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Qualitative study of an eco-epidemiological model with anti-predator and migration presence

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Abstract

In this work, an environmental and epidemiological model has been formulated and analyzed, where it was assumed that there is a disease in the predator society of an incurable type and does not give the predator immunity that leads to the death of the affected predator in the end. On the other hand, it has been assumed that a healthy predator is only capable of predation according to a functional response Holling type – IV, Also, two types of factors are considered behavior against predation and the group's defence to formulate our proposed model. In addition, immigration was taken into consideration for prey society, mathematically and biologically acceptable equilibrium points for this model were found, as well, these points were studied analytically and numerically to know the effect especially, the emigration and behavior against predation to keep both the two types.

Keywords: Eco-epidemiological model, SI-disease, Migration, Anti-predator.

1. Introduction

Biology and mathematics are essential tools for understanding the processes of interaction between organisms in nature, including the processes of predation, competition and coexistence. As biologists collect multiple information about these types to evaluate them and many tools are used to collect that information, on the other hand mathematicians develop mathematical models and these models are based on experiments, observations, then the prediction of the factors that can affect the species, since mathematics is of great importance in understanding the interaction between living organisms in nature by designing mathematical models that describe these interactions, especially the predation model because it is the most important interaction in life, because each of the organisms is food of the other organism.

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In the defense mechanism against predation, getting to know the predator is very important, because most defense operations against predation require the prey to determine whether the predator is dangerous or not, that is, distinguishing between dangerous situations without danger. It is about predation, as in wild buffaloes, where the fabricator is either killed by that group or escapes. Many researchers have proposed models of prey and predator, including anti-predator behavior against predation as a defense of prey for themselves, for example [9], Tang and Qin [11].

Among the important factors in regulating the sizes of living organisms, including humans, are infectious epidemic diseases that can be transmitted through direct contact between the affected person and the healthy person or through external sources such as water, air, etc., most of these diseases are curable and the other is not curable, in addition to that Most diseases can be avoided by taking vaccines, while others give immunity to the infected person after recovery from the disease so that he does not catch it again.

In recent decades, zoonotic diseases - those that have been transmitted from animals to humans - have gained international attention. As diseases such as Ebola, avian influenza, influenza virus, Middle East respiratory syndrome, Rift Valley fever, Sudden acute respiratory syndrome (SARS), West Nile virus, Zika virus - and now, new coronaviruses as reported 19, may or may be massive in infection Major epidemics, with losses from deaths and economic losses in billions.

Dozens of researchers have proposed and studied epidemiological models, some of them suggested the presence of disease in the prey community only , Majeed [3] and [4, 8, 6], some of them assumed the presence of disease in the predatory society only [2], and others studied the presence of disease in both societies, [5, 10]. And no less important than the above , animal migration, the common form of migration in ecology is the distant movement of animals, and this movement or migration is usually seasonal. This type of migration is found in all animal organisms such as birds, fish, reptiles, insects and amphibians. Objects are survival, lack of food or reasons for mating, and these factors or reasons are characteristic of proper migration, otherwise migration is not just a disturbance and spread of these living beings.Many researchers took this factor into consideration , for example, [7, 1].

In this paper an epidemiological model has been studied, where it was assumed that there is a disease in the predator of SI type. On the other hand, it has been assumed that a susceptible predator is only capable of predation the prey according to Holling type – IV a functional response, Also, two types of factors are careful anti-predator and migration of prey to formulate our proposed model. In addition, mathematically and biologically suitable equilibrium points for this model were found, as well, these points were studies analytically and numerically to know the effect of emigration and anti-predator against predation on the dynamics of the proposed model.

2. Mathematical Model

Consider the following eco-epidemiological model:-

$$\begin{aligned}
\frac{dw_1}{\mathrm{dt}} &= \alpha_1 w_1 \left(1 - \frac{w_1}{F}\right) - \frac{\alpha_2 w_1 w_2}{\alpha_3 + \alpha_4 w_1^2} - \alpha_5 w_1, \\
\frac{dw_2}{\mathrm{dt}} &= \frac{\alpha_6 w_1 w_2}{\alpha_3 + \alpha_4 w_1^2} - \alpha_7 w_2 w_3 - \alpha_8 w_1 w_2 - \alpha_9 w_2, \\
\frac{dw_3}{\mathrm{dt}} &= \alpha_7 w_2 w_3 - \alpha_{10} w_3.
\end{aligned} \tag{2.1}$$

The parameters and variables of the above system are illuminated in the next Table.

Table 1: Variables and the parameters of system (2.1)

Parameter	***Representation of the parameter***
$w_1(t)$	***The prey density at time*** t
$w_2(t)$	***The healthy predator density at time*** t
$w_3(t)$	***The sick predator density at time*** t
$\alpha_1 > 0$	***The growth rate of prey***
$F > 0$	***The carrying capacity***
$\alpha_2 > 0$	***Maximum attack rates for prey by healthy predator***
$\alpha_3 > 0$	***The half saturation rate***
$\alpha_4 > 0$	***The inverse measure of inhibitory effect***
$\alpha_5 > 0$	***The migration rate of prey***
$\alpha_6 > 0$	***The uptake rates of food from the prey on to healthy predator***
$\alpha_7 > 0$	***The infection rate by contact***
$\alpha_8 > 0$	$The\ anti-predator\ \ rate$ ***of prey***
$\alpha_9 > 0$	***The decease rate of the predator in the nonappearance of its feeding***
$\alpha_{10} > 0$	***The death rate of sick predator by disease***

Theorem 2.1. *The solutions of system* (2.1) *that start in* R_+^3 *are uniformly bounded.*
Proof . *Let* $(w_1(t),\ w_2(t),\ w_3(t))$ *be any solution of the system* (2.1) *with non-negative initial condition* $(w_1(0),\ w_2(0),\ w_3(0))$ *Let* $W(t) = w_1(t) + w_2(t) + w_3(t)$. *Therefore,*

$$\frac{dW}{dt} < \alpha_1 w_1 \left(1 - \frac{w_1}{F}\right) - (\alpha_2 - \alpha_6)\frac{w_1 w_2}{\alpha_3 + \alpha_4 w_1^2} - \alpha_5 w_1 - \alpha_9 w_2 - \alpha_{10} w_3.$$

Now, hence from the natural fact: $\alpha_6 < \alpha_2$, *thus*

$$\frac{dW}{dt} \leq \frac{\alpha_1 F}{4} - \theta W, \quad where \quad \theta = \min\ \{\ \alpha_5\ , \alpha_9\ , \alpha_{10}\ \}.$$

Now, by the comparison Theorem [12], we get

$$W(t) \leq \frac{\alpha_1 F}{4\theta} + \left(W(0) - \frac{\alpha_1 F}{4\theta}\right)\ e^{-\theta t}$$

Thus $\theta \leq W(t) \leq \frac{\alpha_1 F}{4\theta}$ *as* $t \to \infty$, *and the proof is complete.* □

3. The Equilibrium Points

System (2.1) has five equilibrium points which are given below.

1) The minor equilibrium point $\mathrm{w}^0 = (0, 0, 0)$ always exist.

2) The equilibrium point $\widehat{w} = (\widehat{w}_1\ , 0\ \ , 0\ )$, where, $\widehat{w}_1 = \frac{F(\alpha_1 - \alpha_5)}{\alpha_1}$, exist provided that:

$$\alpha_1 > \alpha_5 \tag{3.1}$$

3) The disease free equilibrium point exists by solving the following set of equations:

$$\alpha_1\left(1-\frac{w_1}{F}\right)-\frac{\alpha_2 w_2}{\alpha_3+\alpha_4 w_1^2}-\alpha_5=0 \tag{3.2}$$

$$\frac{\alpha_6 w_1}{\alpha_3+\alpha_4 w_1^2}-\alpha_8 w_1-\alpha_9=0 \tag{3.3}$$

From equation (3.2) we have,

$$w_2=\frac{1}{\alpha_2 F}\left((\alpha_1-\alpha_5)F(\alpha_4 w_1^2+\alpha_3)\right)-\alpha_1 w_1(\alpha_3+\alpha_4 w_1^2). \tag{3.4}$$

Also from eq.(3.3) we have:

$$-\alpha_4\alpha_8 w_1^3-\alpha_4\alpha_9 w_1^2+(\alpha_6-\alpha_3\alpha_8)\,w_1-\alpha_3\alpha_9=0. \tag{3.5}$$

Clearly, due to discard rule equation (3.5) has either two positive roots or else there are no positive roots depending on the following condition whether it hold or violate respectively, provided that:

$$\alpha_6>\alpha_3\alpha_8, \tag{3.6}$$

That is there are two disease free equilibrium points $\overline{w}=(\overline{w}_1,\overline{w}_2,0)$ and $\bar{\bar{w}}=(\bar{\bar{w}}_1,\bar{\bar{w}}_2,0)$ where: $\overline{w}_2=w_2\left(\overline{w}_1\right)$ *and* $\bar{\bar{w}}_2=w_2\left(\bar{\bar{w}}_1\right)$, if in addition to condition (3.1) the following condition holds:

$$(\alpha_1-\alpha_5)F(\alpha_4 w_1^2+\alpha_3)>\alpha_1 w_1\left(\alpha_3+\alpha_4 w_1^2\right). \tag{3.7}$$

4) The positive equilibrium point exists by solving the following set of equations:

$$\alpha_1\left(1-\frac{w_1}{F}\right)-\frac{\alpha_2 w_2}{\alpha_3+\alpha_4 w_1^2}-\alpha_5=0, \tag{3.8}$$

$$\frac{\alpha_6 w_1}{\alpha_3+\alpha_4 w_1^2}-\alpha_7 w_3-\alpha_8 w_1-\alpha_9=0, \tag{3.9}$$

$$\alpha_7 w_2-\alpha_{10}=0. \tag{3.10}$$

From equation (3.10), we have

$$w_2=\frac{\alpha_{10}}{\alpha_7}. \tag{3.11}$$

Also from eq. (3.9) we have:

$$w_3=\frac{1}{\alpha_7(\alpha_3+\alpha_4 w_1^2)}\left[(\alpha_6-\alpha_3\alpha_8)\,w_1-\alpha_4 w_1\left(\alpha_8 w_1^2+\alpha_9 w_1\right)-\alpha_3\alpha_9\right]$$

Now, by replacing eq. (3.11) in eq.(3.8) we get:

$$\alpha_1\alpha_4\alpha_7 w_1^3-\alpha_4\alpha_7 F\left(\alpha_1-\alpha_5\right)w_1^2+\alpha_1\alpha_3\alpha_7 w_1-F\{\alpha_3\alpha_7\left(\alpha_1-\alpha_5\right)-\alpha_2\alpha_{10}\}=0. \tag{3.12}$$

Clearly, due to discard rule equation (3.12) has either two positive roots or else there are no positive roots or three positive roots depending on the following condition with condition (3.1) whether their hold or violate respectively:

$$\alpha_3\alpha_7\left(\alpha_1-\alpha_5\right)<\alpha_2\alpha_{10}, \tag{3.13}$$

That is there are two equilibrium points $w^*=(w_1^*,w_2^*,w_3^*)$ and $w^{**}=(w_1^{**},w_2^{**},w_3^{**})$ where: $w_2^*=w_2^{**}=\frac{\alpha_{10}}{\alpha_7}$, $w_3^*=w_3\left(w_1^*\right)$ and $w_3^{**}=w_3\left(w_1^{**}\right)$, if in addition to condition (3.6) the following condition holds:

$$(\alpha_6-\alpha_3\alpha_8)\,w_1>\alpha_4 w_1\left(\alpha_8 w_1^2+\alpha_9 w_1\right)+\alpha_3\alpha_9. \tag{3.14}$$

4. Local Stability Analysis

In this section, the stability of system (2.1) has been discussed:-
The Jacobean matrix $G(w_1,\ w_2,\ w_3)$ of system can be written:

$$G=[g_{ij}]_{3\times 3}=\begin{bmatrix} \alpha_1-2\frac{\alpha_1}{F}w_1-\frac{\alpha_2 w_2\left(\alpha_3-\alpha_4 w_1^2\right)}{\left(\alpha_3+\alpha_4 w_1^2\right)^2}-\alpha_5 & -\frac{\alpha_2 w_1}{\alpha_3+\alpha_4 w_1^2}<0 & 0 \\ \frac{\alpha_6 w_2\left(\alpha_3-\alpha_4 w_1^2\right)}{\left(\alpha_3+\alpha_4 w_1^2\right)^2}-\alpha_8 w_2 & \frac{\alpha_6 w_1}{\alpha_3+\alpha_4 w_1^2}-\alpha_7 w_3-\alpha_8 w_1-\ \alpha_9 & -\alpha_7 w_2<0 \\ 0 & \alpha_7 w_2 & -\alpha_{10} \end{bmatrix} \tag{4.1}$$

4.1. *Local stability of* $\mathbf{w^0}$

At w^0 the Jacobean matrix is:

$$G^0=G\left(w^0\right)=\begin{bmatrix} \alpha_1-\alpha_5 & 0 & 0 \\ 0 & -\alpha_9 & 0 \\ 0 & 0 & -\alpha_{10} \end{bmatrix}. \tag{4.2}$$

So, the eigenvalues of G_0 are $\lambda_{0w_1}=\alpha_1-\alpha_5,\ \lambda_{0w_2}=-\alpha_9$ and $\lambda_{0w_3}=-\alpha_{10}$. Therefore, w^0 is stable provided that the following condition

$$\alpha_5>\alpha_1, \tag{4.3}$$

Otherwise it is unstable.

4.2. *Local stability of* $\widehat{\mathbf{w}}$

At $\widehat{w}$ the Jacobian matrix become

$$\widehat{G}=G\left(\widehat{w}\ \right)=\begin{bmatrix} -(\alpha_1-\alpha_5) & -\frac{\alpha_2 \widehat{w}_1}{\alpha_3+\alpha_4 \widehat{w}_1^2} & 0 \\ 0 & \frac{\alpha_6 \widehat{w}_1}{\alpha_3+\alpha_4 \widehat{w}_1^2}-\alpha_8 \widehat{w}_1-\ \alpha_9 & 0 \\ 0 & 0 & -\alpha_{10} \end{bmatrix}. \tag{4.4}$$

So, the eigenvaluesof $\widehat{G}$ are $\lambda_{0w_1}=-(\alpha_1-\alpha_5),\ \lambda_{1w_2}=\frac{\alpha_6 \widehat{w}_1}{\alpha_3+\alpha_4 \widehat{w}_1^2}-\alpha_8 \widehat{w}_1-\ \alpha_9$ and $\lambda_{1w_3}=-\alpha_{10}$. Therefore, $\widehat{w}$ is stable if in addition to condition (3.1) the following condition holds

$$\frac{\alpha_6 \widehat{w}_1}{\alpha_3+\alpha_4 \widehat{w}_1^2}<\alpha_8 \widehat{w}_1+\ \alpha_9, \tag{4.5}$$

Otherwise it is unstable.

4.3. *Local stability of* $\overline{\mathbf{w}}$ *and* $\bar{\bar{w}}$

At $\overline{\mathbf{w}}$ the Jacobian matrix become

$$\overline{G}=G\left(\overline{w}\right)=[\overline{g}_{ij}]_{3\times 3}$$
$$=\begin{bmatrix} \alpha_1-2\frac{\alpha_1}{F}\overline{w}_1-\frac{\alpha_2 \overline{w}_2\left(\alpha_3-\alpha_4 \overline{w}_1^2\right)}{\left(\alpha_3+\alpha_4 \overline{w}_1^2\right)^2}-\alpha_5 & -\frac{\alpha_2 \overline{w}_1}{\alpha_3+\alpha_4 \overline{w}_1^2}<0 & 0 \\ \frac{\alpha_6 \overline{w}_2\left(\alpha_3-\alpha_4 \overline{w}_1^2\right)}{\left(\alpha_3+\alpha_4 \overline{w}_1^2\right)^2}-\alpha_8 \overline{w}_2 & \frac{\alpha_6 \overline{w}_1}{\alpha_3+\alpha_4 \overline{w}_1^2}-\alpha_8 \overline{w}_1-\ \alpha_9 & -\alpha_7 \overline{w}_2<0 \\ 0 & 0 & \alpha_7 \overline{w}_2-\alpha_{10} \end{bmatrix}. \tag{4.6}$$

Then the characteristic equation of $\overline{G}$ is given by:

$$\left[\ \lambda^2 - tr(\overline{B})\ \lambda + Det(\overline{B})\ \right]\ [\alpha_7\overline{w}_2 - \alpha_{10} - \lambda] = 0,$$

where:

$$\mathrm{tr}\left(\overline{B}\right) = \lambda_{2w_1} + \lambda_{2w_2} = \overline{g}_{11} + \overline{g}_{22} = \left(\alpha_1 - 2\frac{\alpha_1}{F}\overline{w}_1 - \frac{\alpha_2\overline{w}_2\left(\alpha_3 - \alpha_4\overline{w}_1^2\right)}{\left(\alpha_3 + \alpha_4\overline{w}_1^2\right)^2}\right) +$$

$$\left(\frac{\alpha_6\overline{w}_1}{\alpha_3 + \alpha_4\overline{w}_1^2} - \alpha_8\overline{w}_1 - \ \alpha_9\right), \mathrm{Det}\left(\overline{B}\right) = \lambda_{2w_1}.\lambda_{2w_2} = \left(\overline{g}_{11}.\overline{g}_{22}\right) - \overline{g}_{12}.\overline{g}_{21} = k_1 - k_2$$

where

$$k_1 = \left(\alpha_1 - 2\frac{\alpha_1}{F}\overline{w}_1 - \frac{\alpha_2\overline{w}_2\left(\alpha_3 - \alpha_4\overline{w}_1^2\right)}{\left(\alpha_3 + \alpha_4\overline{w}_1^2\right)^2}\right)\left(\frac{\alpha_6\overline{w}_1}{\alpha_3 + \alpha_4\overline{w}_1^2} - \alpha_8\overline{w}_1 - \ \alpha_9\right)$$

$$k_2 = \left(\frac{\alpha_2\overline{w}_1}{\alpha_3 + \alpha_4\overline{w}_1^2}\right)\left(\frac{\alpha_6\overline{w}_2\left(\alpha_3 - \alpha_4\overline{w}_1^2\right)}{\left(\alpha_3 + \alpha_4\overline{w}_1^2\right)^2} - \alpha_8\overline{w}_2\right)$$

So, either $\left[\ \lambda^2 - tr(\overline{B})\ \lambda + Det(\overline{B})\ \right] = 0,\ where\ \overline{B} = \begin{bmatrix}\overline{g}_{11} & \overline{g}_{12}\\ \overline{g}_{21} & \overline{g}_{22}\end{bmatrix}$, which gives the first two eigenvalues λ_{2w_1} and λ_{2w_2} are negative provided that

$$\overline{w}_1^2 < \frac{\alpha_3}{\alpha_4}, \tag{4.7}$$

$$\alpha_1 < 2\frac{\alpha_1}{F}\overline{w}_1 + \frac{\alpha_2\overline{w}_2\left(\alpha_3 - \alpha_4\overline{w}_1^2\right)}{\left(\alpha_3 + \alpha_4\overline{w}_1^2\right)^2} + \alpha_5, \tag{4.8}$$

$$\frac{\alpha_6\overline{w}_1}{\alpha_3 + \alpha_4\overline{w}_1^2} < \alpha_8\overline{w}_1 + \ \alpha_9, \tag{4.9}$$

$$\frac{\alpha_6\overline{w}_2\left(\alpha_3 - \alpha_4\overline{w}_1^2\right)}{\left(\alpha_3 + \alpha_4\overline{w}_1^2\right)^2} - \alpha_8\overline{w}_2 \tag{4.10}$$

Or $\alpha_7\overline{w}_2 - \alpha_{10} - \lambda = 0$, which gives $\lambda_{2w_2} = \alpha_7\overline{w}_2 - \alpha_{10}$.
Therefore, E_1 is stable if adding to condition (4.7)-(4.10), the next condition holds:

$$\overline{w}_2 < \frac{\alpha_{10}}{\alpha_7}, \tag{4.11}$$

The opposite of any of the condition above leads to unstable of $\overline{w}$. Similarly for $\bar{\bar{w}}$.

4.4. Local stability of $\mathbf{w}^*$ *and* $\mathbf{w}^{**}$

At $\mathbf{w}^*$ the Jacobian matrix is:

$$G^* = G\left(w^*\right) = [g^*_{ij}]_{3\times3}$$
$$= \begin{bmatrix} \alpha_1 - 2\frac{\alpha_1}{F}w_1^* - \frac{\alpha_2 w_2^*\left(\alpha_3 - \alpha_4 w_1^{*2}\right)}{\left(\alpha_3 + \alpha_4 w_1^{*2}\right)^2} - \alpha_5 & -\frac{\alpha_2 w_1^*}{\alpha_3 + \alpha_4 w_1^{*2}} < 0 & 0 \\ \frac{\alpha_6 w_2^*\left(\alpha_3 - \alpha_4 w_1^{*2}\right)}{\left(\alpha_3 + \alpha_4 w_1^{*2}\right)^2} - \alpha_8 w_2^* & \frac{\alpha_6 w_1^*}{\alpha_3 + \alpha_4 w_1^{*2}} - \alpha_7 w_3^* - \alpha_8 w_1^* - \ \alpha_9 & -\alpha_7 w_2^* < 0, \\ 0 & \alpha_7 w_3^* & \alpha_7 w_2^* - \alpha_{10} \end{bmatrix} \tag{4.12}$$

Then the characteristic equation of G^* is given by:

$$\lambda^3 + \tau_1 \lambda^2 + \tau_2 \lambda + \tau_3 = 0. \tag{4.13}$$

Where:

$$\begin{aligned}
\tau_1 &= -\left(g_{11}^* + g_{22}^* + g_{33}^*\right), \\
\tau_2 &= g_{11}^* \left(g_{22}^* + g_{33}^*\right) + g_{22}^* g_{33}^* - g_{23}^* g_{32}^* - g_{12}^* g_{21}^*, \\
\tau_3 &= -g_{11}^* \left(g_{22}^* g_{33}^* - g_{23}^* g_{32}^*\right) + g_{12}^* g_{21}^* g_{33}^*.
\end{aligned}$$

Now by Routh Hurwitz criterion the roots of eq. (4.13) have negative real parts iff $\tau_i > 0,\ i = 1,3$, and $\Delta = (\tau_1\tau_2 - \tau_3)\tau_3 > 0$.
$Now,\ \tau_i > 0,\ i = 1,3$, if and only if the next conditions hold:

$$w_1^{*2} < \frac{\alpha_3}{\alpha_4}, \tag{4.14}$$

$$\alpha_1 < 2\frac{\alpha_1}{F} w_1^* + \frac{\alpha_2 w_2^* \left(\alpha_3 - \alpha_4 w_1^{*2}\right)}{\left(\alpha_3 + \alpha_4 w_1^{*2}\right)^2} + \alpha_5, \tag{4.15}$$

$$\frac{\alpha_6 w_1^*}{\alpha_3 + \alpha_4 w_1^{*2}} < \alpha_7 w_3^* + \alpha_8 w_1^* + \alpha_9, \tag{4.16}$$

$$w_2^* < \frac{\alpha_{10}}{\alpha_7}, \tag{4.17}$$

$$\frac{\alpha_6 w_2^* \left(\alpha_3 - \alpha_4 w_1^{*2}\right)}{\left(\alpha_3 + \alpha_4 w_1^{*2}\right)^2} > \alpha_8 w_2^*. \tag{4.18}$$

Further, it is easy to check that:

$$\Delta = \left[-\left(g_{22}^* + g_{33}^*\right)\left(g_{11}^{*2} + g_{11}^{*2}\right) - 2g_{11}^* g_{22}^* g_{33}^* - g_{33}^{*2}\left(g_{11}^* + g_{22}^*\right) + g_{23}^* g_{32}^*\left(g_{22}^* + g_{33}^*\right) + \{g_{12}^* g_{22}^* (g_{21}^* + g_{11}^*)\right] \\ \left[-g_{11}^*\left(g_{22}^* g_{33}^* - g_{23}^* g_{32}^*\right) + g_{12}^* g_{21}^* g_{33}^*\right]$$

Hence $\Delta > 0$, if in addition to conditions (4.14)- (4.18), the following condition holds:

$$\frac{\alpha_6 w_2^* \left(\alpha_3 - \alpha_4 w_1^{*2}\right)}{\left(\alpha_3 + \alpha_4 w_1^{*2}\right)^2} - \alpha_8 w_2^* > 2\frac{\alpha_1}{F} w_1^* + \frac{\alpha_2 w_2^* \left(\alpha_3 - \alpha_4 w_1^{*2}\right)}{\left(\alpha_3 + \alpha_4 w_1^{*2}\right)^2} + \alpha_5 - \alpha_1. \tag{4.19}$$

So, w^* is locally stable, reversing any of the above condition leads to unstable e of w^*.
Similarly for w^{**}.

5. Global Stability Analysis

In this section the global stability analysis for system (2.1) for the local stable points is studied

Theorem 5.1. *The point* $w^0 = (0, 0, 0)$ *is globally stable asymptotically with the Basin of attraction of* $Int.R_+^3$ *that satisfies the next condition:*

$$w_1 > F \tag{5.1}$$

Proof . *Consider the following function*

$$M_1(w_1,\ w_2,\ w_3) = w_1(t) + w_2(t) + w_3(t).$$

It is easy to see that $M_1(w_1,\ w_2,\ w_3) \in C^1\left(R^3_+, R\right)$*, and* $M_1\left(w^0\right) = 0$*, and* $M_1(w_1,\ w_2,\ w_3) > 0$ *;* $\forall (w_1,\ w_2,\ w_3) \neq w^0$ *, by differentiating* M_1 *with respect to time* t*,we get:-*

$$\frac{dM_1}{dt} < \alpha_1 w_1\left(1 - \frac{w_1}{F}\right) - (\alpha_2 - \alpha_6)\,\frac{w_1 w_2}{\alpha_3 + \alpha_4 w_1^2} - \alpha_5 w_1 - \alpha_9 w_2 - \alpha_{10} w_3.$$

Therefore, according to the natural facts, $\alpha_2 > \alpha_6$*, and condition* (5.1) *we get:*

$$\frac{dM_1}{dt} < 0.$$

Hence w^0 *is globally stable.* □

Theorem 5.2. *The point* $\widehat{w} = (\widehat{w}_1\ ,0\ \ ,0\ )$ *of system* (2.1) *is globally asymptotically stable with the Basin of attraction of* $Int.R^3_+$ *that satisfy the next conditions:*

$$\frac{\alpha_1}{F}(w_1 - \widehat{w}_1)^2 + \alpha_9 w_2 + \alpha_{10} w_3 > \frac{\alpha_2 \widehat{w}_1 w_2}{\alpha_3 + \alpha_4 w_1^2} \tag{5.2}$$

Proof . *Consider the following function:*

$$M_2(w_1,\ w_2,\ w_3) = (w_1 - \widehat{w}_1 - \widehat{w}_1\ ln\frac{w_1}{\widehat{w}_1}\ ) + w_2\ + w_3.$$

It is easy to see that $M_2(w_1,\ w_2,\ w_3) \in C^1\left(R^3_+, R\right)$*, and* $M_2\left(\widehat{w}\right) = 0$*, and* $M_2(w_1,\ w_2,\ w_3) > 0$ *;* $\forall (w_1,\ w_2,\ w_3) \neq \widehat{w}$ *, by differentiating* M_2 *with respect to time* t*, and make some algebraic manipulations we get*

$$\frac{dM_2}{dt} < -\frac{\alpha_1}{F}\left(w_1 - \widehat{w}_1\right)^2 - \alpha_9 w_2 - \alpha_{10} w_3 - (\alpha_2 - \alpha_6)\frac{\alpha_2 w_1 w_2}{\alpha_3 + \alpha_4 w_1^2} + \frac{\alpha_2 \widehat{w}_1 w_2}{\alpha_3 + \alpha_4 w_1^2}$$

Now, according to the natural facts, $\alpha_2 > \alpha_6$ *and condition*(5.2) *we get*

$$\frac{dM_2}{dt} < 0.$$

Hence $\widehat{w}$ *is globally asymptotically stable.* □

Further more since there are two free disease equilibrium points $\overline{w} = (\overline{w}_1, \overline{w}_2, 0)$, $\bar{\bar{w}} = (\bar{\bar{w}}_1, \bar{\bar{w}}_2, 0)$,and two positive equilibrium points $w^* = (w_1^*, w_2^*, w_3^*)$ and $w^{**} = (w_1^{**}, w_2^{**}, w_3^{**})$ in the interior of R_3^+ having the same local stability conditions but with different neighborhood of starting points then its not possible to studying the global stability of them using Lyapunove function. Therefore we will study it numerically instead of analytically as shown in last section.

6. Numerical Simulation

In this section, The system (2.1) is studied numerically to check our analytic results,for the next set of parameters.

$$\begin{array}{llllll} \alpha_1 = 1, & F = 1, & \alpha_2 = 0.6, & \alpha_3 = 0.2, & \alpha_4 = 0.2, & \alpha_5 = 0.1, \\ \alpha_6 = 0.5, & \alpha_7 = 0.4, & \alpha_8 = 0.1, & \alpha_9 = 0.1, & \alpha_{10} = 0.1 & \end{array} \tag{6.1}$$

with the starting point (0.4,0.2,0.1).

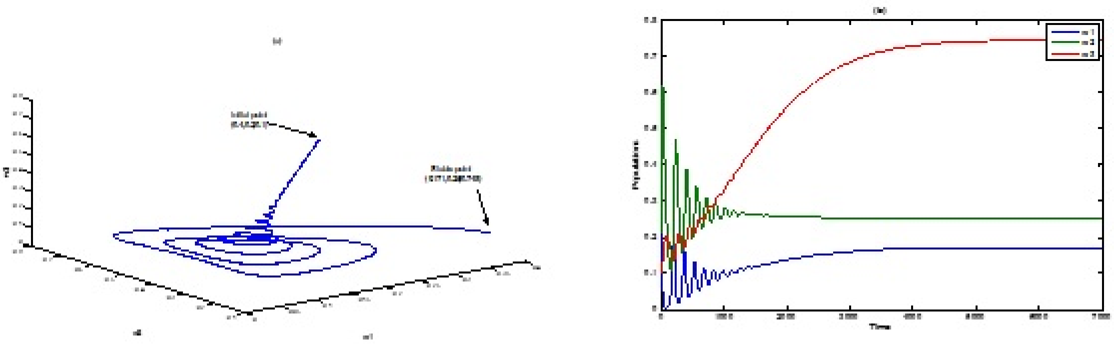

Figure 1: (a) The phase portrait of system (2.1) using data (6.1) started from the point (0.4,0.2,0.1) which approaches to $E_2 = (\mathbf{0.171}, \mathbf{0.25}, \mathbf{0.748})$, (b) The time series of the attractor in (a).

Figure 1 shows that the conditions of the existence and local stability of the positive point are satisfied, and the coexistence fixed point exists and it is detailed as $E_2 = (0.105, 0.232, 0.089)$. Now, in the following table the results of the numerical study of our model were summarized by changing the parameters one by one in relation to the given set of parameters and the initial point (0.4,0.2,0.1) as shown in Table 2.

Table 2: Numerical behavior of the system (2.1) for the data assumed in (6.1) with changing one factor at each time.

Range of parameter	***The stable point***	***The bifurcation point***
$0.01 < \alpha_1 \leq 0.101$	w^0	$\alpha_1 = 0.101$
$0.101 < \alpha_1 \leq 0.108$	$\widehat{w}$	$\alpha_1 = 0.108$
$0.108 < \alpha_1 < 0.885$	$\overline{w}$	$\alpha_1 = 0.885$
$0.108 \leq \alpha_1 < 1$	w^*	
$0.1 \leq F \leq 0.275$	$\overline{w}$	$F = 0.275$
$0.275 < F \leq 1$	w^*	
$0.6 < \alpha_2 \leq 0.688$	w^*	$\alpha_2 = 0.688$
$0.688 < \alpha_2 \leq 1$	w^*	
$0.1 < \alpha_3 \leq 0.173$	$\overline{w}$	$\alpha_3 = 0.173$
$0.173 < \alpha_3 \leq 1$	w^*	
$0.1 < \alpha_4 \leq 0.69$	w^*	$\alpha_4 = 0.69$
$0.69 < \alpha_4 \leq 1$	***Periodic***	
$0.1 \leq \alpha_5 \leq 0.213$	w^*	$\alpha_5 = 0.213$
$0.213 < \alpha_5 \leq 0.951$	$\overline{w}$	$\alpha_5 = 0.951$
$0.951 < \alpha_5 \leq 0.998$	$\overline{w}$	$\alpha_5 = 0.998$
$0.998 < \alpha_5 \leq 1$	w^0	
$0.1 \leq \alpha_6 \leq 0.193$	$\overline{w}$	$\alpha_6 = 0.193$
$0.139 \leq \alpha_6 \leq 0.6$	w^*	
$0.1 \leq \alpha_7 < 0.348$	$\overline{w}$	$\alpha_7 = 0.348$
$0.348 < \alpha_7 \leq 1$	w^*	
$0.1 < \alpha_8 \leq 0.$	w^*	$\alpha_8 = 0.2$
$.2 < \alpha_8 \leq 1$	***Periodic***	
$0.1 < \alpha_9 \leq 1$	w^*	
$0.1 < \alpha_{10} \leq 1$	w^*	

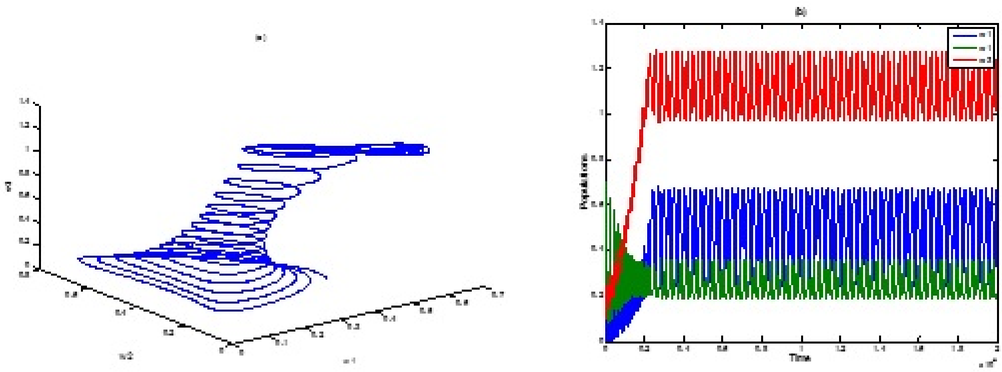

Figure 2: The phase portrait and time series of system (2.1) started from the point (0.4,0.2,0.1) using data (6.1) for typical value α_4=**0.4**: (a) The periodic solution of system (2.1), (b) The time series of the periodic attractor.

7. Conclusions

In this paper an epidemiological model has been studied, where it was assumed that there is a disease in the predator of SI type, also, two types of factors are careful anti-predator and migration of prey to formulate our proposed model, the global dynamics is studied numerically for one hypothetical set of numbers given in eq. (6.1) and from starting point (0.4,0.2,0.1), to know the effect of the parameters on the dynamics of the system, mainly the influence of anti-predation, disease and migration , and the effects have been brief as follows:

1. There is a periodic trajectory of system (2.1) in Int. R^3_+ when the rates of anti-predator and the inverse measure of inhibitory effect are varied.

2. The parameters $\alpha_{i,}$ $i = 1, \ldots, 8$ *and* F played an essential role in the study of system dynamics. As, by changing these parameters made a fundamental change in the behavior of the solution to the system, as some of them gave all points such as the growth and migration rates of prey α_1 and α_5, respectively. But, during the cange the parameters α_9 and α_{10} for the given set of parameters in (6.1), the behavior remains constant and the solution still close to the positive point.

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