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The operations effects on the line graphs of simple graphs

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Abstract

The line graph of the graph Γ denoted by $L(\Gamma)$ is a graph with a vertex set consists of the sets of edges of Γ and two vertices are adjacent in $L(\Gamma)$ if they are incident in Γ. In this article, we discuss and determine the effect of operations on the line graphs of simple graphs.

Keywords: graph, simple, line graph, operations.

1. Introduction

Line graph its properties serve as mathematical models to analyze many particular problems successfully in mathematics, physics, and computer science. One of the most brilliant branches in graph theory is the operations on graphs, which came into existence during the first half of the eighteenth century. The paper written by Leonhard Euler [6] about graph theory and operation on graphs regarded as the first article in this history of graph theory as well as the one written by Vander [3]. A line graph of a simple graph whose vertices are the edges of a graph and where two vertices of the line graph are adjacent if the corresponding edges of the graph are adjacent. This concept was implicitly introduced by Hasler Whitney in [5]. Frank Harrary [9] introduced the operations of graphs and its relations with the graph structure. Norman Biggs [1]) inserted the algebraic properties of line graphs and certain operations of graph in the virtue of linear algebra ([1, 9]).

The following sequel includes basic concepts and definitions which will be used in the article ([9, 1])

Definition 1.1. *A graph Γ is an unordered triple $(V(\Gamma), E(\Gamma), \Psi(\Gamma))$; where $V(\Gamma)$ is a non-empty set of vertices and $E(\Gamma)$ is a set of edges with incident relation $\Psi(\Gamma)$.*

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Definition 1.2. *A simple graph is an undirected graph with no loops or multiple edges.*

Definition 1.3. *Let Γ be a simple graph and let $u \in V(\Gamma)$. The number of edges incident. with u is called degree (or valancy) of u, and if valancy $(u) = k$ for all u in Γ. Then Γ called k- regular.*

Definition 1.4. *A graph Γ is called trivial graph denoted by K, if it consists of one vertex only and it is called edge graph denoted by K_2 if it consists of one edge only.*

Definition 1.5. *A path P_n is a graph consists of one path with n vertices.*

Definition 1.6. *A cycle graph is a graph consists of a closed path with n vertices.*

Definition 1.7. *A star graph $S(1, n)$ is a tree with one root and n end vertices.*

Definition 1.8. *A wheel graph $W(1, n)$ is a graph consists of $S(1, n) \cup C_n$.*

Definition 1.9. *A graph Γ is called a complete graph denoted by K_n if it consists of n vertices in which every vertex adjacent with all other vertices.*

Definition 1.10. *A bigraph $K_{m,n}$ consists of two disjoint sets of vertices and every vertex in the first set adjacent with all vertices in the second set.*

Definition 1.11. *An n-gram graph is a graph consist of n vertices and every vertex adjacent with all other vertices except it's previous and next vertices.*

2. Operations on Graphs

We will introduce the well-known operations on simple graphs, see [9] for more details.

Definition 2.1 (Vertex addition). $\Gamma + \{u\}$ *augmented the vertex (or vertices) with graph Γ.*

Definition 2.2 (Edge addition). $\Gamma + \{e\}$ *adding the edge to the graph Γ, join vertex or two vertices.*

Definition 2.3 (Vertex Removal). $\Gamma - \{u\}$, *removing the vertex (vertices) from the graph Γ and removing all incident edges with the removed vertex.*

Definition 2.4 (Edge removal). $\Gamma - \{u\}$, *removing the edge only with it's ends.*

Definition 2.5 (Graph Union). *The Union of two graphs Γ_1 and Γ_2 denoted by $\Gamma_1 \cup \Gamma_2$ is a graph $V(\Gamma_1 \cup \Gamma_2) = V(\Gamma_1) \cup V(\Gamma_2)$ and $E(\Gamma_1) \cup E(\Gamma_2)$.*

Definition 2.6 (Graph Complement). *Let Γ be a simple graph. Then the complement of graph Γ denoted Γ^c is a graph with $V(\Gamma^c) = V(\Gamma)$ and two vertices are adjacent in Γ^c if and only if they are non-adjacent in Γ and vice versa.*

Definition 2.7 (join of graphs). *Let Γ_1 and Γ_2 be two simple graph , the join of Γ_1 with Γ_2 denoted by $\Gamma_1 + \Gamma_2$ is a graph $\Gamma_1 \cup \Gamma_2$ and every vertex in Γ_2.*

3. Line Graph of well-known Graphs

In the following sequel, we will discuss and investigate line graphs of certain well-known graphs.

Theorem 3.1. *[1] Let P_n be a path. Then $L(P_n) = P_{n-1}$.*

Theorem 3.2. *Let Γ be a star graph $S(1,n)$.Then $L(\Gamma) = K_n$.*

Proof . Let Γ be a star graph $S(1,n)$ so Γ has n edge incident with the roof . Consequently $L(\Gamma)$ consists of n vertices and every vertex adjacent with all other vertices. Hence, the resulted graph is a complete graph with n vertices. □

Theorem 3.3 ([8]). *If $e = (u,v)$ is an edge of graph Γ. Then the valancy of e in $L(\Gamma)$ equals valancy of u +valancy of v -2.*

Theorem 3.4 ([2]). *Let Γ be a k-regular graph. Then $L(\Gamma)$ is $2(k-1)$.*

Theorem 3.5. *Let Γ be a complete graph k_n. Then $L(\Gamma)$ is a graph with $\frac{n(n-1)}{2}$ vertices and $2(n-2)$-regular.*

Proof . Let Γ be a complete graph with n vertices, since Γ is $(n-1)$-regular so Γ has $\frac{n(n-1)}{2}$ vertices and by previous theorem $L(\Gamma)$ is $2(n-2)$-regular. □

Theorem 3.6 ([8]). *The line graph of bipartite graph $K_{m,n}$ is a graph with nm vertices and it is a $(n+m-2)$-regular.*

4. Operations and line graphs

In the following sequel will investigate and determine the effect of certain operations on line graphs.

Theorem 4.1. *Let Γ be a graph. Then $L(\Gamma - u) = L(\Gamma)$.*

Proof . Let Γ be a simple graph and let u be a vertex not in Γ, since addition of a vertex u in Γ has no effect on the set of edges and it's incident relation, so $L(\Gamma + u)$is $L(\Gamma)$. □

Theorem 4.2. *Let Γ be a graph and let e be an edge disjoint from Γ. Then $L(\Gamma + e) = L(\Gamma) \cup \{u\}$.*

Proof . Let Γ be a graph and an edge e not in Γ, so $L(\Gamma + e)$ is $L(\Gamma)$ with an isolated vertex. Then $L(\Gamma + e) = L(\Gamma) \cup \{u\}$; where u represents the edge e. □

Theorem 4.3. *Let Γ be a path graph P_n. Then the line graph of Γ is P_{n-2}.*

Proof . Let Γ be a path graph with n vertices, so removing a vertex of Γ implies the resulted graph is P_{n-1} with $n-2$ edges. Consequently, the line graph of Γ is P_{n-2}. □

Theorem 4.4. *Let Γ be a cycle graph C_n. Then the line graph of Γ a path graph with $n-2$ vertices.*

Proof . Let Γ be a cycle graph with n vertices, so removing a vertex from Γ cause a path graph P_{n-1} so by previous theorem, the line graph of $\Gamma - \{u\}$ is a path graph of P_{n-2}. □

Theorem 4.5. *Let Γ be a complete graph K_n. Then the line graph of Γ is a graph with $\frac{(n^2-3n+2)}{2}$ vertices and $(2n-6)$-regular.*

Proof . Let Γ be a complete graph K_n, so Γ has n vertices and $\frac{n(n-1)}{2}$ edges, removing a vertex u from Γ causes a graph with $(n-1)$ vertices and its $(n-2)$-regular. Consequently, the number of edges in Γ after removing a vertex is $\frac{(n-1)(n-2)}{2}$ vertices and it is $(2(n-2)-2)$-regular, *i.e* $(2n-6)$-regular. □

Theorem 4.6. *Let Γ be a star graph $S(1,n)$, let u be an end vertex. Then let $L(\Gamma-\{u\})$ is a complete graph K_{n-1}.*

Proof . Let Γ be a star graph with $n+1$ vertices, so removing an end vertex caused a star graph with n vertices and $n-1$ edges, all of them have the root as a common vertex, then the resulted line graph is a graph with $n-1$ vertices on which every vertex adjacent with all other vertices. Thus the resulted graph is K_{n-1}. □

Theorem 4.7. *Let Γ be a bigraph $K_{m,n}$ Then, the line graph of $\Gamma-u$ is a graph with $n(m-1)$ vertices and it is$(m+n-3)$-regular.*

Proof . Let Γ be a bigraph $K_{m,n}$, thus removing a vertex from Γ gives a graph with $(m-1)n$ edges, so the line graph of Γ is a graph with $n(m-1)$ vertices and it is $((m-1)+n-2)$- regular. □

Theorem 4.8. *Let Γ_1 and Γ_2 be two graphs. Then $L(\Gamma_1 \cup \Gamma_2) = L(\Gamma_1) U L(\Gamma_2)$.*

Proof . Let Γ_1 and Γ_2 be two disjoint simple graphs, since $E(\Gamma_1 \cup \Gamma_2) = E(\Gamma_1) \cup E(\Gamma_2)$, $V(L(\Gamma_1 \cup \Gamma_2)) = V(L(\Gamma_1)) \cup V(L(\Gamma_2))$. Thus the line graph of $\Gamma_1 \cup \Gamma_2$ is the graph of $L(\Gamma_1) \cup L(\Gamma_2)$. □

Theorem 4.9. *[4] Let Γ be a path graph P_n. Then $|E(L(\Gamma^c))| = |E(K_{n-1})| - |E(P_{n-1})|$*

Theorem 4.10. *The line graph of the complement of the star graph $S(1,n)$, is a graph with $\frac{n(n-1)}{2}$ vertices and it is $2(n-2)$-regular.*

Proof . Let Γ be a star graph $S(1,n)$, the complement of Γ is a complete graph with n vertices and an isolated vertex, thus $L(\Gamma^c)$ is graph with $\frac{n(n-1)}{2}$ vertices and it is $(2(n-1)-2)$-regular. □

Theorem 4.11. *Let Γ be a bigraph $K_{m,n}$. Then the line graph of the complement of Γ is $L(K_m) \cup L(K_n)$.*

Proof . Let Γ be the bigraph $K_{m,n}$, since the complement of Γ consists of $K_m \cup K_n$, $L(\Gamma^c) = L(K_m) \cup L(K_n)$ (previous theorem). □

Theorem 4.12. *Let Γ_1 be k_1- regular with m vertices and Γ_2 be a k_2-regular with n vertices, then $L(\Gamma_1+\Gamma_2)$ is a graph with $\frac{(k_1m+k_2n)}{2}$ vertices and it is (k_1+k_2-2)-regular.*

Proof . Let Γ_1 and Γ_2 be two simple graphs with m vertices and k_2–regular and n vertices with k_2-regular respectively, so the join of Γ_1 and Γ_2 is a graph with $\frac{(k_1m+k_2n)}{2}$ edges. Thus the line graph $\Gamma_1+\Gamma_2$ is a graph with (k_1m+k_2n) vertices and it is (k_1+k_2-2)–regular. □

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