) - \Delta_p(x^*, v_n) \\
&= \Delta_p(v_n, w_n) \\
&\quad + \langle J_{E_1}^p x^* - J_{E_1}^p v_n, v_n - w_n\rangle \\
&\leq \langle J_{E_1}^p v_n - J_{E_1}^p w_n, v_n - w_n\rangle \\
&\quad + \langle J_{E_1}^p x^* - J_{E_1}^p v_n, v_n - w_n\rangle \\
&= \langle J_{E_1}^p x^* - J_{E_1}^p w_n, v_n - w_n\rangle
\end{aligned} \tag{3.15}$$

It follows from (3.14) that

$$\lim_{n\to\infty} \left(\|(I - \mathrm{Q}_{r_n}^B)Aw_n\|^p - \frac{D_q \mu_n^{q-1}}{q} \|A^* J_{E_2}^p (I - \mathrm{Q}_{r_n}^B)Aw_n\|^q \right) = 0. \tag{3.16}$$

From the choice of μ_n in (3.2), we have

$$\mu_n^{q-1} < \frac{q\|(I - \mathrm{Q}_{r_n}^B)Aw_n\|^p}{D_q \|A^* J_{E_2}^p (I - \mathrm{Q}_{r_n}^B)Aw_n\|^q} - \epsilon, \tag{3.17}$$

for small $\epsilon > 0$. This implies that

$$\frac{D_q \mu_n^{q-1} \|A^* J_{E_2}^p (I - \mathrm{Q}_{r_n}^B)Aw_n\|^q}{q} < \|(I - \mathrm{Q}_{r_n}^B)Aw_n\|^p - \frac{\epsilon D_q \|A^* J_{E_2}^p (I - \mathrm{Q}_{r_n}^B)Aw_n\|^q}{q}.$$

Then we have

$$\frac{\epsilon D_q\|A^*J_{E_2}^p(I-Q_{r_n}^B)Aw_n\|^q}{q} < \|(I-Q_{r_n}^B)Aw_n\|^p - \frac{D_q\mu_n^{q-1}\|A^*J_{E_2}^p(I-Q_{r_n}^B)Aw_n\|^q}{q}.$$

Therefore from (3.16), we have

$$\lim_{n\to\infty}\frac{\epsilon D_q}{q}\|A^*J_{E_2}^p(I-Q_{r_n}^B)Aw_n\|^q = 0,$$

hence

$$\lim_{n\to\infty}\|A^*J_{E_2}^p(I-Q_{r_n}^B)Aw_n\| = 0. \tag{3.18}$$

Also from (3.16), we have that

$$\lim_{n\to\infty}\|(I-Q_{r_n}^B)Aw_n\| = 0. \tag{3.19}$$

□

Now, we present a strong convergence theorem for solving the SFP (1.9) using Algorithm 3.1.

Theorem 3.4. *Let E_1, E_2 be p-uniformly convex and uniformly smooth Banach spaces, C and Q be non-empty closed convex subsets of E_1 and E_2 respectively, and $A : E_1 \to E_2$ be a bounded linear operator with $A^* : E_2^* \to E_1^*$. Let $T : C \to C$ be a Bregman weak relatively nonexpansive mapping, and $B : E_2 \to 2^{E_2^*}$ be a maximal monotone operator. Suppose $\Gamma = F(T)\cap C\cap A^{-1}(B^{-1}(0)) \neq \emptyset$. Then, the sequence $\{x_n\}$ generated by Algorithm 3.1 converges strongly to $u \in \Gamma$, where $u = \Pi_\Gamma x_1$.*

Proof . We have already shown in Lemma 3.3(i) that $\lim_{n\to\infty}\Delta_p(x_n, x_1)$ exists. Next, we show that $x_n \to \bar{x} \in \Gamma$. Let $m, n \in \mathbb{N}$, then

$$\Delta_p(x_m, x_n) = \Delta_p(x_m, \Pi_{C_{n-1}\cap H_{n-1}}x_1) \leq \Delta_p(x_m, x_1) - \Delta_p(x_n, x_1) \to 0.$$

Therefore by Lemma 2.9, we get that $\|x_m - x_n\| \to 0$ as $m, n \to \infty$. Thus $\{x_n\}$ is a Cauchy sequence in C. Since C is closed and convex, it implies that there exists $\bar{x} \in C$ such that $x_n \to \bar{x}$ as $n \to \infty$. Since $\|x_n - v_n\| \to 0$, $\|Tv_n - v_n\| \to 0$ and T is a Bregman weak relatively nonexpansive mapping, then $\bar{x} \in F(T)$. More so, since $\|x_n - w_n\| \to 0$, then $w_n \to \bar{x}$ and by the linearity of A, we have $Aw_n \to A\bar{x}$. Also from (3.19), $Q_{r_n}^B Aw_n \to A\bar{x}$. Since $Q_{r_n}^B$ is a resolvent metric of B for $r_n > 0$, then for all $n \in \mathbb{N}$, we have

$$\frac{J_{E_2}^P(Aw_n - Q_{r_n}^B Aw_n)}{r_n} \in BQ_{r_n}^B Aw_n.$$

So for all $(s, s^*) \in B$, we have

$$0 \leq \langle s - Q_{r_n}^B Aw_n, s^* - \tfrac{J_{E_2}^P(Aw_n - Q_{r_n}^B Aw_n)}{r_n}\rangle.$$

It follows from (3.19) that for all $(s, s^*) \in B$, we have

$$0 \leq \langle s^* - 0, s - A\bar{x}\rangle.$$

Since B is maximal monotone, then it implies that $A\bar{x} \in B^{-1}(0)$, hence $\bar{x} \in A^{-1}(B^{-1}0)$. Therefore, $\bar{x} \in \Gamma$.

Finally, we show that $\bar{x} = \Pi_\Gamma x_1$. Suppose there exists $\bar{y} \in \Gamma$ such that $\bar{y} = \Pi_\Gamma x_1$. Then

$$\Delta_p(\bar{y}, x_1) \leq \Delta_p(\bar{x}, x_1). \tag{3.20}$$

We have shown in Lemma 3.2 that $\Gamma \subset C_n$ $\forall n \geq 1$, then $\Delta_p(x_n, x_1) \leq \Delta_p(\bar{x}, x_1)$. By the lower semi-continuity of the norm, we have

$$\begin{aligned}\Delta_p(\bar{x}, x_1) &= \frac{\|\bar{x}\|^p}{q} - \langle J^p_{E_1}\bar{x}, x_1\rangle + \frac{\|x_1\|^p}{p} \\ &\leq \liminf_{n\to\infty}\left\{\frac{\|\bar{x}\|^p}{q} - \langle J^p_{E_1}x_n, x_1\rangle + \frac{\|x_1\|^p}{p}\right\} \\ &= \liminf_{n\to\infty}\Delta_p(\bar{x}, x_1). \\ &\leq \limsup_{n\to\infty}\Delta_p(\bar{x}, x_1) \leq \Delta_p(\bar{y}, x_1). \end{aligned} \tag{3.21}$$

Combining (3.20) and (3.21) we have $\Delta_p(\bar{y}, x_1) \leq \Delta_p(\bar{x}, x_1) \leq \Delta_p(\bar{y}, x_1)$. This implies $\bar{x} = \bar{y}$ and $\bar{x} = \Pi_\Gamma x_1$. Hence $x_n \to \bar{x} = \Pi_\Gamma x_1 \in \Gamma$. This completes the proof. □

The following are consequences of our results.
(i) Taking $B = \partial i_Q$ which is a maximal monotone operator, then $Q^B_{r_n} = P_Q$ (the metric projection on Q). Thus, we obtain the following result from Theorem 3.4 which improve the corresponding results of Suantai et al. [49].

Corollary 3.5. *Let E_1, E_2 be p-uniformly convex and uniformly smooth real Banach spaces, C and Q be non-empty closed convex subsets of E_1 and E_2 respectively, and $A : E_1 \to E_2$ be a bounded linear operator with $A^* : E_2^* \to E_1^*$ and let $T : C \to C$ be a Bregman weak relatively nonexpansive mapping. Suppose $\Gamma = SFP \cap F(T) \neq \emptyset$. Then, the sequence $\{x_n\}$ generated by the following algorithm converges strongly to $u \in \Gamma$, where $u = \Pi_\Gamma x_1$.*

Algorithm 3.6. *Let $\{\alpha_n\}$ be a sequence in (0,1), $x_1 \in C = C_1 = Q_1$ and $\{\theta_n\} \subset (0, 1)$ be a real sequence. Assuming the $(n-1)th$ and nth iterates have been constructed, we calculate the next iterate $(n + 1)th$ via the formula*

$$\begin{cases} w_n = J^q_{E_1^*}\left[J^p_{E_1}(x_n) + \theta_n(J^p_{E_1}(x_n) - J^p_{E_1}(x_{n-1}))\right], \\ v_n = \Pi_C J^q_{E_1^*}\left[J^p_{E_1}(w_n) - \mu_n A^* J^p_{E_2}(I - P_Q)Aw_n\right], \\ u_n = J^q_{E_1^*}\left[\alpha_n J^p_{E_1}(v_n) + (1-\alpha_n)J^p_{E_1}(Tv_n)\right], \\ C_n = \{u \in E_1 : \Delta_p(u, u_n) \leq \Delta_p(u, w_n)\}, \\ H_n = \{u \in E_1 : \langle x_n - u, J^p_{E_1}(x_1) - J^p_{E_1}(x_n)\rangle \geq 0\}, \\ x_{n+1} = \Pi_{C_n\cap H_n}x_1, \quad \forall n \in \mathbb{N}, \end{cases} \tag{3.22}$$

where μ_n is a positive number satisfying

$$\mu_n^{q-1} = \begin{cases} \frac{q\|(I-P_Q)Aw_n\|^p}{D_q\|A^*J^p_{E_2}(I-P_Q)Aw_n\|^q}, & \textit{if } Aw_n \neq P_QAw_n, \\ \epsilon, & \textit{if } Aw_n = P_QAw_n, \end{cases} \tag{3.23}$$

for any $\epsilon > 0$.

(ii) Taking $E_1 = H_1$ and $E_2 = H_2$, where H_1 and H_2 are real Hilbert spaces, we obtain the following result which improve the results of Byrne [8].

Corollary 3.7. *Let H_1, H_2 be real Hilbert spaces, C and Q be non-empty closed convex subsets of H_1 and H_2 respectively, and $A : H_1 \to H_2$ be a bounded linear operator. Let $T : C \to C$ be a Bregman weak relatively nonexpansive mapping, and $B : H_2 \to 2^{H_2}$ be a maximal monotone operator. Suppose $\Gamma = F(T) \cap C \cap A^{-1}(B^{-1}(0)) \neq \emptyset$. Then, the sequence $\{x_n\}$ generated by the following algorithm converges strongly to $u \in \Gamma$, where $u = P_\Gamma x_1$.*

Algorithm 3.8. *Let $\{\alpha_n\}$ be a sequence in (0,1), $x_1 \in C = C_1 = Q_1$, $\{\theta_n\} \subset (0,1)$ be a real sequence and $r_n > 0$. Assuming the $(n-1)$th and nth iterates have been constructed, we calculate the next iterate $(n+1)$th via the formula*

$$\begin{cases} w_n = x_n + \theta_n(x_n - x_{n-1}), \\ v_n = P_C(w_n - \mu_n A^*(I - Q^B_{r_n})Aw_n), \\ u_n = \alpha_n v_n + (1-\alpha_n)Tv_n, \\ C_n = \{u \in H_1 : \|u_n - u\|^2 \leq \|w_n - u\|^2, \\ H_n = \{u \in H_1 : \langle x_n - u, x_1 - x_n\rangle \geq 0\}, \\ x_{n+1} = P_{C_n \cap H_n} x_1, \quad \forall n \in \mathbb{N}, \end{cases} \tag{3.24}$$

where μ_n is a positive number satisfying

$$\mu_n = \begin{cases} \frac{2\|(I-Q^B_{r_n})Aw_n\|^2}{\|A^*(I-Q^B_{r_n})Aw_n\|^2}, & \text{if } Aw_n \neq Q^B_{r_n}Aw_n, \\ \epsilon, & \text{if } Aw_n = Q^B_{r_n}Aw_n, \end{cases} \tag{3.25}$$

for any $\epsilon > 0$.

4. Numerical Examples

In this section, we present two numerical examples to compare the performance of our algorithm with some other algorithms in the literature.

Example 4.1. *Let $E_1 = E_2 = \mathbb{R}^m$ and A be a $m \times m$ randomly generated matrix. Let $C = \{x \in \mathbb{R}^m : \langle a, x\rangle \geq b\}$, where $a = (1, -5, 4, 0, \ldots, 0) \in \mathbb{R}^m$ and $b = 1$. Then*

$$\Pi_C(x) = P_C(x) = \frac{b - \langle a, x\rangle}{\|a\|_2^2} a + x.$$

Let $B : \mathbb{R}^m \to 2^{\mathbb{R}^m}$ be defined by $B(x) = \{2x\}$, and $T = P_C$. We take $\theta_n = \frac{3}{7n}$, $r_n = \frac{1}{2n}$, and $\alpha_n = \frac{n}{5n+1}$. Then our Algorithm (3.1) becomes

$$\begin{cases} w_n = x_n + \frac{3}{7n}(x_n - x_{n-1}), \\ v_n = P_C(w_n - \mu_n A^*(I - Q^B_{r_n})Aw_n), \\ u_n = \frac{n}{5n+1} v_n + \frac{4n+1}{5n+1} P_C(v_n), \\ C_n = \{u \in E_1 : \|u_n - u\|^2 \leq \|w_n - u\|^2\}, \\ H_n = \{u \in E_1 : \langle x_n - u, x_1 - x_n \geq 0\}, \\ x_{n+1} = P_{C_n \cap H_n} x_1, \quad \forall n \in \mathbb{N}, \end{cases}$$

where μ_n is chosen as defined by (3.2) *and* $Q^B_{r_n}(Aw_n) = \left(\frac{n}{n+1}\right) Aw_n$ *for all* $n \geq 1$*. We choose various values of* m *as follows:*

$$\text{Case I: } m = 10, \quad \text{Case II: } m = 20, \quad \text{Case III: } m = 50, \quad \text{Case IV: } m = 40,$$

and use $\frac{\|x_{n+1}-x_n\|_2^2}{\|x_2-x_1\|_2^2} < 10^{-6}$ *as the stopping criterion. We thus, plot the graph of* $\|x_{n+1} - x_n\|_2^2$ *against number of iteration in each case and compare the computation results of our algorithm with Algorithm* (1.4) *and* (1.5) *of Alsulami and Takahashi [6]. We found that Algorithm 3.1 performs better in terms of number of iterations and CPU time-taken for computation than both Algorithms* (1.4) *and* (1.5)*. The computation result can be seen in Figure 1 and Table 1.*

Table 1: Computation result for Example 4.1.

		Algorithm 3.1	Algorithm (1.4)	Algorithm (1.5)
Case I	CPU time (sec)	$7.7660e-4$	0.0024	0.0027
$m = 10$	No. of Iter.	10	144	88
Case II	CPU time (sec)	$7.8468e-4$	0.0013	0.0011
$m = 20$	No. of Iter.	10	150	91
Case III	CPU time (sec)	$7.2380e-4$	0.0063	0.0064
$m = 50$	No. of Iter.	10	155	94
Case IV	CPU time (sec)	$7.2747e-4$	0.0057	0.0061
$m = 100$	No. of Iter.	10	159	97

Example 4.2. *In this second example, we consider the infinite-dimensional space and compare our Algorithm* (3.1) *with Algorithms* (1.7) *and* (1.8) *of Suantai et al. [49]. Let* $E_1 = E_2 = E_3 = L^2([0, 2\pi])$ *with norm* $||x||^2 = \int_0^{2\pi} |x(t)|^2 dt$ *and inner product* $\langle x, y\rangle = \int_0^{2\pi} x(t)y(t)dt$, $x, y \in E$*. Suppose* $C := \{x \in L^2([0, 2\pi]) : \int_0^{2\pi}(t^2 + 1)x(t)dt \leq 1\}$ *and* $Q := \{x \in L^2([0, 2\pi]) : \int_0^{2\pi} |x(t) - \sin(t)|^2 \leq 16\}$ *are subsets of* E_1 *and* E_2 *respectively. Define* $A : L^2([0, 2\pi]) \to L^2([0, 2\pi])$ *by* $A(x)(t) = \int_0^{2\pi} \exp^{-st} x(t)dt$ *for all* $x \in L^2([0, 2\pi])$ *and let* $A = \partial i_Q$*, subdifferential of the indicator function on* Q*, then* $Q_{r_nB} = P_Q$*. Let* $T(x)(t) = \int_0^{2\pi} x(t)dt$ *and choose* $\theta_n = \frac{1}{2(n+1)}$ *and* $\alpha_n = \frac{5n}{8n+7}$*. Then our Algorithm* (3.1) *becomes:*

$$\begin{cases} w_n = x_n + \frac{1}{2(n+1)}(x_n - x_{n-1}), \\ v_n = \Pi_C(w_n - \mu_n A^*(I - P_Q)Aw_n), \\ u_n = \frac{5n}{8n+7}v_n + \frac{3n+7}{8n+7}T(v_n), \\ C_n = \{u \in E_1 : \Delta_p(u, u_n) \leq \Delta_p(u, w_n)\}, \\ H_n = \{u \in E_1 : \langle x_n - u, x_1 - x_n \geq 0\}, \\ x_{n+1} = P_{C_n \cap H_n} x_1, \quad \forall n \in \mathbb{N}, \end{cases}$$

where μ_n is chosen as defined by (3.2) *for all* $n \geq 1$*. We choose various values of the initial point as follows:*
Case (i): $x_1 = 2t \exp(5t)$, $x_0 = \frac{t^2}{2}$,
Case (ii): $x_1 = t^2 \cos(2\pi t)$, $x_0 = \exp(2t)$,

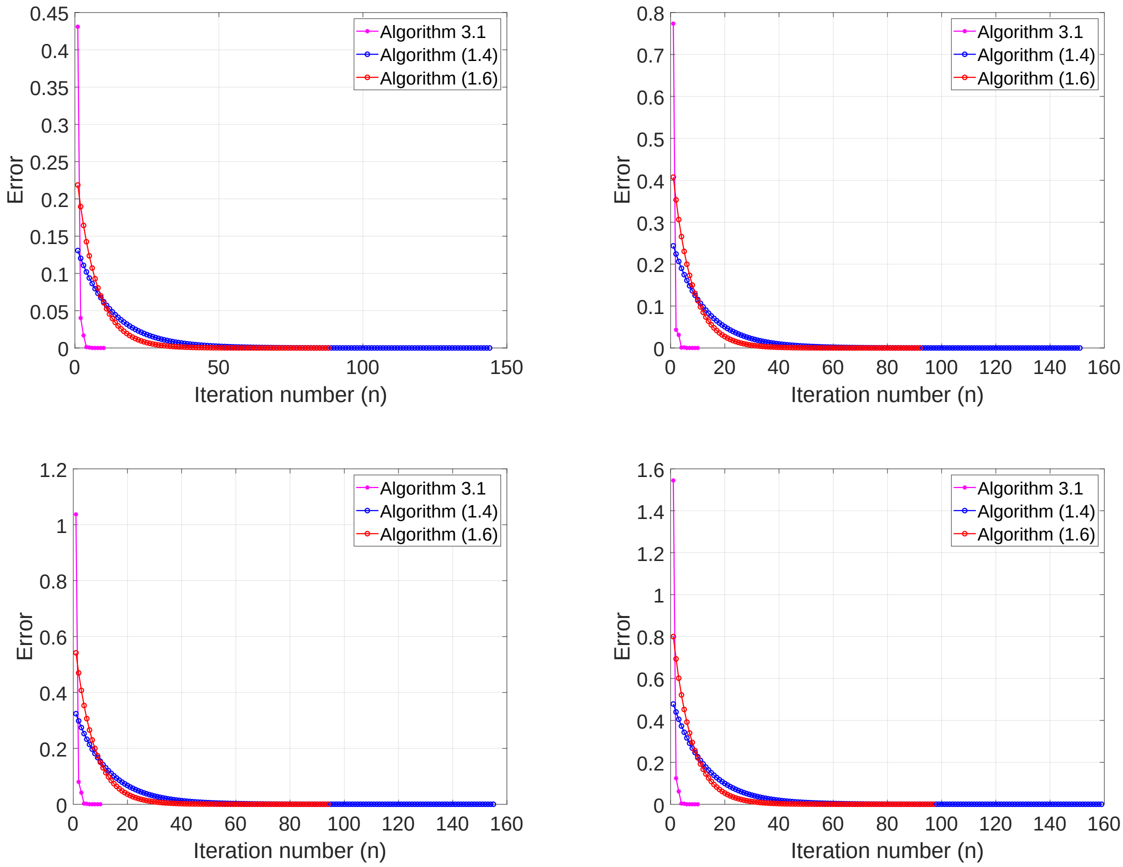

Figure 1: Example 4.1: Top Left Case I; Top Right: Case II; Bottom Left: Case III; Bottom Right: Case IV.

Case (iii): $x_1 = \frac{3}{7}\sin(4t)$, $x_0 = 2t\sin(3t)$,
Case (iv): $x_1 = 5t\cos(2\pi t)$, $x_0 = 2\cos(3\pi t)$.
Using $\frac{\|x_{n+1}-x_n\|^2}{\|x_2-x_1\|^2} < 10^{-4}$ *as stopping criterion, we plot the graph of* $\|x_{n+1} - x_n\|^2$ *against number of iteration and compare the computation results of our algorithm with Algorithm* (1.7) *and* (1.8) *of Suantai et al. [49]. The computational results can be seen in Table 2 and Figure 2.*

Remark 4.3. *From the computation results, it can be inferred that our Algorithm* (3.1) *performs better than Algorithm* (1.7) *and* (1.8) *in terms of number of iterations and cpu-time.*

5. Conclusion

In this paper, we introduced an inertial iterative algorithm for approximating a common solution of split feasibility problem, monotone inclusion problem and fixed point problem for the class of Bregman weak relative nonexpansive mapping in p-uniformly convex and uniformly smooth Banach spaces. Our algorithm is designed in such a way that its implementation does not require a prior information of the norm of the bounded linear operator. We also proved a strong convergence theorem and obtain some consequence results for solving split feasibility problem. We finally give two numerical examples to show the accuracy and efficiency of our algorithm. The results in this paper improve and extend many related results in the literature.

Table 2: Computation result for Example 4.2.

		Algorithm 3.1	Algorithm (1.7)	Algorithm (1.8)
Case I	CPU time (sec)	2.9098	13.5328	3.1893
	No. of Iter.	12	34	25
Case II	CPU time (sec)	2.0835	18.7739	5.8123
	No. of Iter.	10	35	26
Case III	CPU time (sec)	2.3470	7.5584	4.4957
	No. of Iter.	12	34	25
Case IV	CPU time (sec)	2.0908	6.0053	3.0760
	No. of Iter.	10	27	20

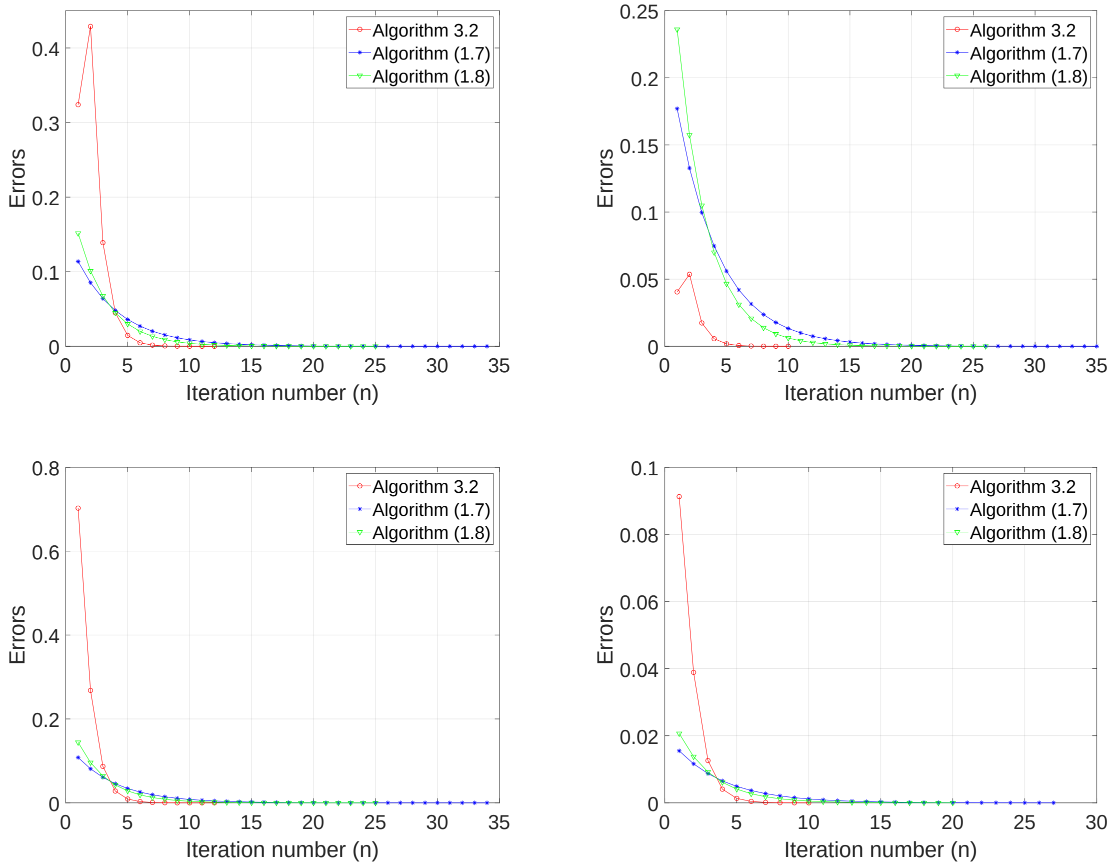

Figure 2: Example 4.1: Top Left Case I; Top Right: Case II; Bottom Left: Case III; Bottom Right: Case IV.

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