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Turán type inequalities for rational functions with prescribed poles

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Abstract

In this paper, we establish some inequalities for rational functions with prescribed poles having t-fold zeros at origin. The estimates obtained generalise as well as refine some known results for rational functions and in turn, produce extensions of some polynomial inequalities earlier proved by Turán, Jain etc.

Keywords: Rational functions, inequalities in complex domain, poles, t-fold zeros.
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1. Introduction

For any arbitrary function f, let $\|f\| = \max_{|z|=1} |f(z)|$, the sup-norm of f on the unit circle $|z| = 1$. Let $\mathcal{P}_n$ denotes the class of all complex polynomials of degree n. For the complex numbers a_i, $i = 0, 1, 2, ..., n$, let

$$w(z) := \prod_{i=1}^{n}(z - a_i); \quad B(z) := \prod_{i=1}^{n}\left(\frac{1 - \overline{a_i}z}{z - a_i}\right)$$

and

$$\mathcal{R}_n := R_n(a_1, a_2, ..., a_n) = \left\{\frac{p(z)}{w(z)} : p \in \mathcal{P}_n\right\}.$$

Then $\mathcal{R}_n$ represents the class of all rational functions with a finite limit at infinity and with at most n poles $a_1, a_2, ..., a_n$. Note that $B(z) \in \mathcal{R}_n$ and $|B(z)| = 1$ for $|z| = 1$. Throughout this paper, we

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shall always assume that all poles $a_1, a_2, ..., a_n$ lie outside the unit disk.

If $p \in \mathcal{P}_n$, then concerning the estimate of $|p'(z)|$ on $|z| = 1$, we have

$$|p'(z)| \leq n\|p\|. \tag{1.1}$$

Inequality (1.1) is a famous result due to Bernstein[5].

If $p \in \mathcal{P}_n$ and all zeros of $p(z)$ lie in $|z| \leq 1$, then it was proved by Turán[10] that

$$\|p'\| \geq \frac{n}{2}\|p\|. \tag{1.2}$$

Concerning the minimum modulus of p and its derivative, Aziz and Dawood[1] proved under the same hypothesis of inequality (1.2) that

$$\min_{|z|=1} |p'(z)| \geq n \min_{|z|=1} |p(z)|. \tag{1.3}$$

In (2000) Jain[6] introduced a complex parameter β and proved the following result which provides refinement as well as generalization of inequality (1.2).

Theorem 1.1 If $p \in \mathcal{P}_n$ and all zeros of p lie in $|z| \leq 1$, then for every $\beta \in \mathcal{C}$ with $|\beta| \leq 1$,

$$\begin{aligned} \left\| zp'(z) + \frac{n\beta}{2}p(z) \right\| &\geq \frac{n}{2}\{1 + Re(\beta)\}\|p\| \\ &\quad + \frac{n}{2}\,|\{1 + Re(\beta)\} - |\beta|| \min_{|z|=1} |p(z)|. \end{aligned} \tag{1.4}$$

Furthermore, Li , Mohapatra, Rodriguez[8](see also [2], [7]) obtained inequalities similar to inequalities (1.1) and (1.2) for rational functions.They replaced polynomial $p(z)$ by a rational function $r(z)$ with prescribed poles $a_1, a_2, ..., a_n$ and z^n by a Blaschke product $B(z)$. In fact, they proved following result.

Theorem 1.2 Suppose $r \in \mathcal{R}_n$ and all zeros of r lie in $|z| \leq 1$, then for $|z| = 1$

$$|r'(z)| \geq \frac{1}{2}|B'(z)||r(z)|. \tag{1.5}$$

Aziz and shah[3] proved the following result which provides an improvement of inequality (1.5).

Theorem 1.3 Suppose $r \in \mathcal{R}_n$ and all zeros of r lie in $|z| \leq 1$, then for $|z| = 1$

$$|r'(z)| \geq \frac{1}{2}|B'(z)| \left\{ |r(z)| + \min_{|z|=1} |r(z)| \right\}. \tag{1.6}$$

As a generalization of inequality (1.5), Aziz and Shah[2] proved the following result:

Theorem 1.4 Suppose $r \in \mathcal{R}_n$ and all zeros of r lie in $|z| \leq k$,$k \leq 1$, then for $|z| = 1$

$$|r'(z)| \geq \frac{1}{2} \left\{ |B'(z)| + n\frac{1-k}{1+k} \right\} |r(z)|. \tag{1.7}$$

2. Main results.

In this paper, we present the following results, which provides generalizations as well as refinements of above inequalities (1.5), (1.6) and (1.7). In fact, we prove:

Theorem 2.1 Suppose $r \in \mathcal{R}_n$ and if $r(z) = \frac{p(z)}{w(z)}$, where $p(z) = \sum_{j=0}^{m} c_j z^j$, $(m \leq n)$ has all its zeros in $|z| \leq k$, $k \leq 1$, with t- fold zeros at origin, then for every complex β with $|\beta| \leq 1$ and $|z| = 1$

$$\left| zr'(z) + \frac{(m-t)\beta}{1+k} r(z) \right| \geq \frac{1}{2} \left\{ |B'(z)| + \frac{1}{1+k} \Big(k(2t-n) + 2m - n + 2(m-t)Re(\beta) \Big) \right\} |r(z)|, \tag{2.1}$$

where m is the number of zeros of $r(z)$.

Equality in (2.1) holds for

$$r(z) = \frac{z^t (z+k)^{m-t}}{(z-a)^n}$$

and $B(z) = (\frac{1-az}{z-a})^n$ evaluated at $z = 1$, $a > 1$ and $\beta = 0$.

Taking $\beta = 0$ and $m = n$ in Theorem 2.1, we get the following refinement as well as generalization of inequality (1.7).

Corollary 2.2 Suppose $r \in \mathcal{R}_n$ is such that r has exactly n zeros and all zeros lie in $|z| \leq k$, $k \leq 1$, with t-fold zeros at origin, then for $|z| = 1$

$$|r'(z)| \geq \frac{1}{2} \left\{ |B'(z)| + \frac{1}{1+k} (2kt + n(1-k) \right\} |r(z)|. \tag{2.2}$$

Equality in (2.2) holds for

$$r(z) = \frac{z^t (z+k)^{n-t}}{(z-a)^n}$$

and $B(z) = (\frac{1-az}{z-a})^n$ evaluated at $z = 1$, $a > 1$.

Remark 2.3 Taking $t = 0$ in inequality (2.2), it reduces to inequality (1.7).

Taking $k = 1$ in Theorem 2.1, we get the following result:

Corollary 2.4 Suppose $r \in \mathcal{R}_n$ and if $r(z) = \frac{p(z)}{w(z)}$, where $p(z) = \sum_{j=0}^{m} c_j z^j$, $(m \leq n)$ has all its zeros in $|z| \leq k$, $k \leq 1$, then for every complex β with $|\beta| \leq 1$ and $|z| = 1$

$$\left| zr'(z) + \frac{(m-t)\beta}{2} r(z) \right| \geq \frac{1}{2} \left\{ |B'(z)| + (t + m - n) + (m-t)Re(\beta) \right\} |r(z)|, \tag{2.3}$$

where m indicates the number of zeros of r.

Equality in (2.3) holds for

$$r(z) = \frac{z^t (z+1)^{m-t}}{(z-a)^n}$$

and $B(z) = (\frac{1-az}{z-a})^n$ evaluated at $z = 1$, $a > 1$ and $\beta = 0$.

Theorem 2.5 Suppose $r \in \mathcal{R}_n$ and if $r(z) = \frac{p(z)}{w(z)}$, where $p(z) = \sum_{j=0}^{m} c_j z^j$, $(m \leq n)$ has all its zeros in $|z| \leq k$, $k \leq 1$, with t- fold zeros at origin, then for every complex β with $|\beta| \leq 1$ and $|z| = 1$

$$\left|zr'(z) + \frac{(m-t)\beta}{1+k} r(z)\right| \geq \frac{1}{2}\left\{|B'(z)| + \frac{k(2t-n) + (2m-n) + 2(m-t)Re(\beta)}{1+k}\right\}|r(z)| + \frac{1}{2}\left\{|B'(z)| + \frac{k(2t-n) + (2m-n) + 2(m-t)Re(\beta) - 2(m-t)|\beta|}{1+k}\right\}\min_{|z|=k}|r(z)|. \tag{2.4}$$

Equality in (2.4) holds for

$$r(z) = \frac{z^t(z+k)^{m-t}}{(z-a)^n}$$

and $B(z) = (\frac{1-az}{z-a})^n$ evaluated at $z = 1$, $a > 1$ and $\beta = 0$.

Taking $t = 0$ in Theorem 2.5, we immediately obtain the following result.

Corollary 2.6 Suppose $r \in \mathcal{R}_n$ and if $r(z) = \frac{p(z)}{w(z)}$, where $p(z) = \sum_{j=0}^{m} c_j z^j$, $(m \leq n)$ has all its zeros in $|z| \leq k$, $k \leq 1$, then for every complex β with $|\beta| \leq 1$ and $|z| = 1$

$$\left|zr'(z) + \frac{m\beta}{1+k} r(z)\right| \geq \frac{1}{2}\left\{|B'(z)| + \frac{(2m-n) - nk + 2mRe(\beta)}{1+k}\right\}|r(z)| + \frac{1}{2}\left\{|B'(z)| + \frac{(2m-n) - nk + 2mRe(\beta) - 2m|\beta|}{1+k}\right\}\min_{|z|=k}|r(z)|. \tag{2.5}$$

Equality in (2.5) holds for

$$r(z) = \frac{(z+k)^m}{(z-a)^n}$$

and $B(z) = (\frac{1-az}{z-a})^n$ evaluated at $z = 1$, $a > 1$ and $\beta = 0$.

Remark 2.7 Taking $m = n$ in Corollary 2.6, it reduces to a result of G.V Milovanović and A. Mir[9].

Taking $k = 1$ in Theorem 2.5, we will get the following result.

Corollary 2.7 Suppose $r \in \mathcal{R}_n$ and if $r(z) = \frac{p(z)}{w(z)}$, where $p(z) = \sum_{j=0}^{m} c_j z^j$, $(m \leq n)$ has all its zeros in $|z| \leq k$, $k \leq 1$ with t fold zeros at origin, then for every complex β with $|\beta| \leq 1$ and $|z| = 1$

$$\left|zr'(z) + \frac{(m-t)\beta}{2} r(z)\right| \geq \frac{1}{2}\Big(|B'(z)| + (t-n+m) + (m-t)Re(\beta)\Big)|r(z)| + \frac{1}{2}\Big(|B'(z)| + (t-n+m) + (m-t)Re(\beta) - (m-t)|\beta|\Big)\min_{|z|=1}|r(z)|. \tag{2.6}$$

Equality in (2.6) holds for

$$r(z) = \frac{z^t(z+1)^{n-t}}{(z-a)^n}$$

and $B(z) = (\frac{1-az}{z-a})^n$ evaluated at $z = 1$, $a > 1$ and $\beta = 0$.

Taking $\beta = 0$ and $m = n$ in theorem 2.5, we immediately get the following result which provides generalization as well as refinement of Theorem 1.3.

Corollary 2.8 If $r \in \mathcal{R}_n$ has exactly n zeros and all zeros lie in $|z| \leq k$, $k \leq 1$, with t-fold zeros at origin, then for $|z| = 1$

$$|r'(z)| \geq \frac{1}{2}\left\{|B'(z)| + \frac{1}{1+k}\{2kt + n(1-k)\}\right\}\left\{|r(z)| + \min_{|z|=k}|r(z)|\right\}. \tag{2.7}$$

Equality in (2.7) holds for

$$r(z) = \frac{z^t(z+k)^{m-t}}{(z-a)^n}$$

and $B(z) = (\frac{1-az}{z-a})^n$ evaluated at $z = 1$, $a > 1$.

3. Lemmas.

For the proof of main results, we need the following lemma due to Aziz and Zargar[4].

Lemma 3.1 If $|z| = 1$, then

$$Re\left(\frac{zw'(z)}{w(z)}\right) = \frac{n - |B'(z)|}{2}, \tag{3.1}$$

where $w(z) = \prod_{i=1}^{n}(z - a_i)$ and $w^*(z) = z^n\overline{w(1/\overline{z})}$.

4. Proofs of the Theorems

Proof of Theorem 2.1 Since $r(z) = \frac{p(z)}{w(z)} \in \mathcal{R}_n$, where $p(z) = \sum_{j=0}^{m} c_j z^j$, $(m \leq n)$ has all its zeros in $|z| \leq k$, $k \leq 1$ with t-fold zeros at origin, therefore we can write

$$r(z) = \frac{z^t h(z)}{w(z)} = z^t \frac{c_{m-t}\prod_{j=1}^{m-t}(z - z_j)}{\prod_{i=1}^{n}(z - a_i)}. \tag{4.1}$$

Let $z_1, z_2, ..., z_{m-t}$ be the zeros of $h(z)$, then $|z_j| \leq k \leq 1$, $j = 1, 2, ..., m-t$. Hence for every complex β with $|\beta| \leq 1$, we obtain from (4.1)

$$\frac{zr'(z)}{r(z)} + \frac{(m-t)\beta}{1+k} = t + \frac{zh'(z)}{h(z)} - \frac{zw'(z)}{w(z)} + \frac{(m-t)\beta}{1+k}.$$

This gives for $0 \leq \theta < 2\pi$, by using Lemma 3.1

$$\begin{aligned} Re\left(\frac{zr'(z)}{r(z)} + \frac{(m-t)\beta}{1+k}\right)\Bigg|_{z=e^{i\theta}} &= t + Re\left(\frac{zh'(z)}{h(z)}\right)\Bigg|_{z=e^{i\theta}} - Re\left(\frac{zw'(z)}{w(z)}\right)\Bigg|_{z=e^{i\theta}} + \frac{(m-t)Re(\beta)}{1+k} \\ &= t + Re\left(\sum_{j=1}^{m-t}\frac{e^{i\theta}}{e^{i\theta} - z_j}\right) - \left(\frac{n - |B'(e^{i\theta})|}{2}\right) + \frac{(m-t)Re(\beta)}{1+k}. \end{aligned} \tag{4.2}$$

It can be easily verified that for $e^{i\theta}$, $0 \le \theta < 2\pi$ and $|z_j| \le k \le 1$,

$$Re\left(\frac{e^{i\theta}}{e^{i\theta} - z_j}\right) \ge \frac{1}{1+k}. \tag{4.3}$$

Using (4.3) in (4.2), we obtain for $0 \le \theta < 2\pi$ that

$$\begin{aligned} Re\left(\frac{zr'(z)}{r(z)} + \frac{(m-t)\beta}{1+k}\right)\Bigg|_{z=e^{i\theta}} &\ge t + \frac{m-t}{1+k} - \left(\frac{n - |B'(e^{i\theta})|}{2}\right) + \frac{(m-t)Re(\beta)}{1+k} \\ &= \frac{1}{2}\left\{|B'(e^{i\theta})| + \frac{1}{1+k}\Big(k(2t-n) + 2m - n + 2(m-t)Re(\beta)\Big)\right\}, \end{aligned}$$

for points $e^{i\theta}$, $0 \le \theta < 2\pi$ other than zeros of $r(z)$. Thus we have

$$\left|e^{i\theta}r'(e^{i\theta}) + \frac{(m-t)\beta}{1+k}r(e^{i\theta})\right| \ge \frac{1}{2}\left\{|B'(e^{i\theta})| + \frac{1}{1+k}\Big(k(2t-n) + 2m - n + 2(m-t)Re(\beta)\Big)\right\}|r(e^{i\theta})|, \tag{4.4}$$

for points $e^{i\theta}$, $0 \le \theta < 2\pi$ other than zeros of $r(z)$.
Since inequality (4.4) is also true for points $e^{i\theta}$, $0 \le \theta < 2\pi$ which are zeros of $r(z)$, therefore it follows that for every complex β with $|\beta| \le 1$ and $|z| = 1$

$$\left|zr'(z) + \frac{(m-t)\beta}{1+k}r(z)\right| \ge \frac{1}{2}\left\{|B'(z)| + \frac{1}{1+k}\Big(k(2t-n) + 2m - n + 2(m-t)Re(\beta)\Big)\right\}|r(z)|. \tag{4.5}$$

This completes the proof of Theorem 2.1.

Proof of Theorem 2.5 If $r(z)$ has a zero on $|z| = k$, then

$$m^* = \min_{|z|=k} |r(z)| = 0$$

and in this case result follows from Theorem 2.1. So, we suppose that all the zeros of $r(z)$ lie in $|z| < k$ with t-fold zeros at origin. Therefore, it follows by Rouche's theorem that all the zeros of rational function $R(z) = r(z) + \lambda m^*$ lie in $|z| < k$ with t- fold zeros at origin for every complex number λ with $|\lambda| \le 1$.
Applying Theorem 2.1 to the rational function $R(z)$, we obtain for $|z| = 1$ and $|\beta| \le 1$ that

$$\left|zR'(z) + \frac{(m-t)\beta}{1+k}R(z)\right| \ge \frac{1}{2}\left\{|B'(z)| + \frac{k(2t-n) + (2m-n) + 2(m-t)Re(\beta)}{1+k}\right\}|R(z)|.$$

Equivalently,

$$\left|zr'(z) + \frac{(m-t)\beta}{1+k}(r(z) + \lambda m^*)\right| \ge \frac{1}{2}\left\{|B'(z)| + \frac{k(2t-n) + (2m-n) + 2(m-t)Re(\beta)}{1+k}\right\}|r(z) + \lambda m^*|. \tag{4.6}$$

Choosing argument of λ suitably on the right hand side of inequality (4.6), we obtain for $|\beta| \leq 1$ and $|z| = 1$

$$\left|zr'(z) + \frac{(m-t)\beta}{1+k}r(z)\right| + \frac{(m-t)m^*}{1+k}|\lambda||\beta| \geq \frac{1}{2}\left\{|B'(z)| + \frac{k(2t-n)+(2m-n)+2(m-t)Re(\beta)}{1+k}\right\}(|r(z)| + |\lambda|m^*). \tag{4.7}$$

Letting $|\lambda| \to 1$ in inequality (4.7), we get for $|\beta| \leq 1$ and $|z| = 1$

$$\left|zr'(z) + \frac{(m-t)\beta}{1+k}r(z)\right| \geq \frac{1}{2}\left\{|B'(z)| + \frac{k(2t-n)+(2m-n)+2(m-t)Re(\beta)}{1+k}\right\}|r(z)| + \frac{1}{2}\left\{|B'(z)| + \frac{k(2t-n)+(2m-n)+2(m-t)-2(m-t)|\beta|}{1+k}\right\}m^*,$$

which gives the desired result.

5. Conclusion

This paper investigates the lower bound estimates of the modulus of the derivative of a rational function $r(z)$ on the unit disk when all the zeros of $r(z)$ lie in $|z| \leq k$, $k \leq 1$, with t-fold zeros at origin. In paticular, our main Theorems generalises and refines the results by Aziz and Shah[2] and Aziz and Shah [3]. The established results for rational functions also yields many polynomial inequalities as a special cases.

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