E-small essential submodules

Mamoon F. Khalf\textsuperscript{a,},\textsuperscript{*} Hind Fadhil Abbas\textsuperscript{b}

\textsuperscript{a} Department of Physics, College of Education, University of Samarra, Iraq
\textsuperscript{b} Directorate of Education Salah Eddin, Khaled Ibn Al Walid School, Tikrit, Iraq

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Abstract

Let $R$ be a commutative ring with identity, and $U_R$ be an $R$-module, with $E = \text{End}(U_R)$. In this work we consider a generalization of class small essential submodules namely E-small essential submodules. Where the submodule $Q$ of $U_R$ is said E-small essential if $Q \cap W = 0$, when $W$ is a small submodule of $U_R$, implies that $N_S(W) = 0$, where $N_S(W) = \{ \psi \in E \mid \text{Im}\psi \subseteq W \}$. The intersection $\overline{B}_R(U)$ of each submodule of $U_R$ contained in $\text{Soc}(U_R)$. The $\overline{B}_R(U)$ is unique largest E-small essential submodule of $U_R$, if $U_R$ is cyclic. Also in this paper we study $\overline{B}_R(U)$ and $\overline{W}_E(U)$. The condition when $\overline{B}_R(U)$ is E-small essential, and $\text{Tot}(U, U) = \overline{W}_E(U) = J(E)$ are given.

Keywords: Small submodule, Small essential submodules, E-small essential submodules, Endomorphism ring.

1. Introduction

Throughout this treatise, all ring $R$ is a associative with identity, and all module over a ring $R$ is unitary right module. Let $U_R$ will always denoted such an $R$-module and $E$ is endomorphism ring and denoted by $E = \text{End}(U_R)$ of ring module. The submodule $W$ is called essential of $U_R$ (denoted by: $W \subseteq U_R$ ) if $0 \neq G \subseteq U_R$, then $W \cap G \neq 0$ (see [1]), where the submodule $G$ of $U_R$ is denoted by $(G \leq U_R$).The submodule $Q$ of $U_R$ is said small submodule (denoted by: $Q \ll U_R$), if $\forall W \ll U_R$ then $Q + W = U_R$ (see [2]). The left annihilator of an submodule $Q$ of $U_R$ is denoted by $\delta_E(Q)$, and the right annihilator of an endomorphism $h$ of $U_R$ is denoted by $k_U(h)$, specifically that $\text{Ker}(h)$. We also denoted $N_E(Q) = \{ \theta \in E \mid \text{Im}\theta \subseteq Q \}$ for each $Q \subseteq U$. Nicholson and Zhou defined annihilator-small right ideals [5]. Also Amouzegar and Keskin introduced and study the right annihilator-small submodules of an R-module. Let $U_R$ be an R-module and $F \leq U_R$, then

*Corresponding author

Email addresses: mamoun42@uosamarra.edu.iq (Mamoon F. Khalf), Hind.f1975@gmail.com (Hind Fadhil Abbas)

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F is said to annihilator- small submodule if \( F + W = U \), where \( W \) is a submodule of \( U_R \), so \( \delta_E(W) = 0 \) [4]. From [7], Zhou had and Zhang give a definition of small- essential submodules. Let \( K \) be a submodule of a module \( U_R \), then \( K \) is called small-essential in \( U_R \) (denoted by \( K \trianglelefteq_E U \), if \( K \cap W = 0 \), with \( W \ll U_R \) implies that \( W = 0 \). In this paper we introduced new concept namely E-small essential submodule, where a submodule \( Q \) of a module \( U \) is E-small essential submodule if \( Q \subseteq E - s U \) if and only if \( Q \) is E-small essential submodule of \( U_R \). Let \( U_R \) be a retractable \( R \)-module and \( R \) be a commutative ring. Then \( Q \trianglelefteq E - s U \), if \( Q \subseteq E - s U \) then \( Q \subseteq E - s U \), if \( Q \subseteq E - s U \). The converse is evident.\( \square \)

\[ \text{Corollary 2.3.} \]

If \( U_R \) is retractable \( R \)-module and \( Q \trianglelefteq E - s U \), then \( Q \trianglelefteq E - s U \).

Proposition 2.4. Let \( U_R \) be a cyclic \( \pi \)-projective module. Then \( Q \) is small essential submodule if and only if \( Q \) is E-small essential submodule of \( U_R \).

Proof. Let \( U_R = uR \) for some \( u \in U_R \), and \( Q \trianglelefteq E - s U \). Let \( V \ll U_R \), we put \( 0 \neq v \in V \), so then there exists \( 0 \neq n \in R \), such that \( v = un \), but \( U_R = uR = unR + u(1 - n)R \), since \( U_R \) is \( \pi \)-projective then there exists \( \beta \in \text{End}(U_R) \), with \( \text{Im} \beta \subseteq \text{un}R \subseteq V \), so \( \text{Im}(1 - \beta) \subseteq (1 - n)uR \), that is \( N_E(V) \neq 0 \). As \( Q \trianglelefteq E - s U \), and \( Q \cap V \neq 0 \). That mean \( Q \trianglelefteq E - s U \). The converse is evident.\( \square \)

Proposition 2.5. Let \( U_R \) be a cyclic \( R \)-module and \( R \) be a commutative ring. Then \( Q \trianglelefteq E - s U \) if and only if \( Q \trianglelefteq E - s U \).

Proof. Is evident.\( \square \)

Lemma 2.6. Let \( U_R \) be an \( R \)-module. If \( V \subseteq Q \subseteq U_R \), and \( Q \trianglelefteq E - s U \), then \( V \trianglelefteq E - s U \).

Proof. Is evident.\( \square \)

Proposition 2.7. Let \( U_R \) be an \( R \)-module. If \( Q \trianglelefteq E - s U \) and \( F \subseteq s U \), then \( Q \cap F \trianglelefteq E - s U \).

Proof. Let \( Q \cap F \cap V = 0 \), where \( V \ll U_R \). Since \( F \subseteq s U \), that is \( Q \cap V = 0 \) and \( N_E(V) = 0 \).\( \square \)

Lemma 2.8. Let \( U_R \) be a module, and \( Q \) be a submodule of \( U_R \) if \( N_E(Q) \subseteq E \), then \( N_E(Q)U_R \trianglelefteq E - s U \). In specially, \( Q \trianglelefteq E - s U \).

Proof. Let \( N_E(Q)U_R \cap V = 0 \), so \( N_E(Q) \cap N_E(V) = 0 \), thus \( N_E(V) = 0 \). But \( N_E(Q) \subseteq E \). So that the last perception by (Lemma 2.6) and since \( N_E(Q)U_R \subseteq Q \subseteq U_R \) always achieve.\( \square \)
Note that the converse of Lemma 2.8 is true if \((N_E(Q) \cap vE)U_R = N_E(Q)U_R \cap vU_R\) verified for each submodule \(Q\) of \(U_R\), and all small element \(v \in E\). And to watch it, let \(N_E(Q) \cap vE = 0\), for any small element \(v \in E\). Thus \(N_E(Q)U_R \cap vU_R = 0\), so \(N_E(vU_R) = 0\). But \(N_E(Q)U_R \leq_{E-s} U_R\) and \(vE \subseteq N_E(vU_R) = 0\), then \(v = 0\). Hence \(N_E(Q) \leq_{s} E_E\).

Recall that an \(R\)-module \(U_R\) is called semi-injective if for each \(\alpha \in E\) such that 
\[
E\alpha = \delta_E(ker(\alpha)) = \delta_E(k_U(\alpha))
\]
(equivalently for any monomorphism \(\alpha : Q \rightarrow U\), where \(Q\) is a factor module of \(U_R\), and for any homomorphism \(\beta : Q \rightarrow U\), then there exists \(\gamma : U \rightarrow U\) such that \(\alpha \gamma = \beta\) [5, p. 261].

**Lemma 2.9.** Let us have the following situation for any \(R\)-module \(U_R\) and \(u \in E\):

\(1\) \(k_U(u) \leq_{E-s} U_R\).

\(2\) \(k_U(u) \leq_{E-s} k_U(wr)\) for all \(0 \neq r \in E\).

\(3\) \(k_E(1_E - au) = 0\) for all \(0 \neq a \in E\).

\(4\) \(k_E(1_E - ua) = 0\) for all \(0 \neq a \in E\).

\(5\) \(k_E(u - au) = k_E(u)\) for all \(0 \neq a \in E\).

Then \((1) \implies (2) \implies (3) \implies (4) \implies (5)\). If \(U_R\) is semi-injective, then \((5) \implies (1)\).

**Proof.** \((1) \implies (2)\) Suppose that \(0 \neq r \in E\), and \(k_U(r) = k_U(wr)\). It is clear that \(k_U(u) \cap rU = 0\). According to \(k_U(u) \leq_{E-s} U_R\), and \(N_E(rU) = 0\), so \(rE \subseteq N_E(rU) = 0\). That is \(r = 0\).

\((2) \implies (3)\) Let \(a \in E\), and \(r \in k_E(1_E - au)\), so \(r = aur\), then \(k_U(u) \subseteq k_U(aur) = k_U(r)\). Then by \((2)\), hence \(r = 0\).

\((3) \implies (4)\) Let \(r \in k_E(1_E - ua)\), for all \(a \in E\), thus \((1_E - ua)r = 0\), that mean \((1_E - au)ar = (a - au)a\) for all \(a \in E\), implies that \(ar = 0\) that by \((3)\), then \(r = ur = 0\).

\((4) \implies (5)\) Let \(r \in k_E(1_E - Au)\), for all \(a \in E\), so by \((4) ur = 0\). Then \(r \in k_E(u)\). Other embedding in a similar way.

\((5) \implies (1)\) Suppose that \(U_R\) is semi-injective. Now, let \(k_U(u) \cap V = 0\) for a small submodule \(V\) of \(U_R\), and let \(r \in N_E(V)\), implies that \(rU \cap k_U(u) = 0\), then \(k_U(r) = k_U(wr)\). But \(U_R\) is semi-injective, then there exists ahomomorphism \(v \in E\) such that \(r = urv\), so \((u - uv) = 0\). Thus \(r \in (u - uuv) = k_E(u)\), then \(ur = 0\), and hence \(r = 0\).

Note that we us define \(\overline{W}_E(U) = \{u \in E \mid ker(v) = k_U(u) \leq_{E-s} U_R\}\) for any module \(U_R\).

**Corollary 2.10.** Let \(U_R\) be a module, and \(u \in \overline{W}_E(U)\). Thus \(Eu \subseteq \overline{W}_E(U)\). If \(U_R\) is semi-injective, then \(uE \subseteq \overline{W}_E(U)\).

**Proof.** Let \(r \in E\), and \(U_R\) is semi-injective, we must show that \(k_U(wr) \leq_{E-s} U_R\). Now let \(v \in E\), since \(k_U(u) \leq_{E-s} U_R\), then by Lemma 2.9(4) \(k_E(1_E - urv) = 0\). Once again form Lemma 2.9(4) \(k_U(urv) \leq_{E-s} U_R\). Thus \(uE \subseteq \overline{W}_E(U)\). Now through the Lemma ??, we get \(Eu \subseteq \overline{W}_E(U)\).

**Corollary 2.11.** We own \(\overline{W}_E(U) \subseteq \delta_E(Soc(E_R))\). Furthermore, \(J(E) \subseteq \overline{W}_E(U)\), if \(U_R\) is a semi-injective.

**Proof.** Let \(w \in \overline{W}_E(U)\), and \(0 \neq u \in Soc(E_R)\), we want to prove that \(0 = wSoc(E_R)\). Now \(u \in E_1 \oplus E_2 \oplus \cdots \oplus E_n\), where \(E_1, E_2, \ldots, E_n\) are simple right ideal of \(E\), and \(n\) is positive integer. Suppose that \(wu \neq 0\) and \(u = u_1 + u_2 + \cdots + u_n\) where as \(u_j \in E_j\) for some \(j \in \{1, 2, \ldots, n\}\), then \(wu_j \neq 0\). As \(E_j\) is simple so \(Euw_j = E_j\). Thus \(u_j = \beta wu_j\) for all \(\beta \in E\). So \(u_j \in k_E(1_E - \beta w)\), but \(k_U(w) \leq_{E-s} U_R\), then from Lemma 2.9 \(k_E(1_E - \beta w) = 0\), that is \(u_j = 0\). This is contradicition. So \(wu = 0\), hence \(\overline{W}_E(U) \subseteq \delta_E(Soc(E_R))\). Now let \(v \in J(E)\) and \(w \in E\). We must prove that \(v \in \overline{W}_E(U)\), we take \(\beta \in k_E(1_E - wv)\). Thus \((1_E - wv) = 0\), but \(1_E - wv\) is invertible, so \(\beta = 0\). Then \(k_E(1_E - wv) = 0\) for all \(w \in E\). Hence from Lemma 2.9 \(v \in \overline{W}_E(U)\), implies that \(J(E) \subseteq \overline{W}_E(U)\).
Corollary 2.12. Let $U_R$ is a semi-injective module and $h \in E$. Then $\text{Ker} h = k_U(\{u\}) \leq_{E-s} U_R$ if and only if $Eh \leq_a E_E$.

**Proof.** Let $h \in E$ and suppose that $k_U(h) \leq_{E-s} U_R$. Now let $E = Eh + P$, where $P$ is an ideal of $E$. So $1_E = rh + q$, where $r \in E$ and $q \in P$, then $k_U(h) \cap k_U(q) = 0$. But $k_U(h) \leq_{E-s} U_R$, then $N_E(k_U(q)) = 0$. That is $N_E(k_U(P)) = 0$, hence $k_U(P) = 0$ implies that $Eh \leq_a E_E$. The converse, suppose $Eh \leq_a E_E$, then from (4) $k_E(h - hrh) = k_E(h)$, for all $r \in E$. Then from Lemma 2.9 $k_U(h) \leq_{E-s} U_R$. □

Corollary 2.13. Let $U_R$ be an $R$-module. If $h^2 = h \in \overline{W}_E(U)$, then $h = 0$.

**Proof.** We can see from the lemma 2.9 (4) and $k_U(h) \leq_{E-s} U_R$, $k_E(1_E - h) = 0$, and since $h \in k_E(1_E - h)$. Implies that $h = 0$. □

Corollary 2.14. Let $P$ be an maximal-ideal of $E$, where $E = \text{End}(U_R)$ and $U_R$ be amodule. Then the following ferries are equivalent:

1. $PU \leq_{E-s} U_R$
2. $P \leq_e E_E$

**Proof.** (1) $\Rightarrow$ (2) Let $PU \leq_{E-s} U_R$ Suppose that $P$ is not essential of $E_E$. Then $P \cap K = 0$, for some $K$ is a non-zero ideal of $E_E$. But $P$ is a maximal ideal, that mean $P$ is direct summand of $E_E$.

So there exists idempotent element $i \in E_E$ such that $P = iE$. Then $PU = iU = k_E(1_E - i) \leq_{E-s} U_R$. Hence $1_E - i \in \overline{W}_E(U)$. Then from (Corollary 2.13) $i = 1$. This is a contradiction.

(2) $\Rightarrow$ (1) Let $P \leq_e E_E$, and $PU \cap V = 0$ for an small submodule $V$ of $U_R$. So $0 = N_E(0) = N_E(PU) \cap N_E(V)$. Then $P \cap N_E(V) = 0$. But $P \leq_e E_E$, then $N_E(V) = 0$. □

Recall that the element $h$ in $E$ is called to be partially invertible if $hE$ contains an non-zero idempotent, where ($hE$ equivalent $Eh$). Where an $R$-module $U_R$ the total of $U_R$ is defined as $\text{Tot}(E) = \text{Tot}(U,U) = \{h \in E|h$ is not partially invertible). Unable to closed the total under addition. In effect, if 0 and 1 are the only idempotent in $E$, then the total of $U_R$ is the set of non-isomorphism.

Proposition 2.15. Let $U_R$ be a module. Then $\overline{W}_E(U) \subseteq \text{Tot}(U,U)$.

**Proof.** If $h \in \overline{W}_E(U)$ and $h \notin \text{Tot}(U,U)$, implies that $h$ is partially invertible then there exists $0 \neq i^2 = i \in Eh$. So by (Corollary 2.10), $i \in \overline{W}_E(U)$. Thus contradicts to (Corollary 2.13). □

Let $P$ is a subset of a ring $R$, then $R$ is called to be $P$-semi-potent if every ideal not contained in $P$ contains an non-zero idempotent, equivalently if every element $q \notin P$ is a partial inverse $R$ is said to be semi-potent if $R$ is $J(R)$-semi-potent.

Lemma 2.16. Let $U_R$ be a module, if $P$ is a subset of $E = \text{End}(U_R)$. Then the following ferries are equivalent:

1. $E$ is $P$-semi-potent.
2. $\text{Tot}(U,U) \subseteq P$.

**Proof.** Is evident from (5), Lemma 20. □

Proposition 2.17. Let $E = \text{End}(U_R)$ for any $R$-module $U_R$. Then $E$ is a semi-potent if and only if $J(E) = \text{Tot}(U,U)$.

**Proof.** Is evident from (5), Theorem 21. □
Proof. It is evident that \( J(E) \subseteq \overline{W}_E(U) \) by (Corollary 2.11). Let \( u \in \overline{W}_E(U) \), if \( u \notin J(E) \) and \( E \) is \( J(E) \)-semi-potent, then \( \overline{W}_E(U) \) have an non-zero idempotent which is a contradiction (we can see corollary 2.13). Then \( J(E) = \overline{W}_E(U) \). Now from Proposition 2.15 \( \overline{W}_E(U) \subseteq \text{Tot}(U, U) \). From other hand, \( E \) is \( \overline{W}_E(U) \)-semi-potent and since \( J(E) = \overline{W}_E(U) \). Hence form Lemma 2.16 \( \text{Tot}(U, U) \subseteq \overline{W}_E(U) \). □

Proposition 2.19. Let \( U_R \) be asemi- injective R-module, and \( E = \text{End}(U_R) \) is a semi-potent. Then \( \overline{W}_E(U) = J(E) = \text{Tot}(U, U) \).

Proof. It is clear that from Corollary 2.11 \( J(E) \subseteq \overline{W}_E(U) \). Let \( x \in \overline{W}_E(U) \), then \( k_U(x) \leq_{E-x} U_R \), hence \( k_E(1_E - u_x) = 0 \), for all \( u \in E \), so from Lemma 2.9 then \( E(1_E - u_x) = E \), thus by hypothesis \( x \in J(E) \). Implies \( \overline{W}_E(U) \subseteq J(E) \). □

A ring \( R \) is said to be right Kasch if every simple right \( R \)-module embeds in \( R \), this is rewarding, if \( k_R(V) \neq 0 \) for every maximal right ideal \( E \) of \( R \). Associated \( R \) alleft ideal \( W_2 \) ring if every left ideal is isomorphic to direct of \( R \) itself is a direct summand of \( R \).

Lemma 2.20. Let \( U_R \) be asemi- injective R-module. In each of the following statements, we have \( \overline{W}_E(U) = J(E) \).

1. \( E \) is semi-potent.
2. \( E \) is right Kasch.
3. \( E \) is a left \( W_2 \) ring.

Proof.

1. Is evident from Proposition 2.18
2. Let \( u \in E \), then \( k_E(u) = 0 \). If \( uE \neq E \), then by (2) \( k_E(uE) = 0 \), that is \( k_E(u) \neq 0 \). This is a contradiction. Hence from Proposition 2.19 \( \overline{W}_E(U) = J(E) \).
3. Let \( v \in E \), then \( k_E(v) = 0 \). If \( Ev = E \), then by (3) \( Ev \) is a direct summand of \( E \), so \( vxx = v \), for some element \( x \in E \). Since \( 0 = k_E(v) = k_E(vx) = E(1_E - vx) \). Hence \( vx = 1_E \) and \( vE = E \), from Proposition 2.19 \( \overline{W}_E(U) = J(E) \).

□

Lemma 2.21. Let \( u = uR \), where \( u \in U \), and \( U \) be a cyclic R-module. Then the following are equivalent for \( w \in U \):

1. \( wR \leq_{E^-} U \)
2. \( g(wR) \subseteq f(U) \), for all \( g \in E \)
3. \( k_E(u - wn) = 0 \), for all \( n \in R \).

Proof. (1) \( \Rightarrow \) (2) Let \( g(wR) = g(U) \), then \( g(wn) = g(u) \), for all \( n \in R \), hence \( g \in k_E(u - wn) \). But \( wR + (u - wn)R = uR = U \), then by (1) \( k_E(u - wn) = 0 \). Therefore \( g = 0 \).

(2) \( \Rightarrow \) (3) Let \( g \in k_E(u - wn) \), for all \( n \in R \), so \( g(u) = g(wn) \subseteq g(wR) \) by (2). Therefore \( g = 0 \).

(3) \( \Rightarrow \) (1) If \( wR + V = U \), where \( V \) is small submodule of \( U_R \), then \( u = wn + v \), for all \( n \in R \) and \( v \in V \). Now let \( g \in k_E(V) \) that mean \( g(u) = g(wn) \). Hence by (3) \( g \in k_E(u - wn) = 0 \). Therefore \( g = 0 \). □

Note: Let \( U_R \) be a module, we can defined \( \overline{B}_R(U) = \cap \{ D \subseteq U_R | D \leq_{E^-} U_R \} \). It is clearly that \( \overline{B}_R(U) \subseteq \text{Soc}(U) \).
Proposition 2.22. If $U_R$ is an retractable and semi-projective $R$-module, then $\overline{R}(U) = \text{Soc}(U) = \text{Soc}(E_E U)$.

Proof. From Corollary 2.3, $\overline{R}(U) = \text{Soc}(U)$. Since $U_R$ is semi-projective, then from (3), Proposition 2.4), $\overline{R}(U) = \text{Soc}(U) = \text{Soc}(E_E U)$.

Let $U_R$ be an $R$-module, an element $c \in U_R$ is called $E$-small essential if $cR \trianglelefteq_{E-s} U_R$. For simplicity, we denoted $C_R(U) = \{c \in U | c \text{ is an } E \text{-small essential in } U\} = \{c \in U | cR \trianglelefteq_{E-s} U_R\}$. It is evident that $C_R(U) \subseteq \overline{R}(U)$. $\Box$

Proposition 2.23. Let $U = aR$ be a cyclic $R$-module, and $X$ be a submodule of $U_R$. Then the following are equivalent:

1. $X \trianglelefteq_{E-s} U_R$
2. $X \subseteq C_R(U)$
3. $k_E(u - a) = 0$, for all $a \in R$.

Proof. (1) $\implies$ (2) Fore Proposition 2.7.

(2) $\implies$ (3) Let $X + Y = U$, where $Y$ is small submodule of $U_R$, $u = x + y$, for all $x \in X$ and $y \in Y$, then $k_E(Y) \subseteq k_E(u - x) = 0$.

(3) $\implies$ (1) According to the hypothesis (3). Therefore $X \trianglelefteq_{E-s} U_R$. $\Box$

Proposition 2.24. Let $U_R$ be an $R$-module, then

1. $\overline{R}(U) = \{c_1 + c_2 + \cdots + c_n | c_j \in C_R(U) \text{ for each } n, j \text{ are positive integer}\}$
2. $\overline{R}(U) = C_R(U) R$.

Proof. (1) Let the set $F = \{c_1 + c_2 + \cdots + c_n | c_j \in C_R(U) \text{ for each } n, j \text{ are positive integer}\}$. If $c \in \overline{R}(U)$, then $c \in F_1 + F_2 + \cdots + F_n$, where $F_j \trianglelefteq_{E-s} U_R$, for each $n, j$ are positive integer. If $c = c_1 + c_2 + \cdots + c_n$, $c_j \in F_j$, implies that from Proposition 2.7, $c_j R \trianglelefteq_{E-s} U_R$. Thus $c_j \in C_R(U)$.

Hence $\overline{R}(U) \subseteq F$. Simply we can note that $F \subseteq \overline{R}(U)$.

(2) Evident by fact, $C_R(U) \subseteq \overline{R}(U)$, and by (1). $\Box$

Proposition 2.25. Let $U_R$ be an $R$-module, consider the following expression:

1. If $F \trianglelefteq_{E-s} U_R$ and $H \trianglelefteq_{E-s} U_R$, then $F + H \trianglelefteq_{E-s} U_R$.
2. $C_R(U)$ is closed under addition.
3. $\overline{R}(U) = C_R(U)$.
4. $\overline{R}(U) \trianglelefteq_{E-s} U_R$.

Can we get (1) $\implies$ (2) $\implies$ (3) and (4) $\implies$ (1) .

But (3) $\implies$ (4) , it can obtained by adding if $U_R$ is cyclic $R$-module. In addition, if $U = uR$, where $u \in U$ one of the above-mentioned condition the following:

(i) $\overline{R}(U)$ is the unique largest $E$-small essential of $U$.

(ii) $\overline{R}(U) \subseteq \{u \in U | k_E(a - uw) = 0, \text{ for all } w \in R\}$

(iii) $\overline{R}(U) = \cap \{G \subseteq_{\text{max}} U | \overline{R}(U) \subseteq G\}$

Proof. (1) $\implies$ (2) Since $(u + v) R \subseteq uR + vR$, so $C_R(U)$ is closed under addition by Prop. 2.7.

(2) $\implies$ (3) It is obvious that $C_R(U) \subseteq \overline{R}(U)$, then from Proposition 2.24 (1), $\overline{R}(U) \subseteq C_R(U)$.

(3) $\implies$ (4) Let $U = uR$, for some $u \in U$, and $\overline{R}(U) + F = U$, where $F$ is a small submodule of $U_R$. Thus by (3) $C_R(U) + F = U$. If $u = v + w$, where $v \in C_R(U)$ and $w \in F$. Thus $U = vR + F$, so $vR \trianglelefteq_{E-s} U_R$. Then $k_E(U) = 0$. Hence $\overline{R}(U) \trianglelefteq_{E-s} U_R$. 

Proposition 2.26. Let $U_R$ be a module. consider the following expression:

1. $\overline{B_R}(U) \leq_{E-s} U_R$
2. If $F \leq_{E-s} U_R$ and $H \leq_{E-s} U_R$, then $F \cap H \leq_{E-s} U_R$

Note $(1) \implies (2)$ verified. As well if $U_R$ finitely cogenerated, hence $(2) \implies (1)$

Proof. $(1) \implies (2)$ Let $F \leq_{E-s} U_R$ and $H \leq_{E-s} U_R$, so $\overline{B_R}(U) \subseteq F \cap H$, then from Lemma 2.8 $F \cap H \leq_{E-s} U_R$.

$(2) \implies (1)$ If $U_R$ finitely cogenerated, and let $\overline{B_R}(U) \cap F = 0$, where $F$ is a small submodule of $U_R$, then $F_1 \cap F_2 \cap \cdots \cap F_n \cap H = 0$, for some $E_j \subseteq \overline{B_R}(U)$. Therefore $N_E(H) = 0$. that by $(1)$. □

References