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E-small essential submodules

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Abstract

Let R be a commutative ring with identity, and U_R be an R-module, with $E = End(U_R)$. In this work we consider a generalization of class small essential submodules namely E-small essential submodules. Where the submodule Q of U_R is said E-small essential if $Q \cap W = 0$, when W is a small submodule of U_R, implies that $N_S(W) = 0$, where $N_S(W) = \{\psi \in E \mid Im\psi \subseteq W\}$. The intersection $\overline{B}_R(U)$ of each submodule of U_R contained in $Soc(U_R)$. The $\overline{B}_R(U)$ is unique largest E-small essential submodule of U_R, if U_R is cyclic. Also in this paper we study $\overline{B}_R(U)$ and $\overline{W}_E(U)$. The condition when $\overline{B}_R(U)$ is E-small essential, and Tot $(U, U) = \overline{W}_E(U) = J(E)$ are given.

Keywords: Small submodule, Small essential submodules, E-small essential submodules, Endomorphism ring.

1. Introduction

Throughout this treatise, all ring R is a associative with identity, and all module over a ring R is unitary right module. Let U_R will always denoted such an R-module and E is endomorphism ring and denoted by $E = End(U_R)$ of ring module. The submodule W is called essential of U_R (denoted by: $W \trianglelefteq U_R$) if $0 \neq G \leq U_R$, then $W \cap G \neq 0$ (see [1]), where the submodule G of U_R is denoted by $(G \leq U_R)$.The submodule Q of U_R is said small submodule (denoted by: $Q \ll U_R$), if $\forall\ W \lneqq U_R$ then $Q + W = U_R$ (*see* [2]). The left annihilator of an submodule Q of U_R is denoted by $\delta_E(Q)$, and the right annihilator of an endomorphism h of U_R is denoted by $k_U(h)$, specifically that Ker(h). We also denoted $N_E(Q) = \{\theta \in E \mid Im\theta \subseteq Q\}$ for each $Q \subseteq U$. Nicholson and Zhou defined annihilator-small right ideals [5]. Also Amouzegar and Keskin introduced and study the right annihilator-small submodules of an R-module. Let U_R be an R-module and $F \leq U_R$, then

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F is a said to annihilator- small submodule if $F + W = U$, where W is a submodule of U_R, so $\delta_E(W) = 0$ [4]. From [7] the Zhouaud and Zhang, give a definition of small- essential submodules. Let K be a submodule of a module U_R, then K is called small-essential in U_R(denoted by $K \trianglelefteq_E U$, if $K \cap W = 0$, with $W \ll U_R$ implies that $W = 0$. In this paper we introduced new concept namely E-small essential submodule, where a submodule Q of a module U_R is called E-small essential (denoted by $Q \trianglelefteq_{E-s} U$) if $Q \cap W = 0$, for each $W \ll U_R$, implies that $N_E(Q) = 0$, where $E = End(U_R)$. In [7], essential submodule is small essential submodule. It is clearly that small essential submodule is E-small essential, so every essential is E-small essential. Also give us the condition that makes every E-small essential is essential (see proposition 2.2), and every E-small essential is small essential (see proposition 2.4 and 2.5). We have verified that the equality is correct for the following statement Tot ($U, U) = \overline{W}_E(U) = J(E)$.

2. Main Results

Definition 2.1. *Let Q be an submodule of a module U_R, then Q is called E-small essential (denoted by $Q \trianglelefteq_{E-s} U$) if $Q \cap W = 0$, where W is small submodule of U_R(or denoted by $W \ll U_R$), implies that $N_E(W) = 0$, where $E = End(U_R)$.*

It is clearly that every small essential submodule is E-small essential submodule, but the opposite is generally not true (meditation the submodule mZ of the Z-module Z).

The left R-module U_R is called retractable if there exists a non-zero homomorphism $\beta : U \to Q$ for each anon-zero submodule Q of U_R.

Proposition 2.2. *Let U_R be an retractable R-module. If $Q \trianglelefteq_{E-s} U_R$, then $Q \leq_e U_R$.*
Proof . *Let $Q \cap F = 0$, for an $F \leq U_R$, then by hypothesis $N_E(F) = 0$. But U_R is retractable, then $F = 0$, that mean $Q \leq_e U_R$.*□

Corollary 2.3. *If U_R is retractable R-module and $Q \trianglelefteq_{E-s} U_R$, then $Q \trianglelefteq_s U_R$.*

Proposition 2.4. *Let U_R be a cyclic and π-projective module. Then Q is small essential submodule if and only if Q is E-small essential submodule of U_R.*
Proof . *Let $U_R = uR$ for some $u \in U_R$, and $Q \trianglelefteq_{E-s} U_R$. Let $V \ll U_R$, we put $0 \neq v \in V$, so then there exists $0 \neq n \in R$, such that $v = un$, but $U_R = uR = unR + u(1-n)R$, since U_R is π-projective then there exists $\beta \in End(U_R)$, with $Im\beta \subseteq unR \subseteq V$, so $Im(1-\beta) \subseteq (1-n)uR$, that is $N_E(V) \neq 0$. As $Q \trianglelefteq_{E-s} U_R$, and $Q \cap V \neq 0$. That mean $Q \trianglelefteq_{E-s} U_R$. The converse is evident.* □

Proposition 2.5. *Let U_R be a cyclic R-module and R be a commutative ring. Then $Q \trianglelefteq_{E-s} U_R$ if and only if $Q \trianglelefteq_s U_R$.*
Proof . *Is evident.* □

Lemma 2.6. *Let U_R be an R-module. If $V \leq Q \leq U_R$, and $Q \trianglelefteq_{E-s} U_R$, then $V \trianglelefteq_{E-s} U_R$.*
Proof . *Is evident.* □

Proposition 2.7. *Let U_R be an R-module. If $Q \trianglelefteq_{E-s} U_R$ and $F \trianglelefteq_s U_R$, then $Q \cap F \trianglelefteq_{E-s} U_R$.*
Proof . *Let $Q \cap F \cap V = 0$, where $V \ll U_R$. Since $F \trianglelefteq_s U_R$, that is $Q \cap V = 0$ and $N_E(V) = 0$.* □

Lemma 2.8. *Let U_R be a module, and Q be a submodule of U_R if $N_E(Q) \trianglelefteq_s E_E$, then $N_E(Q)U_R \trianglelefteq_{E-s} U_R$. In specially, $Q \trianglelefteq_{E-s} U_R$.*
Proof . *Let $N_E(Q)U_R \cap V = 0$, so $N_E(Q) \cap N_E(V) = 0$, thus $N_E(V) = 0$. But $N_E(Q) \trianglelefteq_s E_E$. So that the last perception by (Lemma 2.6) and since $N_E(Q)U_R \subseteq Q \subseteq U_R$ always achieve.* □

Note that the converse of Lemma 2.8 is true if $(N_E(Q) \cap vE) U_R = N_E(Q) U_R \cap vU_R$ verified for each submodule Q of U_R, and all small element $v \in E$. And to watch it, let $N_E(Q) \cap vE = 0$, for any small element $v \in E$. Thus $N_E(Q) U_R \cap vU_R = 0$, so $N_E(vU_R) = 0$. But $N_E(Q)U_R \trianglelefteq_{E-s} U_R$ and $vE \subseteq N_E(vU_R) = 0$, then $v = 0$. Hence $N_E(Q) \trianglelefteq_s E_E$.
Recall that an R-module U_R is called semi-injective if for each $\alpha \in E$ such that

$$E\alpha = \delta_E(ker(\alpha)) = \delta_E(k_U(\alpha))$$

(equivalently for any monomorphism $\alpha : Q \to U$, where Q is a factor module of U_R, and for any homomorphism $\beta : Q \to U$, then there exists $\gamma : U \to U$ such that $\alpha\gamma = \beta$) [5, p. 261].

Lemma 2.9. *Let us have the following situation for any R-module U_R and $u \in E$:*

(1) $k_U(u) \trianglelefteq_{E-s} U_R$.
(2) $k_U(u) \subsetneqq k_U(ur)$ *for all* $0 \neq r \in E$.
(3) $k_E(1_E - au) = 0$ *for all* $0 \neq a \in E$.
(4) $k_E(1_E - ua) = 0$ *for all* $0 \neq a \in E$.
(5) $k_E(u - uau) = k_E(u)$ *for all* $0 \neq a \in E$.

Then $(1) \Longrightarrow (2) \Longrightarrow (3) \Longrightarrow (4) \Longrightarrow (5)$. *If U_R is semi-injective, then* $(5) \Longrightarrow (1)$.
Proof . $(1) \Longrightarrow (2)$ *Suppose that $0 \neq r \in E$, and $k_U(r) = k_U(ur)$. It is clear that $k_U(u) \cap rU = 0$. According to $k_U(u) \trianglelefteq_{E-s} U_R$, and $N_E(rU) = 0$, so $rE \subseteq N_E(rU) = 0$. That is $r = 0$.*
$(2) \Longrightarrow (3)$ *Let $a \in E$, and $r \in k_E(1_E - au)$, so $r = aur$, then $k_U(ur) \subseteq k_U(aur) = k_U(r)$. Then by (2), hence $r = 0$.*
$(3) \Longrightarrow (4)$ *Let $r \in k_E(1_E - ua)$, for all $a \in E$, thus $(1_E - ua) r = 0$, that mean $(1_E - au) ar = (a - aua) r = a(1_E - ua) r = 0$, implies that $ar = 0$ that by (3), then $r = uar = 0$.*
$(4) \Longrightarrow (5)$ *Let $r \in k_E(u - uau)$, for all $a \in E$, so by (4) $ur = 0$. Then $r \in k_E(u)$. Other embedding in a similar way.*
$(5) \Longrightarrow (1)$ *S uppose that U_R is semi-injective. Now, let $k_U(u) \cap V = 0$ for a small submodule V of U_R, and let $r \in N_E(V)$, implies that $rU \cap k_U(u) = 0$, then $k_U(r) = k_U(ur)$. But U_R is semi-injective, then thereesxists ahomomorphism $v \in E$ such that $r = vur$, so $(u - uvu) r = 0$. Thus $r \in (u - uvu) = k_E(u)$, then $ur = 0$, and hence $r = 0$.* □

Note that we us define $\overline{W}_E(U) = \{u \in E |\ker v = k_U(u) \trianglelefteq_{E-s} U_R\}$ for any module U_R.

Corollary 2.10. *Let U_R be a module, and $u \in \overline{W}_E(U)$. Thus $Eu \subseteq \overline{W}_E(U)$. If U_R is semi-injective, then $uE \subseteq \overline{W}_E(U)$.*
Proof . *Let $r \in E$, and U_R is semi-injective, we most show that $k_U(ur) \trianglelefteq_{E-s} U_R$. Now let $v \in E$, since $k_U(u) \trianglelefteq_{E-s} U_R$, then by Lemma 2.9(4) $k_E(1_E - urv) = 0$. Once again form Lemma 2.9(4) $k_U(ur) \trianglelefteq_{E-s} U_R$. Thus $uE \subseteq \overline{W}_E(U)$. Now through the Lemma* **??**, *we get $Eu \subseteq \overline{W}_E(U)$.* □

Corollary 2.11. *We own $\overline{W}_E(U) \subseteq \delta_E(Soc(E_E))$. Furthermore, $J(E) \subseteq \overline{W}_E(U)$, if U_R is a semi-injective.*
Proof . *Let $w \in \overline{W}_E(U)$, and $0 \neq u \in Soc(E_E)$, we want to prove that $0 = wSoc(E_E)$. Now $u \in E_1 \oplus E_2 \oplus \cdots \oplus E_n$, where $E_1, E_2, \ldots, E_n$ are simple right ideal of E, and n is positive integer. Suppose that $wu \neq 0$ and $u = u_1 + u_2 + \cdots + u_n$ where as $u_j \in E_j$ for some $j \in \{1, 2, \ldots, n\}$, then $wu_j \neq 0$. As E_j is simple so $Ewu_j = E_j$. Thus $u_j = \beta wu_j$ for all $\beta \in E$. So $u_j \in k_E(1_E - \beta w)$, but $k_U(w) \trianglelefteq_{E-s} U_R$, then from Lemma2.9 $k_E(1_E - \beta w) = 0$, that is $u_j = 0$. This is acontradiction. So $wu = 0$, hence $\overline{W}_E(U) \subseteq \delta_E(Soc(E_E))$. Now let $v \in J(E)$ and $w \in E$. We must prove that $v \in \overline{W}_E(U)$, we take $\beta \in k_E(1_E - wv)$. Thus $(1_E - wv) = 0$, but $1_E - wv$ is invertible, so $\beta = 0$. Then $k_E(1_E - wv) = 0$ for all $w \in E$. Hence from Lemma2.9 $v \in \overline{W}_E(U)$, implies that $J(E) \subseteq \overline{W}_E(U)$.* □

Corollary 2.12. *Let U_R is a semi- injective module and $h \in E$. Then $Kerh = k_U(u) \trianglelefteq_{E-s} U_R$ if and only if $Eh \ll_a E_E$.*

Proof . *Let $h \in E$ and suppose that $k_U(h) \trianglelefteq_{E-s} U_R$. Now let $E = Eh + P$, where P is an ideal of E. So $1_E = rh + q$, where $r \in E$ and $q \in P$, then $k_U(h) \cap k_U(q) = 0$. But $k_U(h) \trianglelefteq_{E-s} U_R$, then $N_E(k_U(q)) = 0$. That is $N_E(k_U(P)) = 0$, hence $k_U(P) = 0$ implies that $Eh \ll_a E_E$. The converse, suppose $Eh \ll_a E_E$, then from ([4], Corollary 2.8) $k_E(h - hrh) = k_E(h)$, for all $r \in E$. Then from Lemma 2.9 $k_U(h) \trianglelefteq_{E-s} U_R$.* □

Corollary 2.13. *Let U_R be an R-module. If $h^2 = h \in \overline{W}_E(U)$, then $h = 0$.*

Proof . *We can see from the lemma2.9 (4) and $k_U(h) \trianglelefteq_{E-s} U_R$, $k_E(1_E - h) = 0$, and since $h \in k_E(1_E - h)$. Implies that $h = 0$.* □

Corollary 2.14. *Let P be an maximal-ideal of E, where $E = End(U_R)$ and U_R be amodule. Then the following ferries are equivalent:*

1. $PU \trianglelefteq_{E-s} U_R$
2. $P \leq_e E_E$

Proof . (1) $\Longrightarrow$ (2) *Let $PU \trianglelefteq_{E-s} U_R$ Suppose that P is not essential of E_E. Then $P \cap K = 0$, foe some K is a non-zero ideal of E_E. But P is amaximal ideal, that mean P is direct summand of E_E. So there exists idempotent element $i \in E_E$ such that $P = iE$. Then $PU = iU = k_E(1_E - i) \trianglelefteq_{E-s} U_R$. Hence $1_E - i \in \overline{W}_E(U)$. Then from (Corollary 2.13) $i = 1$. This is acontradiction.*
(2) $\Longrightarrow$ (1) *Let $P \leq_e E_E$, and $PU \cap V = 0$ for an small submodule V of U_R. So $0 = N_E(0) = N_E(PU) \cap N_E(V)$. Then $P \cap N_E(V) = 0$. But $P \leq_e E_E$, then $N_E(V) = 0$.* □

Recall that the element h in E is called to be partially invertible if hE contains anon-zero idempotent, where (hE equivalent Eh). Where an R-module U_R the total of U_R is defined as $\mathrm{Tot}(E) = Tot(U,U) = \{h \in E | h \text{ is not partially invertible}\}$.
Unable to closed the total under addition. In effect, if 0 and 1 are the only idempotent in E, then the total of U_R is the set of non-isomorphism.

Proposition 2.15. *Let U_R be a module. Then $\overline{W}_E(U) \subseteq Tot(U,U)$.*

Proof . *If $h \in \overline{W}_E(U)$ and $h \notin Tot(U,U)$, implies that h is partially invertible then there exists $0 \neq i^2 = i \in Eh$. So by (Corollary 2.10), $i \in \overline{W}_E(U)$. Thus acontradicts to (Corollary 2.13).* □

Let P is a subset of a ring R, then R is called to be P-semi-potent if every ideal not contained in P contains anon-zero idempotent, equivalently if every element $q \notin P$ is a partial inverse R is said to be semi-potent if R is $J(R)$-semi-potent.

Lemma 2.16. *Let U_R be a module, if P is a subset of $E = End(U_R)$. Then the following ferries are equivalent:*

1. *E is P-semi-potent.*
2. $Tot(U,U) \subseteq P$.

Proof . *Is evident from ([5], Lemma 20).* □

Proposition 2.17. *Let $E = End(U_R)$ for any R-module U_R. Then E is a semi-potent if and only if $J(E) = Tot(U,U)$.*

Proof . *Is evident from ([5], Theorem 21).* □

Proposition 2.18. *Let U_R be asemi- injective R-module, and $E = End(U_R)$ is a semi-potent. Then $\overline{W}_E(U) = J(E) = Tot(U,U)$.*
Proof . *It is evident that $J(E) \subseteq \overline{W}_E(U)$ by (Corollary 2.11). Let $u \in \overline{W}_E(U)$, if $u \notin J(E)$ and E is $J(E)$-semi-portent, then $\overline{W}_E(U)$ have anon-zero idempotent which is a contradiction (we can see corollary 2.13). Then $J(E) = \overline{W}_E(U)$. Now from Proposition 2.15 $\overline{W}_E(U) \subseteq Tot(U,U)$. From other hand, E is $\overline{W}_E(U)$-semi-portent and since $J(E) = \overline{W}_E(U)$. Hence form Lemma 2.16 $Tot(U,U) \subseteq \overline{W}_E(U)$.* □

Proposition 2.19. *Let U_R be asemi- injective R-module, and $E = End(U_R)$, where $k_E(u) = 0$, for all $u \in E$, such that $Eu = E$. Then $\overline{W}_E(U) = J(E)$.*
Proof . *It is clear that from Corollary 2.11 $J(E) \subseteq \overline{W}_E(U)$. Let $x \in \overline{W}_E(U)$, then $k_U(x) \trianglelefteq_{E-s} U_R$, hence $k_E(1_E - ux) = 0$, for all $u \in E$, so from Lemma 2.9 then $E(1_E - ux) = E$, thus by hypothesis $x \in J(E)$. Implies $\overline{W}_E(U) \subseteq J(E)$.* □

A ring R is said to be right Kasch if every simple right R-module embeds in R, this is rewarding, if $k_R(V) \neq 0$ for every maximal right ideal E of R. Associated R aleft ideal W_2 ring if every left ideal is isomorphic to direct of ${}_R$R is itself a direct summand of ${}_R$R

Lemma 2.20. *Let U_R be asemi- injective R-module. In each of the following statements, we have $\overline{W}_E(U) = J(E)$.*

1. *E is semi-potent.*
2. *E is right Kasch.*
3. *E is a left W_2 ring.*

Proof .

1. *Is evident from Proposition 2.18*
2. *Let $u \in E$, then $k_E(u) = 0$. If $uE \neq E$, then by (2) $k_E(uE) = 0$, that is $k_E(u) \neq 0$. This is a contradiction. Hence from Proposition 2.19 $\overline{W}_E(U) = J(E)$.*
3. *Let $v \in E$, then $k_E(v) = 0$. If $Ev = E$, then by (3) Ev is a direct summand of E, so $vxv = v$, for some element $x \in E$. Since $0 = k_E(v) = k_E(vx) = E(1_E - vx)$. Hence $vx = 1_E$ and $vE = E$, from Proposition 2.19, $\overline{W}_E(U) = J(E)$.*

□

Lemma 2.21. *Let $u = uR$, where $u \in U$, and U be a cyclic R-module. Then the following are equivalent for $w \in U$:-*

1. $wR \trianglelefteq_{E-s} U$
2. *$g(wR) \subsetneqq f(U)$, for all $g \in E$*
3. *$k_E(u - wn) = 0$, for all $n \in R$.*

Proof . (1) $\Longrightarrow$ (2) *Let $g(wR) = g(U)$, then $g(wn) = g(u)$, for all $n \in R$, hence $g \in k_E(u - wn)$. But $wR + (u - wn)R = uR = U$, then by (1) $k_E(u - wn) = 0$. Therefore $g = 0$.*
(2) $\Longrightarrow$ (3) *Let $g \in k_E(u - wn)$, for all $n \in R$, so $g(u) = g(wn) \subseteq g(wR)$ by (2). Therefore $g = 0$.*
(3) $\Longrightarrow$ (1) *If $wR + V = U$, where V is small submodule of U_R, then $u = wn + v$, for all $n \in R$ and $v \in V$. Now let $g \in k_E(V)$* that mean *$g(u) = g(wn)$. Hence by (3) $g \in k_E(u - wn) = 0$. Therefore $g = 0$.* □

Note: Let U_R be a module, we can defined $\overline{B_R}(U) = \cap\{D \subseteq U_R | D \trianglelefteq_{E-s} U_R\}$. It is clearly that $\overline{B_R}(U) \subseteq Soc(U)$.

Proposition 2.22. *If U_R is an retractable and semi- projective R-module, then $\overline{B_R}(U) = Soc(U) = Soc(E_E)U$.*

Proof . *From Corollary 2.3 $\overline{B_R}(U) = Soc(U)$. Since U_R is semi- projective, then from ([3], Proposition 2.4), $\overline{B_R}(U) = Soc(U) = Soc(E_E)U$.*

Let U_R be an R-module, an element $c \in U_R$ is called E-small essential if $cR \trianglelefteq_{E-s} U_R$. For simplicity, we denoted $C_R(U) = \{c \in U | c$ is a E-small essential in $U\} = \{c \in U | cR \trianglelefteq_{E-s} U_R\}$. It is evident that $C_R(U) \subseteq \overline{B_R}(U)$. □

Proposition 2.23. *Let $U = aR$ be a cyclic R-module, and X be a submodule of U_R. Then the following are equivalent:*

1. $X \trianglelefteq_{E-s} U_R$
2. $X \subseteq C_R(U)$
3. $k_E(u-a) = 0$, *for all* $a \in R$.

Proof . (1) $\Longrightarrow$ (2) *Fore Proposition 2.7.*

(2) $\Longrightarrow$ (3) *Let $X + Y = U$, where Y is small submodule of U_R, $u = x + y$, for all $x \in X$ and $y \in Y$, then $k_E(Y) \subseteq k_E(u-x) = 0$.*

(3) $\Longrightarrow$ (1) *According to the hypothesis(3). Therefore $X \trianglelefteq_{E-s} U_R$.* □

Proposition 2.24. *Let U_R be an R-module, Then*

1. $\overline{B_R}(U) = \{c_1 + c_2 + \cdots + c_n | c_j \in C_R(U)$ *for each* n, j *are positive integer*$\}$.
2. $\overline{B_R}(U) = C_R(U) R$.

Proof . (1) *Let the set $F = \{c_1 + c_2 + \cdots + c_n | c_j \in C_R(U)$ for each n, j are positive integer$\}$. If $c \in \overline{B_R}(U)$, then $c \in F_1 + F_2 + \cdots + F_n$, where $F_j \trianglelefteq_{E-s} U_R$, for each n, j are positive integer. If $c = c_1 + c_2 + \cdots + c_n$, $c_j \in F_j$, implies that from Proposition 2.7 $c_j R \trianglelefteq_{E-s} U_R$. Thus $c_j \in C_R(U)$. Hence $\overline{B_R}(U) \subseteq F$. Simply we can note that $F \subseteq \overline{B_R}(U)$.*

(2) *Evident by fact, $C_R(U) \subseteq \overline{B_R}(U)$, and by (1).* □

Proposition 2.25. *Let U_R be an R-module, consider the following expression:*

1. *If* $F \trianglelefteq_{E-s} U_R$ *and* $H \trianglelefteq_{E-s} U_R$, *then* $F + H \trianglelefteq_{E-s} U_R$.
2. $C_R(U)$ *is closed under addition.*
3. $\overline{B_R}(U) = C_R(U)$.
4. $\overline{B_R}(U) \trianglelefteq_{E-s} U_R$

Can we get (1) $\Longrightarrow$ (2) $\Longrightarrow$ (3) *and* (4) $\Longrightarrow$ (1) .

But (3) $\Longrightarrow$ (4) , *it can obtained by adding if U_R is cyclic R-module. In addition, if $U = uR$, where $u \in U$ one of the above-mentioned condition the following:*

(i) $\overline{B_R}(U)$ *is the unique largest E-small essential of* U.

(ii) $\overline{B_R}(U) = \{u \in U | k_E(a - uw) = 0$, *for all* $w \in R\}$

(iii) $\overline{B_R}(U) = \cap\{G \subseteq^{\max} U | \overline{B_R}(U) \subseteq G\}$

Proof . (1) $\Longrightarrow$ (2) *Since $(u+v)R \subseteq uR + vR$, so $C_R(U)$ is closed under addition by Prop. 2.7.*

(2) $\Longrightarrow$ (3) *It is obvious that $C_R(U) \subseteq \overline{B_R}(U)$, then from Proposition 2.24 (1), $\overline{B_R}(U) \subseteq C_R(U)$.*

(3) $\Longrightarrow$ (4) *Let $U = uR$, for some $u \in U$, and $\overline{B_R}(U) + F = U$, where F is a small submodule of U_R. Thus by (3) $C_R(U) + F = U$. If $u = v + w$, where $v \in C_R(U)$ and $w \in F$.Thus $U = vR + F$, so $vR \trianglelefteq_{E-s} U_R$. Then $k_E(U) = 0$. Hence $\overline{B_R}(U) \trianglelefteq_{E-s} U_R$.*

(4) $\Longrightarrow$ (1) *Let* $F \trianglelefteq_{E-s} U_R$ *and* $H \trianglelefteq_{E-s} U_R$. *Thus* $F \subseteq \overline{B_R}(U)$ *and* $H \subseteq \overline{B_R}(U)$, *then* $F + H \subseteq \overline{B_R}(U)$. *Hence from Proposition 2.7 and by (4), implies that* $F + H \trianglelefteq_{E-s} U_R$.
Now, (i) *is evident by (4), and* (ii) *is evident from Lemma 2.21 and by (3). Finally* (iii) *if* $u \in \overline{B_R}(U)$, *so* uR *is not E-small essential by (3), then* $uR + F = U$, *for an small submodule* F *of* U_R, *with* $k_E(U) \neq 0$, *by (4)* $\overline{B_R}(U) \trianglelefteq_{E-s} U_R$, *then we have* $\overline{B_R}(U) + F \neq U$. *If* $\overline{B_R}(U) + F \subseteq G \subseteq^{\max} U$, *thus* $u \notin U$. *This is prove of* (iii). □

Proposition 2.26. *Let* U_R *be a module. consider the following expression:*

1. $\overline{B_R}(U) \trianglelefteq_{E-s} U_R$
2. *If* $F \trianglelefteq_{E-s} U_R$ *and* $H \trianglelefteq_{E-s} U_R$, *then* $F \cap H \trianglelefteq_{E-s} U_R$

Note (1) $\Longrightarrow$ (2) *verified. As well if* U_R *finitely cogenerated, hence* (2) $\Longrightarrow$ (1)

Proof . (1) $\Longrightarrow$ (2) *Let* $F \trianglelefteq_{E-s} U_R$ *and* $H \trianglelefteq_{E-s} U_R$, *so* $\overline{B_R}(U) \subseteq F \cap H$, *then from Lemma 2.8* $F \cap H \trianglelefteq_{E-s} U_R$.
(2) $\Longrightarrow$ (1) *If* U_R *finitely cogenerated, and let* $\overline{B_R}(U) \cap F = 0$, *where* F *is a small submodule of* U_R, *then* $F_1 \cap F_2 \cap \cdots \cap F_n \cap H = 0$, *for some* $E_j \subseteq \overline{B_R}(U)$. *Therefore* $N_E(H) = 0$. *that by (1).* □

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