Int. J. Nonlinear Anal. Appl. 13 (2022) 1, 901-919
ISSN: 2008-6822 (electronic)
http://dx.doi.org/10.22075/ijnaa.2022.5610

# The dynamics and analysis of an SIS disease of a stage-structured prey-predator model with a prey refuge

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*(Communicated by Madjid Eshaghi Gordji)*

**Abstract**

In this paper, the dynamical behaviour of an epidemiological system has been investigated. A stage-structured prey-predator model includes harvest and refuge for only prey, the disease of type (SIS) is just in the immature of the prey and the disease is spread by contact and by external source has been studied. The transmission of infectious disease in the prey populations has been described by the linear type. While Lotka-Volterra functional response is used to describe the predation process of the whole prey population. This model has been represented by a set of nonlinear differential equations. The solution's existence, uniqueness and boundedness have been studied. "The local and global stability conditions of all the equilibrium points" have been confirmed. As a final point, numerical simulation has been used to study the global dynamics of the model.

*Keywords:* A stage-structured, Prey-predator, SIS disease, Refuge, Harvesting.

## 1. Introduction

The first model of prey and predators was a simple hypothesis by the famous Lotka and Volterra in 1925. More realistic predator and prey models have been created by ecologists and mathematicians. Berryman considered in 1992 that dynamic interactions between prey and predator had become one of the most important topics that play a major role in the social, technological sciences, the mathematical, natural environment, particularly in research in ecology and biology. One of the

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*Received:* May 2021 *Accepted:* March 2021

fundamental topics in ecology is the relationship between prey density and reparative density [8]. In the relationship between predator and prey, there is an essential factor, which is the disease that must be investigated in eco-epidemiology, which includes epidemiology and environment, to see how disease affects on the densities of prey and predator. [7] considered the SI disease in prey population which is transmitted from external sources and by contact between the prey individuals. [10] investigated and analyzed the stability of an epidemiological system with diseases of types SI and SIS in the prey population only, the diseases are transmited from external sources and by contact between the preys' individuals. [3] studied the infection dynamics of an SIS disease in predator-prey systems which spread in both species by contact and by predation. and [18] studied prey refuge and harvested modified Leslie-grower predator-prey model with SIS-disease in predator.

Researchers have taken great interest in the prey and predator system to make it more realistic, by including stage structure, different types of functional responses, many types of diseases etc. . The impacts of stage- structure, refuge in the predator- prey ecosystem are the essential topics in the last years. A lot of researchers have been studies stage structure models like [23, 11, 9, 13, 5, 12, 17, 15]. Also a lot of researchers studied the impact of prey refuges on the dynamical system like [4, 24, 12, 1].

Harvesting has a direct and powerful effect on population dynamics, and there are several studies that have dealt with this topic, such as [6, 21, 19, 2, 20]. [22] studied a food chain model with SIS in prey only with harvesting on infected prey, and studied the predation of the first predator of susceptible and the infected prey and the predition of the second predator of the first predator.
In this work "an epidemiological mathematical system" involving of stage-structured in prey-predator model, the prey stages is $U(T) = U_1(T) + U_2(T)$ where $U_1(T)$ is the immature prey and $U_2(T)$ is the mature prey, the model involving SIS disease in the immature prey [susceptible immature species S(T) and infected immature species I(T), where $U_1(T) = S(T) + I(T)$ ] with refuge and harvesting in prey population and a predator $V$ which is predate all kinds of the above preys. In fact, because of the complexity of the proposed model mostly between the disease of immature prey and the stage-structure of prey, it was difficult to find an example

of this model in the environment, but it was not hopeless, so with the help of a biologist, we could find an example represented by prey is Sheep with the SIS disease in Lambs and the predator is a Wolf, where the disease is Hemorrhagic fever.

## 2. The Mathematical Model

In this section, an epidemiological mathematical model has been suggested. The model includes of a stage-structured in "prey whose population density at time $T$ is represented by" $U(T)$ and a predator is represented by $V(T)$. The following assumptions are assumed for this model:

1. The population density of the prey consists of stage structured, the immature represented by $U_1(T)$ and the mature which represented by $U_2(T)$, where $U(T) = U_1(T) + U_2(T)$ .

2. An epidemic of type SIS disease in the immature prey's population which divides the population into two classes, namely S(T) that represents the susceptible immature prey's at time $T$ and I(T) that represents the infected immature prey's at time $T$, where $U_1(T) = S(T) + I(T)$.

3. This disease is transmitted through contact between S and I and through an external source, it does not spread to the mature prey and predator. The proposed disease can be treated and does not give immunity to the immature.

4. The immature prey depends on the mature prey on their feeding.

The predator predate the immature (susceptible and infected) and the mature of prey by Lotka Voltera of functional response. Also, this model involving refuge and harvesting, and the parameters are described in Table 1.

Table 1: The model's parameters

| ***Parameters*** | ***Symbolizing from a biological point of view*** |
|---|---|
| $r > 0$ | ***The growth rate of immature's prey.*** |
| $K > 0$ | ***The carrying capacity of the susceptibleprey.*** |
| $\beta_i$ , ***i=1,2,3.*** | ***The maximum predation rate (MPR) of the predator over the susceptible, infected and the mature of prey respectively which are outside refuge.*** |
| $\gamma_j$,$j$***=1,2.*** | ***The infection rate.*** |
| $m_i, \ i = 1, 2, 3$ | ***The refuge rate of the susceptible, infected and the mature of prey respectively.*** |
| $\gamma_3$ | ***The recovering rate.*** |
| $\alpha > 0$ | ***The grown up rate of the immature into mature (in prey population).*** |
| $\eta_i, \ i = 1, 2, 3$ | ***The conversion rate of food from susceptible, infected and the mature of prey respectively.*** |
| $d$ | ***The natural death rate of the predator.*** |
| $\theta_i, \ i = 1, 2, 3$ | ***The harvesting rate of the susceptible, infected and the mature of prey respectively.*** |

According to these assumptions, we propose the model by "first order non-linear differential equations".

$$\begin{aligned}
\frac{\mathrm{dS}}{\mathrm{dT}} &= rU_2\left(1-\frac{U_2}{K}\right) - \alpha S - \theta_1 S - \beta_1(1-m_1)SV - \gamma_1 SI - \gamma_2 S + \gamma_3 I \\
\frac{\mathrm{dI}}{\mathrm{dT}} &= \gamma_1 SI + \gamma_2 S - \gamma_3 I - \beta_2(1-m_2)IV - \theta_2 I \\
\frac{dU_2}{\mathrm{dT}} &= \alpha S - \beta_3(1-m_3)U_2V - \theta_3 U_2 \\
\frac{dV}{\mathrm{dT}} &= \eta_1(1-m_1)SV + \eta_2(1-m_2)IV + \eta_3(1-m_3)U_2V - dV
\end{aligned} \tag{2.1}$$

Note that, the model has eighteen "parameters which make the analysis difficult, so to simplify it, we reduced the number of them by using dimensionless variables and parameters" as follows:

$$t = rT,\ h_1 = \frac{S}{K}\ ,\ h_2 = \frac{I}{K}\ ,\ h_3 = \frac{U_2}{K}\ ,\ h_4 = \frac{V}{K}\ ,\ p_1 = \frac{\alpha}{r}\ ,\ p_2 = \frac{\gamma_2}{r},\ p_3 = \frac{\theta_1}{r},\ p_4 = \frac{\gamma_1 K}{r}\ ,\ p_5 = \frac{\gamma_3}{r}\ ,$$

$$p_6 = \frac{\beta_1(1-m_1)K}{r},\ p_7 = \frac{\beta_2(1-m_2)K}{r},\ p_8 = \frac{\theta_2}{r}\ ,p_9 = \frac{\beta_3(1-m_3)K}{r}\ ,p_{10} = \frac{\theta_3}{r}\ ,\ \ \mathrm{p}_{11} = \frac{\eta_1(1-m_1)K}{r}\ ,$$

$$\mathrm{p}_{12} = \frac{\eta_2\left(1-m_2\right)K}{r},\ p_{13} = \frac{\eta_3\left(1-m_3\right)K}{r}\ ,\ p_{14} = \frac{d}{r}.$$

So the dimensional system (2.1) can be formulated as:

$$\begin{aligned}
\frac{dh_1}{\mathrm{dt}} &= h_3\left(1-h_3\right) - \left(p_1+p_2+p_3\right)h_1 - p_4h_1h_2 + p_5h_2 - p_6h_1h_4 = \widehat{f_1}\left(h_1,h_2,h_3,h_4\right)\\
\frac{dh_2}{\mathrm{dt}} &= p_2h_1 + p_4h_1h_2 - p_5h_2 - p_7h_2h_4 - p_8h_2 = \widehat{f_2}\left(h_1,h_2,h_3,h_4\right)\\
\frac{dh_3}{\mathrm{dt}} &= p_1h_1 - p_9h_3h_4 - p_{10}h_3 = \widehat{f_3}\left(h_1,h_2,h_3,h_4\right)\\
\frac{dh_4}{\mathrm{dt}} &= p_{11}h_1h_4 + p_{12}h_2h_4 + p_{13}h_3h_4 - p_{14}h_4 = \widehat{f_4}\left(h_1,h_2,h_3,h_4\right)
\end{aligned} \tag{2.2}$$

With $h_1(0) \geq 0$ , $h_2(0) \geq 0$ , $h_3(0) \geq 0$ *and* $h_4(0) \geq 0$. It is noticed that the parameters' number have been reduced from eighteen in system (2.2) to fourteen in system (2.2). Clearly, the "interaction functions of system (2.2) are continuous and have continuous partial derivatives on the following positive four dimensional space".

$R^4_+ = \left\{(h_1,h_2,h_3,h_4) \in R^3_+ :\ h_1(0) \geq 0\ ,\ h_2(0) \geq 0\ ,\ h_3(0) \geq 0\ ,\ h_4(0) \geq 0\right\}$. So, "these functions are lipschitzian on $R^4_+$, and hence the solution of system (2.2) exists and unique. Moreover, all the solutions of system (2.2) with positive initial conditions are uniformly bounded as proven in the following theorem".

**Theorem 2.1.** *The solutions of the system* (2.2) *are uniformly bounded.*
**Proof .** *Let* $H(T) = h_1(t) + h_2(t) + h_3(t) + h_4(t)$

$$\frac{dH}{dt} = \frac{dh_1}{dt} + \frac{dh_2}{dt} + \frac{dh_3}{dt} + \frac{dh_4}{dt}$$

$$= h_3\left(1-h_3\right) - p_3h_1 - p_6h_1h_4 - p_7h_2h_4 - p_8h_2 - p_9h_3h_4 - p_{10}h_3 + p_{11}h_1h_4 + p_{12}h_2h_4 + p_{13}h_3h_4 - p_{14}h_4.$$

$$\frac{dH}{dt} < h_3\left(1-h_3\right) - \left(p_6-p_{11}\right)h_1h_4 - \left(p_7-p_{12}\right)h_2h_4 - \left(p_9-p_{13}\right)h_3h_4 - p_3h_1 - p_8h_2 - p_{10}h_3 - p_{14}h_4.$$

*So,* $\frac{dH}{dt} < \frac{1}{4} - MH$ , *where* $M = min\left\{p_3, p_8, p_{10}, p_{14}\right\}$,

$$\frac{dH}{dt} + MH < \frac{1}{4}.$$

*For the initial value* $H(0) = H_0$ *and by the comparison Theorem [16]. It becomes:*

$$H(t) < \frac{1}{4M} + \left(H_0 - \frac{1}{4M}\right)e^{-Mt}.$$

*Thus,* $\lim_{t\to\infty} H(t) \leq \frac{1}{4M}$ , *and therefore,* $0 \leq H \leq \frac{1}{4M}$ , $\forall t > 0$ . □

## 3. Existence of Equilibrium Points (EPs)

In this section, all the (EPs) of system (2.2) have been found. System (2.2) has six (EPs) as bellow:

1. The trivial (EP) $A_0 = (0,0,0,0)$ always exists.

2. The (EP) $A_1 = \left(\widetilde{h}_1, 0, \widetilde{h}_3, 0\right)$ where $\widetilde{h}_1 = \frac{p_{10}}{p_1^2}\left(p_1\left(1-p_{10}\right)-p_3p_{10}\right)$ and $\widetilde{h}_3 = \frac{p_1(1-p_{10})-p_3p_{10}}{p_1}$ provided that:

$$p_{10} < 1, \tag{3.1}$$

$$p_3 < \frac{p_1\left(1-p_{10}\right)}{p_{10}}. \tag{3.2}$$

Hence $A_1 = \left(\widetilde{h}_1, 0, \widetilde{h}_3, 0\right)$ exist under conditions (3.1) and (3.2)

3. The (EP) $A_2 = \left(\overline{h}_1, \overline{h}_2, \overline{h}_3, 0\right)$ where $\overline{h}_3$ is the unique solution of the equation :

$$a_1 h_3^2 + a_2 h_3 + a_3 = 0, \tag{3.3}$$

where $a_1 = p_1p_4p_{10} > 0$, $a_2 = -p_1\left(p_1\left(p_5+p_8\right)+p_4p_{10}\left[1-p_{10}\right]\right)+p_3p_4p_{10}^2$, $a_3 = p_1\left[\left(p_5+p_8\right)\left[p_1\left(1-p_{10}\right)-p_3p_{10}\right]-p_2p_8p_{10}\right]$, and $\overline{h}_1 = h_1\left(\overline{h}_3\right) = \frac{p_{10}\overline{h}_3}{p_1}$, $\overline{h}_2 = h_2\left(\overline{h}_3\right) = \frac{p_2p_{10}\overline{h}_3}{p_1(p_5+p_8)-p_4p_{10}\overline{h}_3}$ exists if in addition to condition (3.1) and (3.2) the following conditions hold:

$$p_1\left(p_1\left(p_5+p_8\right)+p_4p_{10}\left[1-p_{10}\right]\right) > p_3p_4p_{10}^2 \tag{3.4}$$

$$\frac{\left(p_5+p_8\right)\left[p_1\left(1-p_{10}\right)-p_3p_{10}\right]}{p_{10}} > p_2p_8 \tag{3.5}$$

Hence $A_2\left(\overline{h}_1, \overline{h}_2, \overline{h}_3, 0\right)$ and $A_3\left(\overline{h'}_1, \overline{h'}_2, \overline{h}'_3, 0\right)$ exist under conditions (3.4) and (3.5)

4. The free disease (EP) $A_4 = (\widehat{h}_1, 0, \widehat{h}_3, \widehat{h}_4)$ where $\widehat{h}_4$ is the unique solution of the equation:

$$b_1 h_4^3 + b_2 h_4^2 + b_3 h_4 + b_4 = 0, \tag{3.6}$$

where

$$\begin{aligned}
b_1 &= -p_6p_9\left(p_9p_{11}+p_1p_{13}\right) < 0,\\
b_2 &= \left(p_9p_{11}+p_1p_{13}\right)\left(p_1+p_3+p_6p_{10}\right)+p_6p_9p_{10}p_{11} > 0\\
b_3 &= \left(p_9p_{11}+p_1p_{13}\right)\left[p_1\left(1-p_{10}\right)-p_{10}p_3\right]-p_{10}p_{11}\left(p_9\left(p_1+p_3\right)+p_6p_{10}\right)-p_1^2p_{14},\\
b_4 &= p_{10}p_{11}\left[p_1\left(1-p_{10}\right)-p_{10}p_3\right],
\end{aligned}$$

and $\widehat{h}_3 = h_3\left(\widehat{h}_4\right) = \frac{p_1-\left(p_{10}+p_9\widehat{h}_4\right)\left(p_1+p_3+p_6\widehat{h}_4\right)}{p_1}$ ,$\widehat{h}_1 = h_1\left(\widehat{h}_3, \widehat{h}_4\right) = \frac{\widehat{h}_3\left(p_{10}+p_9\widehat{h}_4\right)}{p_1}$ exists if in addition to condition (3.1) and (3.2) the following conditions hold :

$$p_1 > \left(p_{10}+p_9\widehat{h}_4\right)\left(p_1+p_3+p_6\widehat{h}_4\right), \tag{3.7}$$

$$\left(p_9p_{11}+p_1p_{13}\right)\left[p_1\left(1-p_{10}\right)-p_{10}p_3\right] > p_{10}p_{11}\left(p_9\left(p_1+p_3\right)+p_6p_{10}\right)+p_1^2p_{14}. \tag{3.8}$$

5. The coexistence (EP) $A_5 = (h_1^*, h_2^*, h_3^*, h_4^*)$ exists if and only if the set of the following equations have a positive solution:

$$h_3\left(1-h_3\right)-\left(p_1+p_2+p_3\right)h_1-p_4h_1h_2+p_5h_2-p_6h_1h_4 = 0, \tag{3.9}$$

$$p_2h_1+p_4h_1h_2-p_5h_2-p_7h_2h_4-p_8h_2 = 0, \tag{3.10}$$

$$p_1h_1-p_9h_3h_4-p_{10}h_3 = 0, \tag{3.11}$$

$$p_{11}h_1h_4+p_{12}h_2h_4+p_{13}h_3h_4-p_{14}h_4 = 0 \tag{3.12}$$

From equation (3.12) we have,

$$h_3 = \frac{p_{14} - p_{11}h_1 - p_{12}h_2}{p_{13}}. \tag{3.13}$$

By substituting (3.13) in (3.11) we get

$$h_4 = \frac{p_1 p_{13} h_1 - p_{10}\left[p_{14} - p_{11}h_1 - p_{12}h_2\right]}{p_9\left[p_{14} - p_{11}h_1 - p_{12}h_2\right]}. \tag{3.14}$$

Now by substituting (3.13)and (3.14) in (3.9) and in (3.10) yield the following two isoclines:

$$\begin{aligned} F_1(h_1,h_2) = {} & p_9 p_{11}^3 h_1^3 + p_9 p_{12}^3 h_2^3 + p_9 p_{14}^2(p_{13}-p_{14}) + h_1 h_2[p_9 p_{11} p_{12}(p_{13} - 4p_{14} + 3p_{12}h_2) \\ & + p_{13}^2\left[p_9\left(p_4\left[p_{12}h_2 - 2p_{14}\right] - p_5 p_{11}\right) - p_6 p_{10} p_{12}\right]] + h_1^2[p_9 p_{11} h_2\left(3p_{11}^2 p_{12} + p_4 p_{13}^2\right) \\ & + p_{13}^2[p_9 p_{11}(p_1+p_2+p_3) - p_6(p_1 p_{13} + p_{10}p_{11})] - 3p_9 p_{11}^2 p_{13} p_{14}] + p_{14} h_1[p_9 p_{11}(3p_{14} - 2p_{13}) \\ & - p_{13}^2[p_6 p_{10} - p_9(p_1+p_2+p_3)]] + p_9 p_{12} h_2^2[p_{12}(p_{13} - 3p_{14}) - p_5 p_{13}^2] \\ & + p_9 p_{12} p_{14} h_2(3p_{14} - 2p_{13}) = 0, \end{aligned} \tag{3.15}$$

$$\begin{aligned} F_2(h_1,h_2) = {} & p_2 p_9 h_1(p_{14} - p_{11}h_1) + h_2[h_1(p_9[p_4 p_{14} + p_{11}(p_5+p_7+p_8) - p_7(p_1 p_{13} + p_{10}p_{11})] - p_4 p_9 p_{11} h_1) \\ & + p_{12}h_2(p_9[p_5+p_7+p_8] - p_7 p_{10} - p_9 h_1[p_2+p_4]) + p_{14}[p_7 p_{10} - p_9(p_5+p_7+p_8)]] = 0. \end{aligned} \tag{3.16}$$

Now from (3.15) we observed that, when $h_2 \to 0,\ h_1 \to h_1^{*\prime}$ , where $h_1^{*\prime}$ is the unique solution of the equation:

$$c_1 h_1^3 + c_2 h_1^2 + c_3 h_1 + c_4 = 0, \tag{3.17}$$

where $c_1 = p_9 p_{11}^3 > 0, c_2 = p_{13}^2\left[p_9 p_{11}\left(p_1+p_2+p_3\right) - p_6\left[p_1 p_{13} + p_{10}p_{11}\right]\right] - 3p_9 p_{11}^2 p_{13} p_{14}$ ,
$c_3 = p_{14}\left[p_9 p_{11}\left(3p_{14} - 2p_{13}\right) - p_{13}^2\left[p_6 p_{10} - p_9\left(p_1+p_2+p_3\right)\right]\right]$ , $c_4 = p_9 p_{14}^2\left(p_{13} - p_{14}\right)$.
According to the following conditions

$$p_9 p_{11}\left(p_1+p_2+p_3\right) > p_6\left[p_1 p_{13} + p_{10}p_{11}\right], \tag{3.18}$$

$$p_{13}^2\left[p_9 p_{11}\left(p_1+p_2+p_3\right) - p_6\left[p_1 p_{13} + p_{10}p_{11}\right]\right] > 3p_9 p_{11}^2 p_{13} p_{14}, \tag{3.19}$$

$$3p_{14} > 2p_{13}, \tag{3.20}$$

$$p_6 p_{10} < p_9\left(p_1+p_2+p_3\right), \tag{3.21}$$

$$p_9 p_{11}\left(3p_{14} - 2p_{13}\right) > p_{13}^2\left[p_9\left(p_1+p_2+p_3\right) - p_6 p_{10}\right], \tag{3.22}$$

Further, from eq. (3.16) we notice that, when $h_2 \to 0$, then $h_1 \to h_1^* = \frac{p_{14}}{p_{11}}$.

Now, from eq. (3.15) we have: $\frac{dh_1}{dh_2} = -\frac{\left(\frac{\partial F_1}{\partial h_2}\right)}{\left(\frac{\partial F_1}{\partial h_1}\right)}$. So $\frac{dh_1}{dh_2} < 0$ if one of the following of conditions hold:

$$\left(\frac{\partial F_1}{\partial h_2}\right) > 0,\ \left(\frac{\partial F_1}{\partial h_1}\right) > 0 \qquad or \qquad \left(\frac{\partial F_1}{\partial h_2}\right) < 0,\ \left(\frac{\partial F_1}{\partial h_1}\right) < 0. \tag{3.23}$$

Further, from eq. (3.16) we we have: $\frac{dh_1}{dh_2} = -\frac{\left(\frac{\partial F_2}{\partial h_2}\right)}{\left(\frac{\partial F_2}{\partial h_1}\right)}$. So $\frac{dh_1}{dh_2} > 0$ if one of the following of conditions holds:

$$\left(\frac{\partial F_2}{\partial h_2}\right) > 0,\ \left(\frac{\partial F_2}{\partial h_1}\right) < 0 \qquad or \qquad \left(\frac{\partial F_2}{\partial h_2}\right) < 0,\ \left(\frac{\partial F_2}{\partial h_1}\right) > 0 \tag{3.24}$$

Then $A_4 = (h_1^*, h_2^*, h_3^*, h_4^*)$ where $h_3^* = h_3(h_1^*, h_2^*)$ and $h_4^* = h_4(h_1^*, h_2^*)$ provided

$$p_{14} > p_{11}h_1^* + p_{12}h_2^* \tag{3.25}$$

$$h_1^*(p_1p_{13} + p_{10}p_{11}) + p_{10}p_{12}h_2^* > p_{10}p_{14}. \tag{3.26}$$

$$h_1^* < h_1^{*'}. \tag{3.27}$$

These are presents the conditions of existence of $A_5 = (h_1^*, h_2^*, h_3^*, h_4^*)$ .

## 4. The Local Stability Analysis

In this section, by linearization method the stability analysis of all the (EPs) of system (2.2) has been studied analytically. Note that; $\lambda_{ih_1}$, $\lambda_{ih_2}$, $\lambda_{ih_3}$ and $\lambda_{ih_4}$ denote the "eigenvalues of the Jacobian matrix (JM) $J_i = J(A_i)\,; i = 0, 1, 2, 3, 4, 5$ which describe the dynamics in the direction of $h_1, h_2, h_3$ and $h_4$ respectively. We can write" it for each (EPs)

$$J_i = \begin{bmatrix} -(p_1+p_2+p_3+p_4h_2+p_6h_4) & p_5 - p_4h_1 & 1-2h_3 & -p_6h_1 \\ p_2+p_4h_2 & p_4h_1 - p_7h_4 - (p_5+p_8) & 0 & -p_7h_2 \\ p_1 & 0 & -(p_9h_4+p_{10}) & -p_9h_3 \\ p_{11}h_4 & p_{12}h_4 & p_{13}h_4 & p_{11}h_1+p_{12}h_2+p_{13}h_3-p_{14} \end{bmatrix} \tag{4.1}$$

*4.1. local stability of (EP)* $\mathbf{A_0} = (\mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{0})$

The (JM) at $A_0$ become

$$J_0 = J(A_0) = \begin{bmatrix} -(p_1+p_2+p_3) & p_5 & 1 & 0 \\ p_2 & -(p_5+p_8) & 0 & 0 \\ p_1 & 0 & -p_{10} & 0 \\ 0 & 0 & 0 & -p_{14} \end{bmatrix} \tag{4.2}$$

The characteristic equation of $J_0$ can be given by:

$$(\lambda + p_{14})\left(\lambda^3 + S_1\lambda^2 + S_2\lambda + S_3\right) = 0$$

where $S_1 = p_1+p_2+p_3+p_5+p_8+p_{10} > 0$, $S_2 = p_{10}(p_2+p_3+p_5+p_8) + (p_1+p_3)(p_5+p_8) + p_1(p_{10}-1) + p_2p_8$, $S_3 = p_{10}(p_2p_8 + p_3(p_5+p_8)) + p_1(p_{10}-1)(p_5+p_8)$.
Now either

$$\begin{aligned} &\lambda + p_{14} = 0, \qquad so \quad \lambda_{0h_4} = -p_{14} < 0. \qquad Or \\ &\lambda^3 + S_1\lambda^2 + S_2\lambda + S_3 = 0 \end{aligned} \tag{4.3}$$

Routh-Hurwitz principle have been used to find the roots of eq. (4.3) . So, all the roots of eq.(4.1$a$), have real parts less than zero if and only if $S_i > 0,\ i = 1, 3$ $and$ $\Delta = S_1S_2 - S_3 > 0$.
Through direct calculations shows that $S_3 > 0$ by negating condition (3.1).
Also, $\Delta = p_1(p_{10}-1)(p_1 + (p_5+p_8)(p_2+p_3+p_5+p_8+p_{10}) - p_5 - p_8)$ $+ (p_5+p_8)[p_{10}(p_1-p_3) + p_1(p_1+p_3)] + p_2p_{10}(p_1-p_8) + p_1(1+p_3p_{10}) + p_2p_8 + (p_2+p_3+p_5+p_8$ $+p_{10})[p_{10}(p_2p_8 + p_3(p_5+p_8)) + p_2p_8] > 0$, by negating condition (3.1) and the following condition holds

$$p_1 > max\{p_3\ ,\ p_8\} \tag{4.4}$$

$$p_1 + (p_5+p_8)(p_2+p_3+p_5+p_8+p_{10}) > p_5 + p_8 \tag{4.5}$$

Thus, the (EP) $A_0$ is local asymptotically stable (LAS).

*4.2. Local Stability of (EP)* $\mathbf{A_1}=\left(\widetilde{\mathbf{h}}_\mathbf{1},\mathbf{0},\widetilde{\mathbf{h}}_\mathbf{3},\mathbf{0}\right)$

At $A_1$the (JM) become

$$J_1 = J\left(A_1\right) = \left[a_{ij}\right]_{4\times 4}$$
$$= \begin{bmatrix} -\left(p_1+p_2+p_3\right) & p_5-p_4\widetilde{h}_1 & 1-2\widetilde{h}_3 & -p_6\widetilde{h}_1 \\ p_2 & p_4\widetilde{h}_1-\left(p_5+p_8\right) & 0 & 0 \\ p_1 & 0 & -p_{10} & -p_9\widetilde{h}_3 \\ 0 & 0 & 0 & p_{11}\widetilde{h}_1+p_{13}\widetilde{h}_3-p_{14} \end{bmatrix} \tag{4.6}$$

The characteristic equation of $J_1$ can be given by:

$$\left(a_{44}-\lambda\right)\left(\lambda^3+K_1\lambda^2+K_2\lambda+K_3\right)=0$$

where $K_1=-\left[a_{11}+a_{22}+a_{33}\right]$, $K_2=a_{11}\left[a_{22}+a_{33}\right]-a_{22}a_{33}-a_{12}a_{21}-a_{13}a_{31}$,
$K_3=a_{13}a_{31}a_{22}+a_{33}\left(p_4\widetilde{h}_1-\frac{(p_5+p_8)(p_1+p_3)+p_2p_8}{(p_1+p_3)}\right)$.
Now either $a_{44}-\lambda=0$ so $\lambda_{1h_4}=a_{44}<0$ if the following condition holds

$$p_{11}\widetilde{h}_1+p_{13}\widetilde{h}_3<p_{14}. \tag{4.7}$$
$$Or\ \ \lambda^3+K_1\lambda^2+K_2\lambda+K_3=0 \tag{4.8}$$

Routh-Hurwitz principle have been used to find the roots of eq. (4.8). So, all the roots of eq.(4.8), have real parts less than zero if and only if $K_i>0,\ i=1,3\ \ and\ \Delta=K_1K_2-K_3>0$.
The direct calculations shows that $K_i>0,\ i=1,3$ if the following conditions hold:

$$p_4\widetilde{h}_1<min\left\{p_5,\frac{\left(p_5+p_8\right)\left(p_1+p_3\right)+p_2p_8}{\left(p_1+p_3\right)}\right\}, \tag{4.9}$$
$$1<2\widetilde{h}_3 \tag{4.10}$$
$$\Delta=\left(a_{11}+a_{22}+a_{33}\right)\left(a_{12}a_{21}+a_{13}a_{31}+a_{22}a_{33}-a_{11}\left(a_{22}+a_{33}\right)\right)+a_{33}\left(\frac{\left(p_5+p_8\right)\left(p_1+p_3\right)+p_2p_8}{\left(p_1+p_3\right)}-p_4\widetilde{h}_1\right)$$
$$-a_{13}a_{31}a_{22}>0,$$

if (4.9) and (4.10) and the following conditions hold

$$a_{12}a_{21}+a_{22}a_{33}<a_{11}\left(a_{22}+a_{33}\right)-a_{13}a_{31}, \tag{4.11}$$
$$\left(a_{11}+a_{22}+a_{33}\right)\left(a_{12}a_{21}+a_{13}a_{31}+a_{22}a_{33}-a_{11}\left(a_{22}+a_{33}\right)\right)+a_{13}a_{31}a_{22}$$
$$>a_{33}\left(p_4\widetilde{h}_1-\frac{\left(p_5+p_8\right)\left(p_1+p_3\right)+p_2p_8}{\left(p_1+p_3\right)}\right), \tag{4.12}$$

Thus the (EP) $A_1$ becomes (LAS).

*4.3. Local Stability of (EP)* $\mathbf{A_2}\left(\overline{\mathbf{h}}_\mathbf{1},\overline{\mathbf{h}}_\mathbf{2},\overline{\mathbf{h}}_\mathbf{3},\mathbf{0}\right)$ *and* $\mathbf{A_3}\left(\overline{\mathbf{h}'}_\mathbf{1},\overline{\mathbf{h}'}_\mathbf{2},\overline{\mathbf{h}'}_\mathbf{3},\mathbf{0}\right)$

The (JM) at $\mathrm{A_2}$ is the same for $\mathrm{A_3}$ is become:

$$J_2=J\left(A_2\right)=\left[n_{\mathrm{ij}}\right]_{4\times 4}$$
$$= \begin{bmatrix} -\left(p_1+p_2+p_3+p_4\overline{h}_2\right) & p_5-p_4\overline{h}_1 & 1-2\overline{h}_3 & -p_6\overline{h}_1 \\ p_2+p_4\overline{h}_2 & p_4\overline{h}_1-\left(p_5+p_8\right) & 0 & -p_7\overline{h}_2 \\ p_1 & 0 & -p_{10} & -p_9\overline{h}_3 \\ 0 & 0 & 0 & p_{11}\overline{h}_1+p_{12}\overline{h}_2+p_{13}\overline{h}_3-p_{14} \end{bmatrix} \tag{4.13}$$

The characteristic equation of $J_2$ can be given by:

$$(n_{44}-\lambda)\left(\lambda^3+D_1\lambda^2+D_2\lambda+D_3\right)=0$$

where $D_1=-(n_{11}+n_{22}+n_{33})$ , $D_2=n_{22}n_{33}+n_{11}(n_{22}+n_{33})-n_{12}n_{21}-n_{13}n_{31}$,
$D_3=n_{13}n_{31}n_{22}-n_{33}\left(p_4\overline{h}_1-\frac{(p_5+p_8)(p_1+p_3)+p_8(p_2+p_4\overline{h}_2)}{(p_1+p_3)}\right)$.
Now either $n_{44}-\lambda=0$ so $\lambda_{2h_4}=n_{44}<0$ if the following condition holds

$$p_{11}\overline{h}_1+p_{12}\overline{h}_2+p_{13}\overline{h}_3<p_{14} \tag{4.14}$$

$$Or \quad \lambda^3+D_1\lambda^2+D_2\lambda+D_3=0 \tag{4.15}$$

Routh-Hurwitz principle have been used to find the roots of eq. (4.15) . So, all the roots of eq. (4.15), have real parts less than zero if and only if $D_i>0, i=1,3$ *and* $\Delta=D_1D_2-D_3>0$.
The direct calculations shows that $D_i>0, i=1,3$ if the following conditions hold:

$$p_4\overline{h}_1<min\left\{p_5,\frac{(p_5+p_8)(p_1+p_3)+p_8(p_2+p_4\overline{h}_2)}{(p_1+p_3)}\right\} \tag{4.16}$$

$$2\overline{h}_3>1, \tag{4.17}$$

$\Delta=(n_{11}+n_{22}+n_{33})(n_{12}n_{21}+n_{13}n_{31}-n_{11}(n_{22}+n_{33}))>0$, if (4.14) and (4.15) and the following condition holds:

$$n_{11}(n_{22}+n_{33})-n_{13}n_{31}>n_{12}n_{21}. \tag{4.18}$$

Thus, the (EP) $A_2$ becomes (LAS) .

*4.4. Local Stability of (EP)* $\mathbf{A_4}=(\widehat{\mathbf{h}}_1,\mathbf{0},\widehat{\mathbf{h}}_3,\widehat{\mathbf{h}}_4)$

At $A_4$ the Jacobian matrix become:

$$J_4=J(A_4)=[m_{ij}]_{4\times4}$$

$$=\begin{bmatrix}\left(p_1+p_2+p_3+p_6\widehat{h}_4\right) & p_5-p_4\widehat{h}_1 & 1-2\widehat{h}_3,\ m_{14}=-p_6\widehat{h}_1 & \\ p_2 & p_4\widehat{h}_1-(p_5+p_8)-p_7\widehat{h}_4 & 0 & 0\\ p_1 & 0 & -\left(p_{10}+p_9\widehat{h}_4\right) & -p_9\widehat{h}_3\\ p_{11}\widehat{h}_4 & p_{12}\widehat{h}_4 & p_{13}\widehat{h}_4 & 0\end{bmatrix} \tag{4.19}$$

Then the characteristic equation of $J_4$ is given by:

$$\lambda^4+L_1\lambda^3+L_2\lambda^2+L_3\lambda+L_4=0, \tag{4.20}$$

where,

$L_1=-(m_{11}+m_{22}+m_{33})$,
$L_2=m_{11}(m_{22}+m_{33})+m_{22}m_{33}-m_{12}m_{21}-m_{13}m_{31}-m_{14}m_{41}-m_{43}m_{34}$,
$L_3=m_{43}m_{34}(m_{11}+m_{22})+m_{14}[m_{41}(m_{33}+m_{22})-m_{21}m_{42}-m_{31}m_{43}]-m_{33}m_{11}m_{22}+m_{12}m_{21}m_{33}$
$+m_{13}[p_1p_4\widehat{h}_1+p_9p_{11}\widehat{h}_3\widehat{h}_4-p_1\left(p_5+p_8+p_7\widehat{h}_4\right)]$,

$L_4=m_{14}[m_{33}(m_{21}m_{42}-m_{22}m_{41})+m_{22}m_{31}m_{43}]+m_{22}m_{34}(\frac{p_{11}+p_{13}\left(p_1+p_2+p_3+p_6\widehat{h}_4\right)}{p_{11}}-2\widehat{h}_3)$
$+m_{21}m_{34}(m_{12}m_{43}-m_{13}m_{42})$.

Routh-Hurwitz principle can be used to find the roots of eq.(4.20). So, all the roots of eq.(4.20) , have real parts less than zero if and only if $L_i > 0,\ i = 1,3,4\ and\ \Delta = (L_1L_2 - L_3)\,L_3 - L_1^2L_4 > 0$. Through direct calculations shows that $L_i > 0,\ i = 1,3,4$ if the following condition hold:

$$p_4\widehat{h}_1 < p_5\ , \tag{4.21}$$

$$p_1p_4\widehat{h}_1 + p_9p_{11}\widehat{h}_3\widehat{h}_4 < p_1(p_5 + p_8 + p_7\widehat{h}_4), \tag{4.22}$$

$$m_{12}m_{21}m_{33} > m_{11}m_{22}m_{33} + m_{14}\left[m_{21}m_{42} + m_{31}m_{43} - m_{41}\left(m_{33} + m_{22}\right)\right] - m_{43}m_{34}\left(m_{11} + m_{22}\right) + m_{13}\left(p_1(p_5 + p_8 + p_7\widehat{h}_4) - p_1p_4\widehat{h}_1 - p_9p_{11}\widehat{h}_3\widehat{h}_4\right), \tag{4.23}$$

$$1 < 2\widehat{h}_3 < \frac{p_{11} + p_{13}\left(p_1 + p_2 + p_3 + p_6\widehat{h}_4\right)}{p_{11}}, \tag{4.24}$$

$$m_{14}\left[m_{33}\left(m_{21}m_{42} - m_{22}m_{41}\right) + m_{22}m_{31}m_{43}\right] + m_{34}m_{22}\left(\frac{p_{11} + p_{13}\left(p_1 + p_2 + p_3 + p_6\widehat{h}_4\right)}{p_{11}} - 2\widehat{h}_3\right) > m_{21}m_{34}\left(m_{13}m_{42} - m_{12}m_{43}\right) \tag{4.25}$$

$\Delta = q_1 - q_2$ , where

$$q_1 = (m_{11} + m_{22} + m_{33})(\Big[\left(p_1 + p_3 + p_{10} + \widehat{h}_4\left[p_6 + p_9\right]\right)\left(p_4\widehat{h}_1 - p_5 - p_8 - p_7\widehat{h}_4\right) - p_2p_8 - \widehat{h}_4\left(p_2p_7 + p_6p_{11}\widehat{h}_1\right) - \left(p_1 + p_2 + p_3 + p_6\widehat{h}_4\right)\left(p_{10} + p_9\widehat{h}_4\right) - \widehat{h}_3\left(2p_1 + p_9p_{13}\widehat{h}_4\right)\Big]\left[m_{43}m_{34}(m_{11} + m_{22}) - m_{33}(m_{11}m_{22} - m_{12}m_{21}) + m_{14}[m_{41}\left(m_{33} + m_{22}\right) - m_{21}m_{42} - m_{31}m_{43}] + m_{13}(m_{31}m_{22} - m_{41}m_{34})\right] - (m_{11} + m_{22} + m_{33})(m_{34}\Big[\left[p_{11} + p_{13}\left(p_1 + p_2 + p_3 + p_6\widehat{h}_4\right)\right]\left[p_5 + p_8 + p_7\widehat{h}_4 - p_4\widehat{h}_1\right] + p_4\widehat{h}_1\left[2p_{11}\widehat{h}_3 + p_2p_{13}\right] - p_2\left[p_5p_{13} + p_{12}\left(2\widehat{h}_4 - 1\right)\right] + 2p_{11}\widehat{h}_4\left(p_5 + p_8 + p_7\widehat{h}_4\right)\Big] + m_{14}(m_{33}\left[m_{21}m_{42} - m_{22}m_{41}\right] + m_{22}m_{31}m_{43}))),$$

$$q_2 = \left[m_{43}m_{34}(m_{11} + m_{22}) + m_{14}\left[m_{41}(m_{33} + m_{22}) - m_{21}m_{42} - m_{31}m_{43}\right] - m_{33}(m_{11}m_{22} - m_{12}m_{21}) + m_{13}(m_{31}m_{22} - m_{41}m_{34})\right]^2$$

$\Delta > 0$ if conditions (4.21)-(4.24) and the following conditions hold

$$\left(p_1 + p_3 + p_{10} + \widehat{h}_4\left[p_6 + p_9\right]\right)\left(p_4\widehat{h}_1 - p_5 - p_8 - p_7\widehat{h}_4\right) < p_2p_8 + \widehat{h}_4\left(p_2p_7 + p_6p_{11}\widehat{h}_1\right) + \left(p_1 + p_2 + p_3 + p_6\widehat{h}_4\right)\left(p_{10} + p_9\widehat{h}_4\right) + \widehat{h}_3\left(2p_1 + p_9p_{13}\widehat{h}_4\right), \tag{4.26}$$

$$\left[p_{11} + p_{13}\left(p_1 + p_2 + p_3 + p_6\widehat{h}_4\right)\right]\left[p_5 + p_8 + p_7\widehat{h}_4 - p_4\widehat{h}_1\right] + p_4\widehat{h}_1\left[2p_{11}\widehat{h}_3 + p_2p_{13}\right] > p_2\left[p_5p_{13} + p_{12}\left(2\widehat{h}_4 - 1\right)\right] + 2p_{11}\widehat{h}_4\left(p_5 + p_8 + p_7\widehat{h}_4\right), \tag{4.27}$$

$$m_{34}\Big[\left[p_{11} + p_{13}\left(p_1 + p_2 + p_3 + p_6\widehat{h}_4\right)\right]\left[p_5 + p_8 + p_7\widehat{h}_4 - p_4\widehat{h}_1\right] + p_4\widehat{h}_1\left[2p_{11}\widehat{h}_3 + p_2p_{13}\right] - p_2\left[p_5p_{13} + p_{12}\left(2\widehat{h}_4 - 1\right)\right] + 2p_{11}\widehat{h}_4\left(p_5 + p_8 + p_7\widehat{h}_4\right)\Big] > m_{14}\left(m_{33}\left[m_{22}m_{41} - m_{21}m_{42}\right] - m_{22}m_{31}m_{43}\right) \tag{4.28}$$

$$q_1 > q_2 \tag{4.29}$$

Thus, the (EP) $A_4(\widehat{h}_1, 0, \widehat{h}_3, \widehat{h}_4)$ becomes (LAS) .

*4.5. local stability of (EPS)* $\mathbf{A_5}$=($\mathbf{h_1^*}$,$\mathbf{h_2^*}$,$\mathbf{h_3^*}$,$\mathbf{h_4^*}$)

The JM at $A_5$, can be written as:

$$J_5 = J(A_5) = [r_{ij}]_{4\times 4} = \begin{bmatrix} -(p_1+p_2+p_3+p_4h_2^*+p_6h_4^*) & p_5-p_4h_1^* & 1-2h_3^* & -p_6h_1^* \\ p_2+p_4h_2^* & p_4h_1^*-(p_5+p_8)-p_7h_4^* & 0 & -p_7h_2^* \\ p_1 & 0 & -(p_9h_4^*+p_{10}) & -p_9h_3^* \\ p_{11}h_4^* & p_{12}h_4^* & p_{13}h_4^* & 0 \end{bmatrix} \quad (4.30)$$

Then the characteristic equation of $J_4$ is given by:

$$\lambda^4 + E_1\lambda^3 + E_2\lambda^2 + E_3\lambda + E_4 = 0, \quad (4.31)$$

where

$$\begin{aligned} E_1 &= -(r_{11}+r_{22}+r_{33}), \\ E_2 &= r_{11}(r_{22}+r_{33}) + r_{22}r_{33} - r_{12}r_{21} - r_{13}r_{31} - r_{34}r_{43} - r_{14}r_{41} - r_{24}r_{42}, \\ E_3 &= r_{34}r_{43}(r_{11}+r_{22}) + r_{24}r_{42}(r_{11}+r_{33}) + r_{14}r_{41}(r_{22}+r_{33}) + r_{33}[(p_2+p_4h_2^*)(p_8+p_7h_4^*) \\ &\quad - (p_1+p_3+p_6h_4^*)(p_4h_1^*-p_5-p_8-p_7h_4^*)] + r_{31}(r_{13}r_{22}-r_{14}r_{43}) - r_{14}r_{21}r_{42} - r_{41}(r_{12}r_{42}+r_{13}r_{34}) \\ E_4 &= r_{34}r_{43}([(p_2+p_4h_2^*)(p_8+p_7h_4^*) - (p_1+p_3+p_6h_4^*)(p_4h_1^*-p_5-p_8-p_7h_4^*)) + r_{42}r_{33}(r_{14}r_{21}-r_{11}r_{24}) \\ &\quad + r_{12}r_{24}(r_{41}r_{33}-r_{31}r_{43}) + r_{14}r_{22}(r_{31}r_{43}-r_{41}r_{33}) + r_{13}r_{24}r_{42}r_{41} + r_{13}r_{34}(r_{22}r_{41}-r_{21}r_{42}). \end{aligned}$$

Routh-Hurwitz principle can be used to find the roots of eq. (4.31) . So, all the roots of eq.(4.31), have real parts less than zero if and only if $E_i > 0,\ i = 1,3,4$ *and* $\Delta = (E_1E_2 - E_3)E_3 - E_1^2E_4 > 0$. Through direct calculations shows that $E_i > 0,\ i = 1,3,4$ if the following conditions hold:

$$p_5 > p_4h_1^*, \quad (4.32)$$

$$1 < 2h_3^*, \quad (4.33)$$

$$(p_2+p_4h_2^*)(p_8+p_7h_4^*) > (p_1+p_3+p_6h_4^*)(p_4h_1^*-p_5-p_8-p_7h_4^*), \quad (4.34)$$

$$\begin{aligned} &r_{34}r_{43}[r_{11}+r_{22}] + r_{24}r_{42}(r_{11}+r_{33}) + r_{14}r_{41}(r_{22}+r_{33}) + r_{33}[(p_2+p_4h_2^*)(p_8+p_7h_4^*) - (p_1+p_3 \\ &+ p_6h_4^*)(p_4h_1^*-p_5-p_8-p_7h_4^*)] + r_{31}(r_{13}r_{22}-r_{14}r_{43}) - r_{14}r_{21}r_{42} > r_{41}(r_{12}r_{42}+r_{13}r_{34}), \end{aligned} \quad (4.35)$$

$$\begin{aligned} &r_{34}r_{43}((p_2+p_4h_2^*)(p_8+p_7h_4^*) - (p_1+p_3+p_6h_4^*)(p_4h_1^*-p_5-p_8-p_7h_4^*)) + r_{42}r_{33}(r_{14}r_{21}-r_{11}r_{24}) \\ &+ r_{12}r_{24}(r_{41}r_{33}-r_{31}r_{43}) + r_{14}r_{22}(r_{31}r_{43}-r_{41}r_{33}) + r_{13}r_{24}r_{42}r_{41} > r_{13}r_{34}(r_{21}r_{42}-r_{22}r_{41}) \end{aligned} \quad (4.36)$$

$\Delta = u_1 - u_2$, where

$$\begin{aligned} u_1 &= (r_{11}+r_{22}+r_{33})[(r_{13}r_{31}+r_{34}r_{43}+r_{12}r_{21}+r_{14}r_{41}+r_{24}r_{42}-r_{11}(r_{22}+r_{33})-r_{22}r_{33})(r_{34}r_{43})(r_{11}+r_{22}) \\ &\quad + r_{42}r_{24}(r_{11}+r_{33}) - r_{14}r_{21} + r_{41}(r_{14}(r_{22}+r_{33}) - r_{12}r_{42} - r_{13}r_{34}) \\ &\quad + r_{33}((p_2+p_4h_2^*)(p_8+p_7h_4^*) - (p_1+p_3+p_6h_4^*)(p_4h_1^*-p_5-p_8-p_7h_4^*))((p_2+p_4h_2^*) \\ &\quad (p_8+p_7h_4^*) - (p_1+p_3+p_6h_4^*)(p_4h_1^*-p_5-p_8-p_7h_4^*)) + r_{31}(r_{13}r_{22}-r_{14}r_{43}) \\ &\quad + (r_{11}+r_{22}+r_{33})(r_{34}[r_{43}((p_2+p_4h_2^*)(p_8+p_7h_4^*) - (p_1+p_3+p_6h_4^*)(p_4h_1^* \\ &\quad - p_5-p_8-p_7h_4^*)) + r_{13}(r_{21}r_{42}-r_{22}r_{41})] \\ &\quad + r_{42}[r_{33}(r_{11}r_{24}-r_{14}r_{21}) - r_{13}r_{24}r_{14}] + r_{12}r_{24}(r_{31}r_{43}-r_{41}r_{33}) + r_{14}r_{22}[r_{41}r_{33}-r_{31}r_{43}])]], \\ u_2 &= [r_{34}r_{43}(r_{11}+r_{22}) + r_{42}(r_{24}(r_{11}+r_{33}) - r_{14}r_{21}) + r_{41}(r_{14}(r_{22}+r_{33}) - r_{12}r_{42} - r_{13}r_{34}) \\ &\quad + r_{33}((p_2+p_4h_2^*)(p_8+p_7h_4^*) - (p_1+p_3+p_6h_4^*)(p_4h_1^*-p_5-p_8-p_7h_4^*)) + r_{31}(r_{13}r_{22}-r_{14}r_{43})]^2 \end{aligned}$$

$\Delta > 0$ if in addition to conditions (4.32)- (4.36), the following condition hold

$$u_1 > u_2 \tag{4.37}$$

Thus, the (EP) $A_5\left(h_1^*, h_2^*, h_3^*, h_4^*\right)$ is (LAS) .

## 5. Global Stability Analysis (GS)

In this section, by Lyapunov method the (GS) analysis for the (LAS) (EPs) have been considered analytically as in the next theorems

**Theorem 5.1.** *Assume that $A_0 = (0, 0, 0, 0)$ is the (LAS) in the $R_+^4$. Then $A_0$ is global asymptotically stabile (GAS) if the following condition hold:*

$$h_3 \geq 1 \tag{5.1}$$

**Proof .** *Suggest the following function*

$$\widehat{V}_0\left(h_1, h_2, h_3, h_4\right) = h_1 + h_2 + h_3 + h_4$$

*Clearly $\widehat{V}_0 : R_+^4 \to R$ is a $C^1$ "positive definite function. Then by differentiating $\widehat{V}_0$ for time $t$ and by some algebraic manipulation " we get:*

$$\begin{aligned}\frac{d\widehat{V}_0}{dt} &= \frac{dh_1}{dt} + \frac{dh_2}{dt} + \frac{dh_3}{dt} + \frac{dh_4}{dt}\\ &= h_3\left(1 - h_3\right) - p_3 h_1 - p_6 h_1 h_4 - p_7 h_2 h_4 - p_8 h_2 - p_9 h_3 h_4 - p_{10} h_3 + p_{11} h_1 h_4 + p_{12} h_2 h_4 + p_{13} h_3 h_4 - p_{14} h_4,\\ \frac{d\widehat{V}_0}{dt} &< h_3\left(1 - h_3\right) - p_3 h_1 - p_8 h_2 - p_{10} h_3 - p_{14} h_4 - \left(p_6 - p_{11}\right) h_1 h_4 - \left(p_7 - p_{12}\right) h_2 h_4 - \left(p_9 - p_{13}\right) h_3 h_4,\end{aligned} \tag{5.2}$$

*By the biological facts $p_6 > p_{11}$ , $p_7 > p_{12}$ and $p_9 > p_{13}$ we get*

$$\frac{d\widehat{V}_0}{dt} < h_3\left(1 - h_3\right) - p_3 h_1 - p_8 h_2 - p_{10} h_3 - p_{14} h_4 \tag{5.3}$$

*Now according to the condition* (5.1)

$$\frac{d\widehat{V}_0}{dt} < 0. \tag{5.4}$$

*Hence $A_0$ is a (GAS).* □

**Theorem 5.2.** *Assume that $A_1 = \left(\widetilde{h}_1, 0, \widetilde{h}_3, 0\right)$ is (LAS) in the $R_+^4$. Then $A_1$ is (GAS) if that the* *following conditions hold:*

$$\left[\sqrt{\frac{\widetilde{h}_3\left(1-\widetilde{h}_3\right)}{h_1\widetilde{h}_1}}\left(h_1-\widetilde{h}_1\right)-\sqrt{\frac{p_1\widetilde{h}_1}{h_3\widetilde{h}_3}}\left(h_3-\widetilde{h}_3\right)\right]^2+p_8h_2+h_4\left(p_{14}-p_9\widetilde{h}_3\right)+\left(\frac{p_4h_1-p_5}{h_1}\right)h_2\widetilde{h}_1>$$

$$p_2h_1+p_6\widetilde{h}_1h_4, \tag{5.5}$$

$$p_4h_1<p_5\ , \tag{5.6}$$

$$p_9\widetilde{h}_3<p_{14}, \tag{5.7}$$

$$1>\widetilde{h}_3, \tag{5.8}$$

$$\frac{1-\left(h_3+\widetilde{h}_3\right)}{h_1}+\frac{p_1}{h_3}<2\sqrt{\frac{p_1\left(1-\widetilde{h}_3\right)}{h_1h_3}}. \tag{5.9}$$

**Proof .** *Suggest the following function*

$$\widehat{V}_1\left(h_1,h_2,h_3,h_4\right)=\left(h_1-\widetilde{h}_1-\widetilde{h}_1\ln\frac{h_1}{\widetilde{h}_1}\right)+h_2+\left(h_3-\widetilde{h}_3-\widetilde{h}_3\ln\frac{h_3}{\widetilde{h}_3}\right)+h_4$$

*Clearly* $\widehat{V}_1 : R^4_+ \to R$ *is a* $C^1$ *"positive definite function. Then by differentiating* $\widehat{V}_1$ *for time* $t$ *and by some algebraic manipulation " we get:*

$$\frac{d\widehat{V}_1}{dt}=-\left(h_1-\widetilde{h}_1\right)^2\left(\frac{\widetilde{h}_3\left(1-\widetilde{h}_3\right)}{h_1\widetilde{h}_1}\right)+\left(h_1-\widetilde{h}_1\right)\left(h_3-\widetilde{h}_3\right)\left(\frac{1-\left(h_3+\widetilde{h}_3\right)}{h_1}+\frac{p_1}{h_3}\right)-\frac{p_1\widetilde{h}_1}{h_3\widetilde{h}_3}\left(h_3-\widetilde{h}_3\right)^2$$
$$-h_1h_4\left(p_6-p_{11}\right)-h_2h_4\left(p_7-p_{12}\right)-h_3h_4\left(p_9-p_{13}\right)+p_2h_1-p_8h_2+h_4\left(p_9\widetilde{h}_3-p_{14}\right)+\widetilde{h}_1h_2\left[p_4-\frac{p_5}{h_1}\right]$$
$$+p_6\widetilde{h}_1h_4, \tag{5.10}$$

*By the biological facts* $p_6>p_{11}$ *,* $p_7>p_{12}$ *and* $p_9>p_{13}$ *we get*

$$\frac{d\widehat{V}_1}{dt}<-\left(h_1-\widetilde{h}_1\right)^2\left(\frac{\widetilde{h}_3\left(1-\widetilde{h}_3\right)}{h_1\widetilde{h}_1}\right)+\left(h_1-\widetilde{h}_1\right)\left(h_3-\widetilde{h}_3\right)\left(\frac{1-\left(h_3+\widetilde{h}_3\right)}{h_1}+\frac{p_1}{h_3}\right)-\frac{p_1\widetilde{h}_1}{h_3\widetilde{h}_3}\left(h_3-\widetilde{h}_3\right)^2$$
$$+p_2h_1-p_8h_2+h_4\left(p_9\widetilde{h}_3-p_{14}\right)+\widetilde{h}_1h_2\left[p_4-\frac{p_5}{h_1}\right]+p_6\widetilde{h}_1h_4, \tag{5.11}$$

*Now according to the conditions* (5.8) *and* (5.9)

$$\frac{d\widehat{V}_1}{dt}<-\left[\sqrt{\frac{\widetilde{h}_3\left(1-\widetilde{h}_3\right)}{h_1\widetilde{h}_1}}\left(h_1-\widetilde{h}_1\right)-\sqrt{\frac{p_1\widetilde{h}_1}{h_3\widetilde{h}_3}}\left(h_3-\widetilde{h}_3\right)\right]^2+p_2h_1-p_8h_2+h_4\left(p_9\widetilde{h}_3-p_{14}\right)$$
$$+\left(\frac{p_4h_1-p_5}{h_1}\right)\widetilde{h}_1h_2+p_6\widetilde{h}_1h_4 \tag{5.12}$$

*Now according to the condition* (5.5) - (5.7) *we have* $\frac{d\widehat{V}_1}{dT}<0$.
*Hence* $A_1$ *is a (GAS).* □

Moreover since there are two (EPs) $A_2\left(\overline{h}_1,\overline{h}_2,\overline{h}_3,0\right)$ and $A_3\left(\overline{h'}_1,\overline{h}'_2,\overline{h'}_3,0\right)$ in the interior of $R_4^+$ having exactly the same conditions of local stability but with various neighborhood of starting points then it is impossible to studying the global stability of them using Lyapunove function. So we will study it numerically instead of analytically as shown in the next section.

**Theorem 5.3.** *Assume that the (LAS) $A_4=(\widehat{h}_1,0,\widehat{h}_3,\widehat{h}_4)$ in the $R_+^4$. Then $A_4$ is (GAS) if the following conditions hold :*

$$p_8+p_{12}\widehat{h}_4+\frac{p_5\widehat{h}_1}{h_1}>p_4\widehat{h}_1 \tag{5.13}$$

$$\left[\sqrt{\frac{\widehat{h}_3\left(1-\widehat{h}_3\right)}{h_1\widehat{h}_1}}\left(h_1-\widehat{h}_1\right)-\sqrt{\frac{p_1\widehat{h}_1}{h_3\widehat{h}_3}}\left(h_3-\widehat{h}_3\right)\right]^2+h_2\left(p_8+p_{12}\widehat{h}_4-\frac{\widehat{h}_1\left(p_4h_1-p_5\right)}{h_1}\right)>p_2h_1, \tag{5.14}$$

$$1>\widehat{h}_3>1-h_3, \tag{5.15}$$

$$\frac{1-\left(h_3+\widehat{h}_3\right)}{h_1}+\frac{p_1}{h_3}<2\sqrt{\frac{p_1\left(1-\widehat{h}_3\right)}{h_1h_3}}. \tag{5.16}$$

**Proof .** *Suggest the following function*

$$\widehat{V}_2\left(h_1,h_2,h_3,h_4\right)=\left(h_1-\widehat{h}_1-\widehat{h}_1\ln\frac{h_1}{\widehat{h}_1}\right)+h_2+\left(h_3-\widehat{h}_3-\widehat{h}_3\ln\frac{h_3}{\widehat{h}_3}\right)+\left(h_4-\widehat{h}_4-\widehat{h}_4\ln\frac{h_4}{\widehat{h}_4}\right).$$

*Clearly $\widehat{V}_2:R_+^4\to R$ is a $C^1$ "positive definite function. Then by differentiating $\widehat{V}_2$ for time $t$ and by some algebraic manipulation " we get:*

$$\begin{aligned}\frac{d\widehat{V}_2}{dt}=&-\left(h_1-\widehat{h}_1\right)^2\left(\frac{\widehat{h}_3\left(1-\widehat{h}_3\right)}{h_1\widehat{h}_1}\right)+\left(h_1-\widehat{h}_1\right)\left(h_3-\widehat{h}_3\right)\left[\frac{1-\left(h_3+\widehat{h}_3\right)}{h_1}+\frac{p_1}{h_3}\right]-\frac{p_1\widehat{h}_1}{h_3\widehat{h}_3}\left(h_3-\widehat{h}_3\right)^2\\&-\left(p_7-p_{12}\right)h_2\widehat{h}_4-\left(p_9-p_{13}\right)\left(h_3-\widehat{h}_3\right)\left(h_4-\widehat{h}_4\right)-\left(p_6-p_{11}\right)\left(h_1-\widehat{h}_1\right)\left(h_4-\widehat{h}_4\right)\\&-h_2\left(p_8+p_{12}\widehat{h}_4\right)+p_2h_1+h_2\widehat{h}_1\left(\frac{p_4h_1-p_5}{h_1}\right).\end{aligned} \tag{5.17}$$

*By the biological facts $p_6>p_{11}$ , $p_7>p_{12}$ and $p_9>p_{13}$ we get*

$$\begin{aligned}\frac{d\widehat{V}_2}{dt}<&-\left(h_1-\widehat{h}_1\right)^2\left(\frac{\widehat{h}_3\left(1-\widehat{h}_3\right)}{h_1\widehat{h}_1}\right)+\left(h_1-\widehat{h}_1\right)\left(h_3-\widehat{h}_3\right)\left[\frac{1-\left(h_3+\widehat{h}_3\right)}{h_1}+\frac{p_1}{h_3}\right]-\frac{p_1\widehat{h}_1}{h_3\widehat{h}_3}\left(h_3-\widehat{h}_3\right)^2\\&-h_2\left(p_8+p_{12}\widehat{h}_4\right)+p_2h_1+h_2\widehat{h}_1\left(\frac{p_4h_1-p_5}{h_1}\right)\end{aligned} \tag{5.18}$$

*Now according to the conditions* (5.15) *and* (5.16) *we have*

$$\frac{d\widehat{V}_2}{dt}\le-\left[\sqrt{\frac{\widehat{h}_3\left(1-\widehat{h}_3\right)}{h_1\widehat{h}_1}}\left(h_1-\widehat{h}_1\right)-\sqrt{\frac{p_1\widehat{h}_1}{h_3\widehat{h}_3}}\left(h_3-\widehat{h}_3\right)\right]^2-h_2\left(p_8+p_{12}\widehat{h}_4-\frac{\widehat{h}_1\left(p_4h_1-p_5\right)}{h_1}\right)+p_2h_1 \tag{5.19}$$

*Now according to the conditions* (5.13) *and* (5.14) *we have* $\frac{d\widehat{V}_2}{dT} < 0$. *Hence* $A_4$ *is a (GAS).* □

**Theorem 5.4.** *Assume that the (LAS)* $A_5 = (h_1^*, h_2^*, h_3^*, h_4^*)$ *in the* $R_+^4$. *Then* $A_5$ *is (GAS) if the following conditions hold :*

$$h_1 > h_1^*, \tag{5.20}$$

$$h_3 > h_3^*, \tag{5.21}$$

$$h_3 (h_3 + h_3^* - 1) > p_1 h_1, \tag{5.22}$$

$$1 > h_3^* > 1 - h_3, \tag{5.23}$$

$$\frac{p_5 h_2 + p_2 h_1}{h_1 h_2} < 2\sqrt{\frac{p_2 \left[h_3^* (1 - h_3^*) + p_5 h_2^*\right]}{h_1 h_2 h_2^*}}. \tag{5.24}$$

**Proof .** *Suggest the following function*

$$\widehat{V}_3 (h_1, h_2, h_3, h_4) = \left(h_1 - h_1^* - h_1^* \ln \frac{h_1}{h_1^*}\right) + \left(h_2 - h_2^* - h_2^* \ln \frac{h_1}{h_2^*}\right) + \left(h_3 - h_3^* - h_3^* \ln \frac{h_3}{h_3^*}\right) + \left(h_4 - h_4^* - h_4^* \ln \frac{h_4}{h_4^*}\right). \tag{5.25}$$

*Clearly* $\widehat{V}_3 : R_+^4 \to R$ *is a* $C^1$ *"positive definite function. Then by differentiating* $\widehat{V}_3$ *for time* $t$ *and by some algebraic manipulation" we get:*

$$\frac{d\widehat{V}_3}{dt} = -(h_1 - h_1^*)^2 \left(\frac{h_3^* (1 - h_3^*) + p_5 h_2^*}{h_1 h_1^*}\right) + \frac{p_5 h_2 + p_2 h_1}{h_1 h_2} (h_1 - h_1^*)(h_2 - h_2^*) - \frac{p_2 h_1^*}{h_2 h_2^*} (h_2 - h_2^*)^2 + \left[\frac{1 - (h_3 + h_3^*)}{h_1} + \frac{p_1}{h_3}\right] (h_1 - h_1^*)(h_3 - h_3^*) - \frac{p_1 h_1^*}{h_3 h_3^*} (h_3 - h_3^*)^2 - (p_6 - p_{11})(h_1 - h_1^*)(h_4 - h_4^*) - (p_7 - p_{12})(h_2 - h_2^*)(h_4 - h_4^*) - (p_9 - p_{13})(h_3 - h_3^*)(h_4 - h_4^*). \tag{5.26}$$

*By the biological facts* $p_6 > p_{11}$ , $p_7 > p_{12}$ *and* $p_9 > p_{13}$ *we get*

$$\frac{d\widehat{V}_3}{dt} < -(h_1 - h_1^*)^2 \left(\frac{h_3^* (1 - h_3^*) + p_5 h_2^*}{h_1 h_1^*}\right) + \frac{p_5 h_2 + p_2 h_1}{h_1 h_2} (h_1 - h_1^*)(h_2 - h_2^*) - \frac{p_2 h_1^*}{h_2 h_2^*} (h_2 - h_2^*)^2 + \left[\frac{1 - (h_3 + h_3^*)}{h_1} + \frac{p_1}{h_3}\right] (h_1 - h_1^*)(h_3 - h_3^*) - \frac{p_1 h_1^*}{h_3 h_3^*} (h_3 - h_3^*)^2 \tag{5.27}$$

*Now according to the conditions* (5.23) *and* (5.24), *we have*

$$\frac{d\widehat{V}_3}{dt} < -\left[\sqrt{\frac{h_3^* (1 - h_3^*) + p_5 h_2^*}{h_1 h_1^*}} (h_1 - h_1^*) - \sqrt{\frac{p_2 h_1^*}{h_2 h_2^*}} (h_2 - h_2^*)\right]^2 - \frac{p_1 h_1^*}{h_3 h_3^*} (h_3 - h_3^*)^2 + \left[\frac{1 - (h_3 + h_3^*)}{h_1} + \frac{p_1}{h_3}\right] (h_1 - h_1^*)(h_3 - h_3^*) \tag{5.28}$$

*Now according to the conditions* (5.20)- (5.22), *we have* $\frac{d\widehat{V}_3}{dT} < 0$. *Hence* $A_5$ *is a (GAS).* □

## 6. Numerical Simulation

In this section, the earlier results are proven " numerically by Runge-Kutta method with predictor-corrector method. Note that, we used MATLAB for plotting and turbo C++ for programming and then the results obtained were discussed. For one set of parameters and different initial points system (2.2) has been studied numerically. It is observed that, for the set of parameters eq. (6.1) that is satisfies the conditions of existence of the positive (EP) system (2.2) has a (GAS) positive (EP)".

$$\left.\begin{array}{lllllll} p_1 = 0.5, & p_2 = 0.6, & p_3 = 0.002, & p_4 = 0.4, & p_5 = 0.9, & p_6 = 0.1, & p_7 = 0.2, \\ p_8 = 0.1, & p_9 = 0.1, & p_{10} = 0.06, & p_{11} = 0.09, & p_{12} = 0.1, & p_{13} = 0.09, & p_{14} = 0.1 \end{array}\right\} \tag{6.1}$$

The solution of system (2.2) is a (GAS) which converges to $A_5(0.259, 0.136, 0.720, 1.243)$. It starts from four different initial points $(0.9, 0.3, 1, 1)$ , $(1.3, 0.5, 0.4, 0.9)$ , $(0.1, 0.7, 0.5, 1)$ and $(0.6, 0.1, 0.3, 4)$, this approves our analytical result that was achieved.

To discuss the behaviour of the dynamicl system and the effect of the parameters on it, we change only one parameter at a time from the given data in (6.1) and Table 2, shows the results for the affected parameters $p_i,\ i = 1, 2, 3, 8, 10, 11, 12, 13\ and\ 14$ .

Table 2: The affected parameters

| ***Parameter's Range*** | ***Converge to*** | ***Parameter's Range*** | ***Converge to*** |
|---|---|---|---|
| $0.01 \le p_1 < 0.63$ | $A_5$ | $0.63 \le p_1 < 0.98$ | $A_2$ |
| $0.01 \le p_2 < 0.25$ | $A_2$ | $0.25 \le p_2 \le 1$ | $A_5$ |
| $0.001 \le p_3 < 0.2$ | $A_5$ | $0.2 \le p_3 < 1$ | $A_2$ |
| $0.01 \le p_8 < 0.36$ | $A_5$ | $0.36 \le p_8 < 1$ | $A_2$ |
| $0.001 \le p_{10} < 0.55$ | $A_5$ | $0.55 \le p_{10} < 0.9$ | $A_2$ |
| $0.9 \le p_{10} < 1$ | $A_0$ | | |
| $0.01 < p_{11} \le 0.064$ | $A_2$ | $0.064 < p_{11} \le 0.09$ | $A_5$ |
| $0.01 \le p_{12} < 0.057$ | $A_2$ | $0.057 \le p_{12} \le 0.1$ | $A_5$ |
| $0.01 \le p_{13} < 0.085$ | $A_2$ | $0.085 \le p_{13} \le 0.09$ | $A_5$ |
| $0.01 \le p_{14} < 0.105$ | $A_5$ | $0.105 \le p_{14} < 1$ | $A_2$ |

But Table 3 shows that the unaffected parameters $p_i,\ i = 4, 5, 6, 7$ *and* 9.

Table 3: The unaffected parameters

| ***Parameter's Range*** | ***Converge to*** | ***Parameter's Range*** | ***Converge to*** |
|---|---|---|---|
| $0.01 \le p_4 \le 1$ | $A_5$ | $0.2 \le p_7 \le 1.5$ | $A_5$ |
| $0.01 \le p_5 < 1$ | $A_5$ | $0.1 \le p_9 \le 1.5$ | $A_5$ |
| $0.1 \le p_6 \le 1.5$ | $A_5$ | | |

Now Figure 1. shows changing the harvesting rate of the mature of prey $p_{10}$. As it is noticed in Table 2.

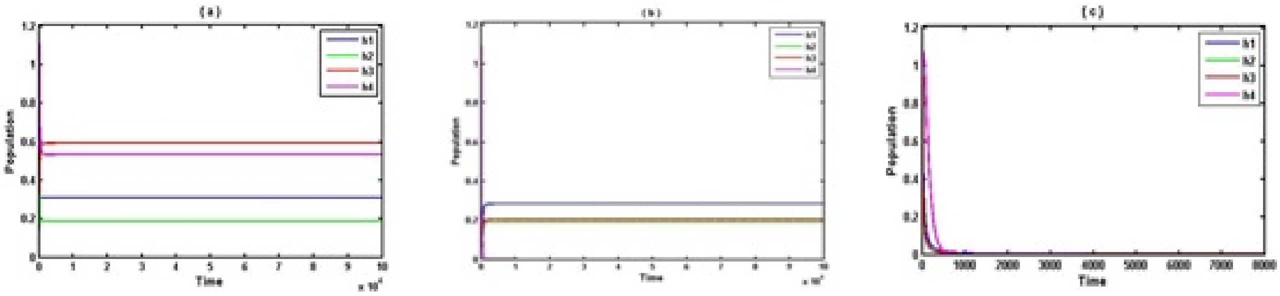

Figure 1: (a) Time series (TS) of the solution converges to $A_5 = (0.309, 0.189, 0.592, 0.532)$, for typical value $p_{10} = 0.1$. (b) (TS) of the solution converges to $A_2 = (0.283, 0.192, 0.202, 0)$ , for typical value $p_{10} = 0.7$. (c) (TS) of the solution converges to $A_0 = (0, 0, 0, 0)$ , for typical value $p_{10} = 0.99$.

Now, changing only the parameters $p_2, p_4, p_7, p_8, p_{12}$ and $p_{14}$ at the same time with the rest of parameters as in equation (6.1), it is noticed that for $0.001 \le p_2 < 0.006$ , $0.001 \le p_4 < 0.14$ ,$0.85 \le p_7 < 1$, $0.71 \le p_8 < 1$, $0.001 \le p_{12} < 0.7$ and $0.01 \le p_{14} < 0.093$ "the solution converges to" $A_4$ as seen in Figure 2., for typical value $p_2 = 0.01$ , $p_4 = 0.1$, $p_7 = 0.9$, $p_8 = 0.9$ , $p_{12} = 0.01$ *and* $p_{14} = 0.09$ .

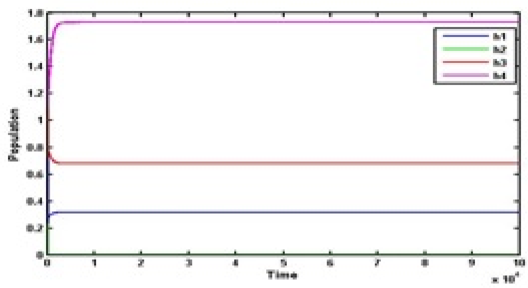

Figure 2: (TS) of the solution converges to $A_4 = (0.318, 0, 0.682, 1.729)$ for typical value $p_2 = 0.01$ , $p_4 = 0.1$, $p_7 = 0.9$, $p_8 = 0.9, p_{12} = 0.01$ and $p_{14} = 0.09$

Now, varying only the parameters $p_2, p_4, p_8$ , $p_{11}$and $p_{13}$ at the same time with the rest of parameters as in equation (6.1), it is observed that for $0.001 \le p_2 < 0.0013$, $0.001 \le p_4 < 0.0013$, $0.23 \le p_8 < 1$, $0.001 \le p_{11} < 0.05$ and $0.01 \le p_{13} < 0.106$ "the solution converges to" $A_1$ as seen in Figure 3. for typical value $p_2 = 0.001$, $p_4 = 0.01$, $p_8 = 0.5$, $p_{11} = 0.01$ *and* $p_{13} = 0.1$ .

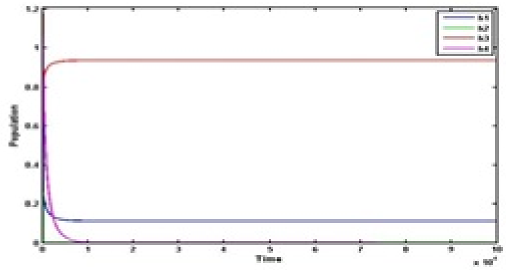


Figure 3: (TS) of the solution converges to $A_1 = (0.113, 0, 0.940, 0)$ for typical value $p_2 = 0.001,\ p_4 = 0.01,\ p_8 = 0.5,\ p_{11} = 0.01$ *and* $p_{13} = 0.1$.

## 7. Discussion and Conclusions

In this work, a stage-structured prey-predator model includes harvest and refuge for only prey has been studied. The disease of type (SIS) is just in the immature of the prey, and the disease is spread by contact and by external source has been suggested. The transmission of infectious disease in prey population has been described by the linear type. While Lotka-Volterra functional response is used to describe the predation process of the whole prey population. System (2.2) has been solved numerically for one initial point and one set of parameters given by (6.1). This model by a set of differential nonlinear equations has been represented, and we obtained that:

1. For the set of parameters given that we have proposed in (6.1) the system (2.2) has no periodic solution.

2. For the set of parameters given in equation (6.1), the most effectiveness parameters on the stability of system (2.2) are $p_1,\ p_2, p_3,\ p_8\ , p_{10}\ , p_{12}, p_{11}, p_{13}$and $p_{14}$. Varying only the parameters $p_2, p_4 p_7,\ p_8,\ p_{12}$ and $p_{14}$ at the same time with the rest of parameters as in equation (6.1) it is noticed that for $0.001 \leq p_2 < 0.006$ ,$0.001 \leq p_4 < 0.14$ ,$0.85 \leq p_7 < 1,\ 0.71 \leq p_8 < 1,\ 0.001 \leq p_{12} < 0.7$ and $0.01 \leq p_{14} < 0.093$ "the solution converges to" $A_4$ .

3. Changing only the parameters $p_2, p_4, p_8,\ , p_{11}, p_{13}$ and $p_{14}$ at the same time with the rest of parameters as in equation (6.1), it is observed that for $0.001 \leq p_2 < 0.0013$ ,$0.001 \leq p_4 < 0.0013,\ 0.23 \leq p_8 < 1,\ 0.001 \leq p_{11} < 0.05$ and $0.01 \leq p_{13} < 0.106$ "the solution converges to" $A_1$.

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