Int. J. Nonlinear Anal. Appl. 14 (2023) 1, 1–10
ISSN: 2008-6822 (electronic)
http://dx.doi.org/10.22075/ijnaa.2021.22864.2423

F-contractions fixed point results in extended rectangular b-metric space

Patel Mittal[a,*], Jaita Sharma[a], Deheri Gunamani[b]

[a]*Department of Applied Mathematics, Faculty of Technology & Engineering, The M. S. University of Baroda, Vadodara-390 001, India*

[b]*Department of Mathematics, Sardar Patel University, Vallabh Vidyanagar-388 120, India*

(Communicated by Farshid Khojasteh)

Abstract

In the present paper, we prove the common fixed point theorem for F-contraction by considering one map as an orbitally continuous map in $\Omega-$Extended rectangular b-metric space. In addition, we find a fixed point for Banach and Kannan type contraction inequality without consideration of orbital continuity. Also, the map does not force to be continuous at a fixed point. An example is also provided for the utility of our results.

Keywords: Extended rectangular b-metric space, F-contraction, Orbitally continuous, Common Fixed point, Fixed point
2020 MSC: Primary 47H10; Secondary 54E50

1 Introduction

In 1993, the notion of b-metric space was introduced by Czerwik [10] as a generalization of a metric space, by modifying the third condition in the metric space. In recent years Goswami, Mohanta and Kamran derived some fixed point theorems in that space ([15],[22],[27]). Recently, Kamran [17] generalized b-metric space to become extended b-metric space. In 2000, Branciari [7] introduced rectangular (generalized) metric space (RMS) by replacing triangular inequality with a rectangular one in the context of fixed point theorem.

In 2015, George [14] introduced the concept of rectangular b-metric space and proved Banach contractions Fixed point theorem for this space. Then Mitrovic and Radenovic [21] established a common fixed point in such space. Recently in 2019, Mustafa [23] introduced the $\Omega-$Extended rectangular b-metric space and proved fixed point results. Also, Pant [25] and Bisht [6] derived some fixed point results at which the map does not force to be continuous at the fixed point.

Motivated by the idea of Mustafa [23] and Lukacs et al. [20], we derive common fixed point results in complete $\Omega-$Extended rectangular b-metric space. Consequently, we prove Banach and Kannan type F-contraction and find a unique fixed point. Our results we are deal with the discontinuity of metric space. Also, an example is given to strengthen our new results.

*Corresponding author
Email addresses: mittal.maths49@yahoo.com (Patel Mittal), jaita.sharma-appmath@msubaroda.ac.in (Jaita Sharma), gm.deheri@rediff.com (Deheri Gunamani)

Received: March 2021 *Accepted:* July 2021

2 Preliminaries

In this section, we need to recall some basic definitions and necessary results from existing literature.

Definition 2.1. [4, 10] Let X be a non-empty set with the coefficient $s \geqslant 1$, and the mapping $d : X \times X \to [0, \infty)$ satisfies the following:

(1) $d(x, y) = 0$ if and only if $x = y$;

(2) $d(x, y) = d(y, x)$ for all $x, y \in X$;

(3) $d(x, y) \leq s[d(x, z) + d(z, y)]$; $\forall x, y, z \in X$.

Then d is called a b-metric on X and (X, d) is called a b-metric space with coefficient s.

Definition 2.2. [14] Let X be a nonempty set with the coefficient $s \geqslant 1$, and the mapping $d : X \times X \to [0, \infty)$ satisfies the following:

(1) $d(x, y) = 0$ if and only if $x = y$;

(2) $d(x, y) = d(y, x)$ for all $x, y \in X$;

(3) $d(x, y) \leq s[d(x, w) + d(w, z) + d(z, y)]$ for all x,y $\in X$ and for all distinct points w,z $\in X - \{x, y\}$.

Then (X, d) is called a rectangular b- metric space(in short RbMS).

In 2012, Wardowski [29] introduced a new type of contraction called Wardowski F-contraction and obtained a fixed point for complete metric space. Then several papers have dealt with the F-contraction mappings and their extensions (see [1, 13, 15, 19]). Cosentino et al. [9] introduced the following condition (F_4) and obtained some results in b-metric spaces. Then Lukacs and Kajanto [20] defined F-Contraction as follows:

Definition 2.3. [20] Let (X, d) be a b-metric space with constant $s \geqslant 1$and $T : X \to X$ is said to be a F-contraction if there exists $\tau > 0$ such that $d(Tx, Ty) > 0$ implies

$$\tau + F(s.d(Tx, Ty)) \leq F(d(x, y)); \qquad \text{for all} \quad x, y \in X. \tag{2.1}$$

where, $F : (0, \infty) \to R$ belongs to $\mathcal{F}_{s,\tau}$ satisfying the following conditions:

(F_1) F is strictly increasing.

(F_2) For each sequence $\{\alpha_n\}_{n \in N}$ of positive numbers $\lim\limits_{n\to\infty} \alpha_n = 0$ iff $\lim\limits_{n\to\infty} F(\alpha_n) = -\infty$.

(F_3) There exists $k \in (0, 1)$ such that $\lim\limits_{\alpha\to 0^+} \alpha^k F(\alpha) = 0$.

(F_4) Let $s \geqslant 1$ be a real number. For each sequence $\{\alpha_n\}_{n \in N}$ of positive numbers such that $\tau + F(s\alpha_n) \leq F(\alpha_{n-1})$, for all $n \in N$ and some $\tau > 0$, then $\tau + F(s^n\alpha_n) \leq F(s^{n-1}\alpha_{n-1})$, for all $n \in N$

In our results, (F_3) and (F_4) is not required and denote class of all function satisfies (F_1) and (F_2) by $\mathcal{F}_{\Omega,\tau}$. In 1971, Ciric [8] gave the notion of orbital continuity as below:

Definition 2.4. If f is a self-mapping of a metric space (X, d), then the set $O(x, f) = \{f^n x : n = 1, 2, ...\}$ is called the orbit of f at x and f is called orbitally continuous if $\lim_{k\to\infty} f^{n_k} x = x$, for some $x \in X$ implies $\lim_{k\to\infty} f(f^{n_k} x) = fx$.

Remark: It is obvious that a continuous function is always orbitally continuous but the converse may not be true. The following examples illustrate this fact.

Example 2.5. [5] Let $X = [0, 2]$, the map $f : X \to X$ defined by

$$f(x) = 1 \quad if \quad x \in [0, 1], f(x) = 0 \quad if \quad x \in (1, 2].$$

It is clear that f is orbitally continuous but not continuous at $x = 1$.

Example 2.6. Let $X = [0, 1)$ and the map $f : X \to X$ defined by

$$f(x) = \begin{cases} 0 & ; x = 0, \\ \frac{1}{2} - x, & ; 0 < x \le \frac{1}{2} \\ \frac{3}{2} - x, & ; \frac{1}{2} < x < 1. \end{cases}$$

Sequence $\{\frac{1}{2^n}\} \to 0$, but $f(\frac{1}{2^n}) \nrightarrow f(0)$, So f is not orbitally continuous.

Example 2.7. Let $X = A \cup B$, where $A = \{\frac{1}{n}; n = 2, 3, 4, ...\}$ and $B = \{0, 1, 2, 3, ...\}$. The map $f : X \to X$defined by

$$f(x) = \begin{cases} 1 & ; x = 0, \\ \frac{x}{2}, & ; x \in A \\ \frac{1}{x}, & ; x \in B - \{0\}. \end{cases}$$

Here $\{\frac{1}{n}\} \to 0$, but $f(\frac{1}{n}) \nrightarrow f(0)$, hence f is not orbitally continuous.

Definition 2.8. [23] Let X be a nonempty set, $\Omega : [0, \infty) \to [0, \infty)$ be a strictly increasing continuous function with $t \le \Omega(t)$, for all $t > 0$ and $0 = \Omega(0)$ and let $\tilde{r} : X \times X \to [0, \infty)$ be a mapping such that for all $x, y \in X$ and all distinct points $u, v \in X$, each distinct from x and y satisfies the following conditions:

(1) $\tilde{r}(x, y) = 0$ if and only if $x = y$;

(2) $\tilde{r}(x, y) = \tilde{r}(y, x)$;

(3) $\tilde{r}(x, y) \le \Omega[\tilde{r}(x, u) + \tilde{r}(u, v) + \tilde{r}(v, y)]$.

Then $(X, \tilde{r})$ is called an extended rectangular b- metric space (in short ERbMS).

The concepts of convergence, Cauchy sequence, and completeness in a ERbMS are defined in a standard way. In [14, Example 1.7], it is seen that sequences in ERbMS may have more than one limit. However, there is a special situation where this is not possible, and this will be used in some proofs.

Theorem 2.9. [23] Let $(X, \tilde{r})$ be an ERbMS and let $\{x_n\}$ be a Cauchy sequence in X such that $x_n \ne x_m$ whenever $n \ne m$. Then $\{x_n\}$ can converge to at most one point.

While proving our results discontinuity of the $\Omega-$ERbMS can be managed by the following lemma.

Lemma 2.10. [23] Let $(X, \tilde{r})$ be an ERbMS with the function Ω, then we have the following:

(i) Suppose that $\{x_n\}$ and $\{y_n\}$ are sequences in X such that $x_n \to x, y_n \to y$ and the elements of $\{x, y, x_n, y_n : n \in N\}$ are totally distinct. Then, we have

$$\Omega^{-1}(\tilde{r}(x, y)) \le \lim_{n\to\infty} \inf \tilde{r}(x_n, y_n) \le \lim_{n\to\infty} \sup \tilde{r}(x_n, y_n) \le \Omega(\tilde{r}(x, y))$$

(ii) Let $\{x_n\}$ be a Cauchy sequence in X converging to x. If x_n has infinitely many distinct terms, then

$$\Omega^{-1}(\tilde{r}(x, y)) \le \lim_{n\to\infty} \inf \tilde{r}(x_n, y) \le \lim_{n\to\infty} \sup \tilde{r}(x_n, y) \le \Omega(\tilde{r}(x, y))$$

for all $y \in X$ with $x \ne y$.

3 Main results

Theorem 3.1. Let $(X, \tilde{r})$ be a complete ERbMS with non-trivial function $\Omega(i.e., \Omega(t) \ne t)$. Let f and g be commuting mappings into itself which satisfies the following:

(i) if there exists $\tau > 0$ such that

$$\tau + F(\Omega[\tilde{r}(fx, fy)]) \leq F(\tilde{r}(gx, gy)); \quad \forall x, y \in X. \tag{3.1}$$

(ii) either f or g is orbitally continuous and the range of g contains the range of f.

Then f and g have a unique common fixed point, say z. Moreover, f and g are continuous at common fixed point z if and only if $\lim_{x\to z} \max\{\tilde{r}(x, fx) + \tilde{r}(x, gx), \tilde{r}(z, fz)\} = 0$.

Proof . Let $x_0 \in X$ be arbitrary. Then fx_0 and gx_0 are well defined. Since $fx_0 \in g(X)$, there exists $x_1 \in X$ such that $gx_1 = fx_0$. Continuing this process, if x_n is chosen, we choose a point in X such that $gx_{n+1} = fx_n$.

Step I : We will prove that $\lim_{n\to\infty} \tilde{r}(gx_{n+1}, gx_n) = 0$. From our contractive condition (3.1) one can have

$$\begin{aligned} \tau + F(\Omega[\tilde{r}(gx_{n+1}, gx_n)]) &= \tau + F(\Omega[\tilde{r}(fx_n, fx_{n-1})]) \\ &\leq F(\tilde{r}(gx_n, gx_{n-1})). \end{aligned}$$

This implies

$$F(\Omega[\tilde{r}(gx_{n+1}, gx_n)]) \leq F(\tilde{r}(gx_n, gx_{n-1})) - \tau. \tag{3.2}$$

Since, $t \leq \Omega(t)$, F is strictly increasing, one can observe with the use of (3.2)

$$\begin{aligned} F(\tilde{r}(gx_{n+1}, gx_n)) &< F(\Omega[\tilde{r}(gx_{n+1}, gx_n)]) \\ &\leq F(\tilde{r}(gx_n, gx_{n-1})) - \tau \\ &\vdots \\ &< F(\tilde{r}(gx_1, gx_0)) - n\tau. \end{aligned}$$

Then

$$\lim_{n\to\infty} \sup F(\tilde{r}(gx_{n+1}, gx_n)) = \lim_{n\to\infty} \inf F(\tilde{r}(gx_{n+1}, gx_n)) = \lim_{n\to\infty} F(\tilde{r}(gx_{n+1}, gx_n)) = -\infty$$

which together with (2.3 F2) gives

$$\lim_{n\to\infty} \tilde{r}(gx_{n+1}, gx_n) = 0. \tag{3.3}$$

Step 2 : We will show that $gx_n \neq gx_m$ for $n \neq m$.

Case (i) If $gx_n = gx_{n+1}$ for some n, then $fx_n = gx_n = u$, for some n. This yields

$$fu = fgx_n = gfx_n = gu \tag{3.4}$$

Now our claim is to prove $\tilde{r}(u, fu) = 0$. On the contrary, let $\tilde{r}(u, fu) > 0$. Using contractive condition (3.1), one comes across

$$F(\ \tilde{r}(fx_n, fu)) < F(\Omega\ \tilde{r}(fx_n, fu)) \leq F(\tilde{r}(gx_n, gu)) - \tau$$

From equation (3.4), one has

$$F(\ \tilde{r}(fx_n, fu)) < F(\tilde{r}(fx_n, fu))$$

which is absurd. Hence our assumption is wrong.

$$fu = u = gu.$$

Hence, u is the common fixed point of f and g.

Case (ii): If $gx_n \neq gx_{n+1}$ for all $n \geqslant 0$, then $gx_n \neq gx_{n+k}$ for all $n \geqslant 0, k \geqslant 1$. If $gx_n = gx_{n+k}$ for some $n \geqslant 0, k \geqslant 1$, then

$$\begin{aligned} F(\ \tilde{r}(gx_{n+1}, gx_n)) < F(\Omega[\tilde{r}(gx_{n+1}, gx_n)]) &= F(\Omega[\tilde{r}(gx_{n+k+1}, gx_{n+k})]) \\ &\leq F(\tilde{r}(gx_{n+k}, gx_{n+k-1})) - \tau \\ &< F(\Omega[\tilde{r}(gx_{n+k}, gx_{n+k-1})]) - \tau \\ &\leq F(\tilde{r}(gx_{n+k-1}, gx_{n+k-2})) - 2\tau \\ &\vdots \\ &< F(\tilde{r}(gx_{n+1}, gx_n)) - k\tau \end{aligned}$$

$F(\tilde{r}(gx_{n+1}, gx_n)) < F(\ \tilde{r}(gx_{n+1}, gx_n))$, which is a contradiction. Hence $gx_n \neq gx_{n+k}$ for all $n \geqslant 0, k \geqslant 1$. Therefore, we can assume that $gx_n \neq gx_m$ for $n \neq m$.

Step 3 : Now it is shown that $\{gx_n\}$ is an $\tilde{r}$-Cauchy sequence. Suppose to the contrary that there exists $\epsilon > 0$ for which we can find two subsequences $\{gx_{m_i}\}$ and $\{gx_{n_i}\}$ of $\{gx_n\}$ such that m_i is the smallest index, where

$$m_i > n_i > i \ \ \textit{and} \ \ \tilde{r}(gx_{n_i}, gx_{m_i}) \geqslant \epsilon. \tag{3.5}$$

It means that

$$\tilde{r}(gx_{n_i}, gx_{m_i-2}), \tilde{r}(gx_{n_i}, gx_{m_i-1}) < \epsilon. \tag{3.6}$$

By using the $\Omega-$rectangular inequality and (3.5), one obtains

$$\epsilon \leq \tilde{r}(gx_{n_i}, gx_{m_i}) \leq \Omega[\tilde{r}(gx_{n_i}, gx_{n_i+1}) + \tilde{r}(gx_{n_i+1}, gx_{m_i-1}) + \tilde{r}(gx_{m_i-1}, gx_{m_i})]$$

which together with (3.3) and taking the upper limit as $i \to \infty$, we have

$$\Omega^{-1}(\epsilon) \leq \limsup_{i\to\infty} \tilde{r}(gx_{n_i+1}, gx_{m_i-1}). \tag{3.7}$$

Again,from the $\Omega-$rectangular inequality, one finds that

$$\tilde{r}(gx_{n_i+1}, gx_{m_i-2}) \leq \Omega[\tilde{r}(gx_{n_i+1}, gx_{n_i}) + \tilde{r}(gx_{n_i}, gx_{m_i-1}) + \tilde{r}(gx_{m_i-1}, gx_{m_i-2})].$$

Taking the upper limit as $i \to \infty$, From (3.3) and (3.6), one arrives at

$$\limsup_{i\to\infty} \tilde{r}(gx_{n_i+1}, gx_{m_i-2}) \leq \Omega(\epsilon). \tag{3.8}$$

Since F is strictly increasing and with the use of inequalities (3.6) and (3.7), one gets

$$\begin{aligned} F(\epsilon) &= F(\Omega[\Omega^{-1}(\epsilon)]) \\ &\leq F(\Omega[\limsup_{n\to\infty} \tilde{r}(gx_{n_i+1}, gx_{m_i-1})]) \\ &\leq F(\limsup_{n\to\infty} \tilde{r}(gx_{n_i}, gx_{m_i-2})) \\ &< F(\epsilon) \end{aligned}$$

a contradiction. Thus, $\{gx_n\}$ is a $\tilde{r}-$Cauchy sequence in X. Since $(X, \tilde{r})$ is a complete $\Omega-$ERbMS. So, there exists $z \in X$ such that

$$\lim_{n\to\infty} gx_n = z \tag{3.9}$$

which yields $\lim_{n\to\infty} gx_n = \lim_{n\to\infty} fx_{n-1} = z$.

Step4 : In this we will prove that z is the coincidence point of f and g. i.e. $fz = gz$. With the use of $\Omega-$rectangular inequality, one finds that

$$\tilde{r}(fz, gz) \leq \Omega[\tilde{r}(fz, fgx_n) + \tilde{r}(fgx_n, fgx_{n-1}) + \tilde{r}(fgx_{n-1}, gz)].$$

Letting limit supremum, one has

$$\limsup_{n\to\infty} \tilde{r}(fz, gz) \leq \Omega[\limsup_{n\to\infty} \tilde{r}(fz, fgx_n) + \limsup_{n\to\infty} \tilde{r}(fgx_n, fgx_{n-1}) + \limsup_{n\to\infty} \tilde{r}(fgx_{n-1}, gz)].$$

Without loss of generality, we can assume that f is orbitally continuous. Also, we have, f and g are commutative. Applying Lemma (2.2) and equation (3.9), we get

$$\lim_{n\to\infty} \tilde{r}(fz, gz) = \limsup_{n\to\infty} \tilde{r}(fz, gz) \leq \Omega(0),$$

one conclude that $fz = gz$.

Step 5 : At last, we will prove that, z is the unique common fixed point of f and g. At first, we will prove that $gz = z$.

$$\begin{aligned} F[(\tilde{r}(gx_n, gz))] = F[(\tilde{r}(fx_{n-1}, fz))] &< F[\Omega(\tilde{r}(fx_{n-1}, fz))] \\ &\leq F(\tilde{r}(gx_{n-1}, gz)) - \tau \\ &< F[\Omega(\tilde{r}(gx_{n-2}, gz))] - 2\tau \\ &\vdots \\ &< F[\tilde{r}(gx_0, gz)] - n\tau. \end{aligned}$$

Taking limit as $n \to \infty$ and from definition (2.3), we have

$$\lim_{n\to\infty} F[\tilde{r}(gx_n, gz))] = -\infty.$$

This implies

$$\lim_{n\to\infty} \tilde{r}(gx_n, gz) = 0.$$

Since the Cauchy sequence $\{gx_n\}$ converges to both z and gz, it is clear that $gz = z$. Thus, $gz = z = fz$. It is easy to check that z is the unique common fixed point. For the second part, let f and g be continuous at fixed point z, then for the sequence $\{gx_n\}$ of (3.9), we have $\lim_{n\to\infty} fgx_n = fz = z$ and $\lim_{n\to\infty} ggx_n = gz = z$. That is

$$\tilde{r}(gx_n, fgx_n) = 0 \quad \text{and} \quad \tilde{r}(gx_n, ggx_n) = 0.$$

So,

$$\lim_{x\to z} \max\{\tilde{r}(gx_n, fgx_n) + \tilde{r}(gx_n, ggx_n), \tilde{r}(z, fz)\} = 0.$$

Conversely, let $\lim_{x\to z} \max\{\tilde{r}(x, fx) + \tilde{r}(x, gx), \tilde{r}(z, fz)\} = 0$. Then,

$$\lim_{n\to\infty} \{\tilde{r}(gx_n, fgx_n) + \tilde{r}(gx_n, ggx_n)\} = 0.$$

Now it is obvious that

$$\lim_{n\to\infty} \tilde{r}(gx_n, fgx_n) = 0, \lim_{n\to\infty} \tilde{r}(gx_n, ggx_n) = 0.$$

It is clear that f and g are continuous at fixed point z. □

Example 3.2. Let $X = A \cup B$, where $A = [0, \frac{1}{3}]$ and $B = (\frac{1}{3}, 1)$. Define $\tilde{r} : X \times X \to [0, \infty)$ such that $\tilde{r}(x, y) = \tilde{r}(y, x)$ for all $x, y \in X$ and

$$\tilde{r}(x, y) = \begin{cases} 0, & ; x = y \\ \frac{1}{16} & ; x, y \in A \\ 1 & ; x, y \in B \\ \frac{1}{4} & ; \text{otherwise} \end{cases} \tag{3.10}$$

Then $(X, \tilde{r})$ is a $\Omega-$ERbMS with $\Omega(t) = 2t$, which is not a rectangular metric space. The mappings $f, g : X \to X$ defined by $f(x) = \frac{1}{3} \;\; ; x \in A \cup B$ and

$$g(x) = \begin{cases} \frac{1}{3} & ; x \in A - \{0\} \\ \frac{4}{3} - x & ; x \in B \\ 0 & ; x = 0 \end{cases}$$

Here we have $R(f) \subset R(g)$, f and g are commutative and f is orbitally continuous. A sequence $\{\frac{1}{3^n}\} \to 0$, but $g(\frac{1}{3^n}) \nrightarrow g(0)$, so g is not orbitally continuous map. For all $x, y \in X$, we have $\tilde{r}(fx, fy) = \tilde{r}(\frac{1}{3}, \frac{1}{3}) = 0$, which is trivially hold. We conclude that the equation (3.1) is satisfied. Thus f and g have unique common fixed point $\frac{1}{3}$.

If we put $gx = I_x$ (the identity map), then equation (3.1) turns into Banach type contractive condition. To prove the below theorem, orbital continuity is not required.

Theorem 3.3. Let f be a self-map of a complete ERbMS with the non-trivial function Ω, and if there exists $\tau > 0$ such that

$$\tau + F(\Omega[\tilde{r}(fx, fy)]) \leq F(\tilde{r}(x, y)); \ \forall x, y \in X. \tag{3.11}$$

Then f has a unique fixed point. Moreover, f is continuous at fixed point z if and only if

$$\lim_{x \to z} \max\{\tilde{r}(x, fx), \tilde{r}(z, fz)\} = 0.$$

Proof . Let the sequence $\{x_n\}$ defined by $fx_n = x_{n+1}$. It is easy to prove that

$$\lim_{n \to \infty} \tilde{r}(x_{n+1}, x_n) = 0. \tag{3.12}$$

Next, we will prove $x_n \neq x_m$ for $n \neq m$. Suppose to the contrary, $x_n = x_m$ for some $n > m$ then $x_{n+1} = fx_n = fx_m = x_{m+1}$. By continuing this process, one have $x_{n+k} = x_{m+k}$, for all $k \in N$. Then from inequality (3.11),

$$\begin{aligned} F(\tilde{r}(x_n, x_{n+1})) < F(\Omega[\tilde{r}(x_m, x_{m+1})]) &\leq F(\tilde{r}(x_n, x_{n+1})) - \tau \\ &< F(\tilde{r}(x_n, x_{n+1})) \end{aligned}$$

contradiction, hence $x_n \neq x_m$ for $n \neq m$. In a similar way, as the previous theorem, one can easily prove that $\{x_n\}$ is a Cauchy sequence and hence convergent to z.

$$\lim_{n \to \infty} x_n = z. \tag{3.13}$$

Next, one arrives at

$$\begin{aligned} F(\Omega[\tilde{r}(fx_{n-1}, fz)]) &\leq F(\tilde{r}(x_{n-1}, fz)) - \tau \\ &< F(\Omega[\tilde{r}(fx_{n-2}, fz)]) - \tau \\ &\leq F(\tilde{r}(x_{n-2}, fz)) - 2\tau \\ &\vdots \\ &< F[\tilde{r}(x_0, fz)] - n\tau. \end{aligned}$$

Taking limit as $n \to \infty$ and from definition (2.3), we have

$$\lim_{n \to \infty} F[\tilde{r}(fx_{n-1}, fz))] = -\infty.$$

This implies

$$\lim_{n \to \infty} \tilde{r}(x_n, fz) = 0.$$

Since the Cauchy sequence $\{x_n\}$ converges to both z and fz, it must be the case $fz = z$. It is easy to check that z is the unique fixed point. The second part can be similarly proved as the previous theorem (3.1). □

Example 3.4. Let $X = A \cup B$,where $A = [0, \frac{1}{3}]$ and $B = (\frac{1}{3}, 1)$. Define $\tilde{r} : X \times X \to [0, \infty)$ such that $\tilde{r}(x, y) = \tilde{r}(y, x)$ for all $x, y \in X$ and

$$\tilde{r}(x, y) = \begin{cases} 0, & ; x = y \\ \frac{1}{16} & ; x, y \in A \ ; x \ or \ y \neq 0. \\ 1 & ; x, y \in B \\ \frac{1}{4} & ; \text{otherwise} \end{cases} \tag{3.14}$$

Then $(X, \tilde{r})$ is a $\Omega-$ERbMS with $\Omega(t) = 2t$, which is not a rectangular metric space. The mappings $f : X \to X$ defined by

$$f(x) = \begin{cases} \frac{1}{3} & ; x \in A - \{0\} \\ \frac{x}{3} & ; x \in B \\ \frac{1}{4} & ; x = 0. \end{cases}$$

Here f is not orbitally continuous. For $F(x) = lnx (x > 0)$ and $\tau = \ln 2$, all the conditions required in theorem (3.3) are satisfied. Hence $\{\frac{1}{3}\}$ is the unique fixed point at which the map is discontinuous.

In the next theorem, we have taken Kannan type contractive condition.

Theorem 3.5. Let f be a self-map of a complete ERbMS with non-trivial function Ω $(i.e., \Omega(t) \neq t)$. and if their exists $\tau > 0$ such that

$$\tau + F(\Omega[\tilde{r}(fx, fy)]) \leq \frac{1}{2}\{F(\tilde{r}(x, fx)) + F(\tilde{r}(y, fy))\};\ \forall x, y \in X. \tag{3.15}$$

Then f has a unique fixed point. Moreover, f is continuous at fixed point z if and only if

$$\lim_{x \to z} \max\{\tilde{r}(x, fx), \tilde{r}(z, fz)\} = 0.$$

Proof . Let the sequence $\{x_n\}$ defined by $fx_n = x_{n+1}$. It is clear that

$$\lim_{n \to \infty} \tilde{r}(x_{n+1}, x_n) = 0. \tag{3.16}$$

Now we will show that $x_n \neq x_m$ for $n \neq m$. Suppose to the contrary, $x_n = x_m$ for some $n > m$ then $x_{n+1} = fx_n = fx_m = x_{m+1}$. By continuing this process, one have $x_{n+k} = x_{m+k}$ for all $k \in N$. Let $\mu_n = \tilde{r}(x_n, x_{n+1})$, Then from inequality (3.15),

$$\begin{aligned} F(\mu_m) = F(\mu_n) < F(\Omega[\mu_n]) &\leq F(\mu_{n-1}) - 2\tau \\ &< F(\mu_{n-2}) - 4\tau \\ &\vdots \\ &< F(\mu_m) \end{aligned} \tag{3.17}$$

contradiction, hence $x_n \neq x_m$ for $n \neq m$. From (3.15),

$$\tau + F(\Omega[\tilde{r}(x_{n+1}, x_{m+1})]) \leq \frac{1}{2} F((\mu_n) + (\mu_m)).$$

In the limit as $n \to \infty$ we get

$$F(\tilde{r}(x_{n+1}, x_{m+1})) = -\infty$$

or

$$\tilde{r}(x_{n+1}, x_{m+1}) = 0$$

which means $\{x_n\}$ is a Cauchy sequence and hence convergent to z i.e.

$$\lim_{n \to \infty} x_n = z.$$

Next, one arrives at

$$F(\tilde{r}(fx_{n-1}, fz)) < F(\Omega[\tilde{r}(fx_{n-1}, fz)]) \leq \frac{1}{2}\{F(\tilde{r}(x_{n-1}, x_n)) + F(\tilde{r}(z, fz))\} - \tau.$$

Taking limit as $n \to \infty$ and from definition (2.3), we have

$$\lim_{n \to \infty} F[\tilde{r}(fx_{n-1}, fz))] = -\infty.$$

This implies

$$\lim_{n \to \infty} \tilde{r}(x_n, fz) = 0.$$

Since, the Cauchy sequence $\{x_n\}$ converges to both z and fz, it must be the case $fz = z$. It is easy to check that z is the unique common fixed point. As same as Theorem (3.1), one can easily prove the second part of the theorem. □

Example 3.6. Let $X = A \cup B$,where $A = [0, \frac{1}{3}]$ and $B = (\frac{1}{3}, 1)$. Define $\tilde{r} : X \times X \to [0, \infty)$ such that $\tilde{r}(x, y) = \tilde{r}(y, x)$ for all $x, y \in X$ and

$$\tilde{r}(x,y) = \begin{cases} 0, & ; x = y \\ \frac{1}{12} & ; x, y \in A \; ; x \;\; or \; y \neq 0. \\ 1 & ; x, y \in B \\ \frac{1}{2} & ; \text{otherwise} \end{cases} \tag{3.18}$$

Then $(X, \tilde{r})$ is a $\Omega-$ERbMS with $\Omega(t) = 2t$, which is not a rectangular metric space. Let the mapping $f : X \to X$ defined as in previous example (3.4), then for $F(x) = \ln x$ $(x > 0)$ and $\tau = 0.20$, all the conditions required in theorem (3.5) are satisfied. Hence $\{\frac{1}{3}\}$ is the unique fixed point at which the map is discontinuous.

References

[1] H.H. Alsulami, E. Karapınar and H. Piri , *Fixed points of generalized F-Suzuki type contraction in complete b-metric spaces*, Discrete Dyn. Nat. Soc. **2015** (2015).

[2] M. Asim, M. Imdad and S. Radenovic, *Fixed point results in extended in rectangular b-metric spaces with an application*, U. P. B. Sci. Bull. **81** (2019), no. 2, 11–20.

[3] A. Azam and M. Arshad, *Kannan fixed point theorem on generalized metric spaces*, J. Nonlinear Sci. Appl. **1** (2008), 45–48.

[4] I. A. Bakhtin, *The contraction mapping principle in almost metric spaces*, Funct. Anal. **30** (1989), 26–37.

[5] R.K. Bisht and R.P. Pant, *A remark on discontinuity at fixed point*, J. Math. Anal. Appl. **445** (2017), no. 2, 1239–1242.

[6] R.K. Bisht and V. Rakocevic, *Discontinuity at fixed point and metric completeness*, Appl. Gen. Topol. **21** (2020), no. 2, 349–362.

[7] A. Branciari, *A Fixed point theorem of Banach-Caccippoli type on a class of generalized metric spaces*, Publ. Math. Debrecen **57** (2000), 31–37.

[8] Lj.B. Ciric, *On contraction type mappings*, Math. Balkanica **1** (1971), 52–57.

[9] M. Cosentino, M. Jleli, B. Samet, and C. Vetro, *Solvability of integrodifferential problems via fixed point theory in b-metric spaces*, Fixed Point Theory Appl. **2015** (2015), no. 1.

[10] S. Czerwik, *Contraction mappings in b-metric spaces*, Acta Math. Inf. Univ. Osraviensis **1** (1993), no. 1, 5–11.

[11] P. Das, *A fixed point theorem on a class of generalized metric spaces*, Korean J. Math. Sci. **9** (2002), 29–33.

[12] H.S. Ding, V. Ozturk and S. Radenovic, *On some new fixed point results in b-rectangular metric spaces*, J. Nonlinear Sci. Appl. **8** (2015), no. 4, 378–386.

[13] I.M. Erhan, E. Karapinar and T. Sekulic, *Fixed points of (psi, phi) contractions on rectangular metric spaces*, Fixed Point Theory Appl. **138** (2012).

[14] R. George, S. Radenovic, K.P. Reshma and S. Shukla, *Rectangular b-metric space and contraction principles*, J. Nonlinear Sci. Appl. **8** (2015), no. 6, 1005–1013.

[15] N. Goswami, N. Haokip and V. Mishra, *F-contractive type mappings in b-metric spaces and some related fixed point results*, Fixed Point Theory Appl. **13** (2019).

[16] N. Haokip and N. Goswami, *Some fixed point theorems for generalized Kannan type mappings in b-metric spaces*, Proyecciones J. Math. **38** (2019), no. 4, 763–782.

[17] T. Kamran, M. Samreen,and Q.UL. Ain, *A generalization of b-Metric space and some fixed point theorems*, Math. **5** (2017), no. 19.

[18] R. Kannan, *Some results on fixed points*, Bull. Cal. Math. Soc. **60** (1968), 71–76.

[19] S.U. Khan, M. Arshad, A. Hussain and M. Nazam, *Two new types of fixed point theorems for F-contraction*, J. Adv. Stud. Topology **7** (2016), no. 4, 251–260.

[20] A. Lukacs and S. Kajanto, *Fixed point theorems for various types of F-contractions in complete b-metric spaces*, Fixed Point Theory **19** (2018), no. 1, 321–334.

[21] Z. D. Mitrovic and S. Radenovic, *A common fixed point theorem of Jungck in rectangular b-metric spaces*, Acta Math. Hungarica **153** (2017), no. 2, 401–407.

[22] S. Mohanta, *Coincidence points and common fixed points for expansive type mappings in b-metric spaces*, Iran. J. Math. Sci. Inf. **11** (2016), no. 1, 101-–113.

[23] Z. Mustafa, V. Parvaneh, M. Jaradat and Z. Kadelburg, *Extended rectangular b-Metric spaces and some fixed point theorems for contractive mappings*, Symmetry **11** (2019), no. 4, 594.

[24] B. Nurwahyu, *Common fixed point theorems on generalized ratio contraction mapping in extended rectangular b-metric spaces*, Int. J. Math. Math. Sci. **2019** (2019).

[25] R.P. Pant and N. Ozgur, *New results on discontinuity at fixed point*, J. Fixed Point Theory Appl. **22** (2020), no. 39.

[26] J.R. Roshan, N. Hussain and V. Parvaneh, *New fixed point results in rectangular b-metric spaces*, Nonlinear Anal.: Model. Control **21** (2016), no. 5, 614–634.

[27] M. Samreen, T. Kamran and N. Shahzad, *Some fixed point theorems in b-Metric space endowed with graph*, Abstr. Appl. Anal. **2013** (2013), 1–9.

[28] N. Ullah and M. Shehu Shagari, *Fixed point results in complex-valued rectangular extended b-metric spaces with applications*, Math. Anal. Convex Optim. **1** (2020), no. 2, 109–122.

[29] D. Wardowski, *Fixed point theory of a new type of contractive mappings in complete metric spaces*, Fixed Point Theory Appl. **2012** (2012), 94.