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Solvability of infinite systems of fractional differential equations in the space of tempered sequence space $m^\beta(\phi)$

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Abstract

The purpose of this article, is to establish the existence of solution of infinite systems of fractional differential equations in space of tempered sequence $m^\beta(\phi)$ by using techniques associated with Hausdorff measures of noncompactness. Finally, we provide an example to highlight and establish the importance of our main result.

Keywords: Fractional differential equations, Hausdorff measure of noncompactness, Meir-Keeler condensing operator, Space of tempered sequence.
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1. Introduction and Preliminaries

The degree of noncompactness of a set is measured by means of functions called measures of noncompactness. The first measure of noncompactness, the function α, was defined and studied by Kuratowski [16] for purely topological considerations. In 1955, Darbo [9] used a measure of noncompactness to investigate the operators whose properties can be characterized as an intermediate between those of contraction and compact mappings. Darbo's fixed point theorem is useful in establishing the existence of solutions of various classes of differential equations, especially for implicit differential equations, integral equations and integro-differential equations, (see [5, 7, 14]).

The fractional calculus, an active branch of mathematics analysis, is as old as the classical calculus which we know today. The original ideas of fractional calculus can be traced back to the end of

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the seventeenth century when the classical differential and integral calculus theories were created and developed by Newton and Leibniz; see [11]. Fractional integral and differential equation have applications in different topics such as control theory [24] , image processing [8] etc. Certain types modeling of real life problems leads to the fractional differential equation found in Deng [10] and such types of equations have been studied by many authors [3, 23] and reference there in.

The Hausdorff measure of noncompactness χ was introduced by Goldenstein et al. [12] in the year 1957, and it was further studied by Goldenstein and Markus [13]. Recently, measures of noncompactness have also been used in defining geometric properties of Banach spaces and in characterizing compact operators between sequence spaces. The study of sequence spaces has been of great interest recently. A number of books have been published in this area over the last few years (see, for example, [7]). Sequence spaces have various applications in several branches of functional analysis, in particular, the theory of locally convex spaces, matrix transformations, as well as the theory of summability invariably depends upon the study of sequences and series.

In recent years, a lot of scholars (see e.g. [1, 6, 17]) studied the existence of solutions of integral equations in one or two variables on some spaces. The sequence space $m(\phi)$, introduced and studied by W.L.C. Sargent in 1960, is closely related to the space l^p. Mursaleen obtained an explicit formula for the Hausdorff measure of noncompactness of any bounded subset in $m(\phi)$ [20]. Also, Mursaleen et al. [23] established the solvability of an infinite systems of fractional differential equations in the spaces c_0 and l_p then M. Rabbani et al. [26] discuss the existence of solutions of an infinite system of fractional differential equations in tempered sequence spaces c_0^β and l_p^β.

The aim of this paper is to investigate the solvability of the following infinite systems of nonlinear fractional integral equation

$$\begin{cases} D^\alpha u_i(t) = f_i(t, u(t)), \ t \in (0, T) \\ u_i(0) = u_i^0 = 0, \ u_i(T) = a u_i(\xi); \ i = 1, 2, \ldots \\ 1 < \alpha < 2, \ a\xi^{\alpha-1} < T^{\alpha-1}, \end{cases} \tag{1.1}$$

where each $u_i(t)$ is a differentiable function of class $C^{[\alpha]+1}$. Also, we will denote the sequence $\{u_i(t)\}_{i=1}^\infty = u(t)$, $\{u_i(0)\}_{i=1}^\infty = u_0$, $\{u_i(\xi)\}_{i=1}^\infty = u(\xi)$ and $\{f_i(t, u(t))\}_{i=1}^\infty = f(t, u(t))$ which is an element of some Banach sequence space m^β, for each $i \in \mathbb{N}$.

Also, we construct the Hausdorff measures of noncompactness in space of tempered sequence $m^\beta(\phi)$ and we give an example to verify the effectiveness and applicability of our results.

In the following, we give a few auxiliary facts, which will be used in our further considerations.

By the symbol $\mathbb{R}$ we will denote the set of real numbers, and by $\mathbb{N}$ the set of natural numbers (positive integers). We write $\mathbb{R}_+$ to denote the interval $[0, +\infty)$. Assume that $\mathbb{E}$ is a Banach space with the zero element θ. Denote by $B(x, r)$ the closed ball in $\mathbb{E}$ centered at x and with radius r and $B_r = B(\theta, r)$.

Suppose $\mathfrak{M}_{\mathbb{E}}$ is the family of all nonempty bounded subsets of the space $\mathbb{E}$ and let $\mathfrak{N}_{\mathbb{E}}$ be its subfamily consisting of all relatively compact sets. If $\mathcal{A}$ is a nonempty subset of $\mathbb{E}$ then by $\overline{\mathcal{A}}$ and $\mathrm{Conv}(\mathcal{A})$ we denote the closure and convex closure of $\mathcal{A}$, respectively.

In what follows we will accept the following axiomatic definition of the concept of a measure of noncompactness.

Definition 1.1. *[5] A mapping $\mu : \mathfrak{M}_{\mathbb{E}} \to \mathbb{R}_+$ is called a measure of noncompactness (MNC for short) if*

(*i*) $\ker \mu$ *is nonempty and a subset of* $\mathfrak{N}_{\mathbb{E}}$.

(*ii*) $\mu(X) \leq \mu(Y)$ *for* $X \subset Y$.

(iii) $\mu(\overline{X}) = \mu(X)$.

(iv) $\mu(CovX) = \mu(X)$.

(v) *For all* $\lambda \in [0,1]$, $\mu(\lambda X + (1-\lambda)Y) \leq \lambda\mu(X) + (1-\lambda)\mu(Y)$.

(vi) *If* $(X_n)_{n\in\mathbb{N}}$ *is a sequence of closed sets from* $\mathfrak{M}_{\mathbb{E}}$ *satisfying* $X_{n+1} \subset X_n$ *for all* $n \in \mathbb{N}$ *and* $\mu(X_n) \to 0$ *as* $n \to \infty$, *then*

$$X_\infty = \bigcap_{n=1}^{\infty} X_n \neq \varnothing.$$

Definition 1.2. *[5] Let* (X,d) *be a metric space and let* $Q \in \mathfrak{M}_X$. *Then the Kuratowski measure of noncompactness of* Q, *denoted by* $\alpha(Q)$, *is the infimum of the set of all numbers* $\varepsilon > 0$ *such that* Q *can be covered by a finite number of sets with diameters* ε, *that is,*

$$\alpha(Q) = \inf\Big\{\varepsilon > 0 : Q \subset \bigcup_{i=1}^{n} S_i, S_i \subset X, diam(S_i) < \varepsilon \ (i = 1,2,\ldots,n);\ n \in \mathbb{N}\Big\},$$

where $diam(S_i) = \sup\{d(x,y) : x,y \in S_i\}$.

The Hausdorff measure of noncompactness for a bounded set Q is defined by

$$\chi(Q) = \inf\Big\{\varepsilon > 0 : Q \subset \bigcup_{i=1}^{n} B(x_i,r_i), x_i \in X, r_i < \varepsilon \ (i = 1,2,\ldots,n);\ n \in \mathbb{N}\Big\}.$$

The Hausdorff measure of noncompactness is often called the ball measure of noncompactness.

Lemma 1.3. *[4] Let* Q, Q_1 *and* Q_2 *be bounded subsets of a metric space* $(X;d)$. *Then*

1° $\mathcal{X}(Q) = 0$ *if and if* Q *is totally bounded,*

2° $Q_1 \subset Q_2$ *implies that* $\mathcal{X}(Q_1) \leq \mathcal{X}(Q_2)$,

3° $\mathcal{X}(\overline{Q}) = \mathcal{X}(Q)$,

4° $\mathcal{X}(Q_1) \cup \mathcal{X}(Q_2) = \max\{\mathcal{X}(Q_1), \mathcal{X}(Q_2)\}$.

In the case of a normed space $(X, \|.\|)$, the function $\mathcal{X}_X : \mathfrak{M} \to \mathbb{R}_+$ has some additional properties connected with the linear structure for example, we have

(i) $\mathcal{X}(Q_1 + Q_2) \leq \mathcal{X}(Q_1) + \mathcal{X}(Q_2)$,

(ii) $\mathcal{X}(Q + x) = \mathcal{X}(Q)$ for all $x \in X$,

(iii) $\mathcal{X}(\lambda Q) = |\lambda|\mathcal{X}(Q)$ for all $\lambda \in \mathbb{C}$,

(iv) $\mathcal{X}(Q) = \mathcal{X}(ConvQ)$.

In 1969, Meir and Keeler [18] introduced the concept of Meir–Keeler contractive mapping and proved some fixed point theorems for this kind of mappings. Thereafter, Aghajani, Mursaleen, and Haghighi [2] generalized some fixed point and coupled fixed point theorems for Meir–Keeler condensing operators via measures of noncompactness.

Definition 1.4. *[2] Let C be a nonempty subset of a Banach space E and let μ be an arbitrary measure of noncompactness on E. An operator $T : C \to C$ is called a Meir–Keeler condensing operator if for any $\varepsilon > 0$, there exists $\delta > 0$ such that*

$$\varepsilon \leq \mu(X) < \varepsilon + \delta \quad \textit{implies} \quad \mu(T(X)) < \varepsilon,$$

for any bounded subset X of C.

Theorem 1.5. *[2] Let C be a nonempty, bounded, closed, and convex subset of Banach space E and let μ be an arbitrary measure of noncompactness on E. If $T : C \to C$ is a continuous and Meir–Keeler condensing operator, then T has at least one fixed point and the set of all fixed points of T in C is compact.*

Let $I = [0, S]$ and let $C(I, E)$ be the Banach space of all continuous functions defined on I with values in the space E. The space $C(I, E)$ is furnished with the standard norm

$$\|x\|_c := \sup\{\|x(t)\| : t \in I\}, \qquad x \in C(I, E).$$

Proposition 1.6. *[5] If $W \subseteq C(I, E)$ is bounded and equicontinuous, then the function $\chi(W(.))$ is continuous on I and*

$$\chi(W) = \sup_{t \in I} \chi(W(t)), \quad \chi\big(\int_0^t W(s)ds\big) \leq \int_0^t \chi(W(s))ds. \tag{1.2}$$

In the sequel, we shortly recall some basic facts about fractional calculus (for more details see [15, 25]).

Definition 1.7. *([25]) The fractional integral of order α is defined as*

$$I^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \frac{f(s)}{(t-s)^{1-\alpha}} ds, \ \alpha > 0,$$

where $\Gamma(.)$ is the gamma function, provided that the integral exists.

Definition 1.8. *([25]) For at least n-times continuously differentiable function $f : [0, \infty) \to \mathbb{R}$, the Caputo fractional derivative of order $\alpha > 0$ is defined as*

$${}^cD^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{f^{(n)}(s)}{(t-s)^{\alpha-n+1}} ds,$$

where $n = [\alpha] + 1$, $[\alpha]$ denotes the integer part of the real number α.

Proposition 1.9. *[23] Let $f \in C[0, T]$ be a given function and $1 < \alpha < 2$. Then the unique solution of*

$${}^cD^\alpha u(t) = f(t), \ u(0) = 0, \ u(T) = au(\xi),$$

is given by

$$u(t) = \int_0^T K(t, s) f(s) ds,$$

where $K(t,s)$ is the Green's function, given by

$$K(t,s) = \frac{1}{\Gamma(\alpha)(T^{\alpha-1} - a\xi^{\alpha-1})}\begin{cases} K_1(t,s), & 0 \le t \le \xi, \\ K_2(t,s), & \xi \le t \le T. \end{cases}$$

$$K_1(t,s) = \begin{cases} (t-s)^{\alpha-1}(T^{\alpha-1} - a\xi^{\alpha-1}) - t^{\alpha-1}[(T-s)^{\alpha-1} - a(\xi - s)^{\alpha-1}]; & 0 \le s \le \xi, \\ -t^{\alpha-1}[(T-s)^{\alpha-1} - a(\xi-s)^{\alpha-1}]; & t \le s \le \xi, \\ -(t(T-s)^{\alpha-1}); & \xi \le s \le T. \end{cases}$$

$$K_2(t,s) = \begin{cases} (t-s)^{\alpha-1}(T^{\alpha-1} - a\xi^{\alpha-1}) - t^{\alpha-1}[(T-s)^{\alpha-1} - a(\xi - s)^{\alpha-1}]; & 0 \le s \le \xi, \\ (t-s)^{\alpha-1}(T^{\alpha-1} - a\xi^{\alpha-1}) - (t(T-s))^{\alpha-1}; & \xi < s \le t, \\ -(t(T-s))^{\alpha-1}; & t < s \le T. \end{cases}$$

Remark 1.10. *[23] It can be verified that the Green's function $K(t,s)$ defined on rectangle $[0,T] \times [0,T]$ as $K_1(t,s) : [0,\xi] \times [0,T] \to \mathbb{R}$ and $K_2(t,s) : [\xi,T] \times [0,T] \to \mathbb{R}$ is continuous w.r.t to t and s.*

2. Hausdorff Measure of noncompactness of tempered sequence space $m^\beta(\phi)$

In this section, we introduce and formulate the Hausdorff measure of noncompactness in the tempered sequence space $m^\beta(\phi)$.

The theory of FK spaces is the most powerful and widely used tool in the characterization of matrix mappings between sequence spaces, and the most important result was that matrix mappings between FK spaces are continuous.

A sequence space X is called an FK space if it is a locally convex Frechet space with continuous coordinates $p_n : X \to \mathbb{C}$ $(n \in \mathbb{N})$, where $\mathbb{C}$ denotes the complex field and $p_n(x) = x_n$ for all $x = (x_k) \in X$ and every $n \in \mathbb{N}$. A normed space FK is called a BK space, that is, a BK space is a Banach sequence space with continuous coordinates.

On the other hand, the classical sequence spaces are BK spaces with their natural norms. More precisely, the spaces l_∞, c and c_0 are BK spaces with the sup-norm given by $\|x\|_{l_\infty} = \sup_k |x_k|$. Also, the space $l_p (1 \le p < \infty)$ is a BK space with the usual l_p-norm defined by $\|x\|_{l_p} = (\sum_k |x_k|^p)^{\frac{1}{p}}$. (see [19, 21, 22] and the references therein).

Let l^0 be the set of all real sequences and let $\mathcal{C}$ denote the space whose elements are finite sets of distinct positive integers. Given any element σ of $\mathcal{C}$, we denote by $c(\sigma)$ the sequence $c_n(\sigma)$ for which $c_n(\sigma) = 1$ if $n \in \sigma$, and $c_n(\sigma) = 0$ otherwise. Further, let

$$\mathcal{C}_r = \Big\{\sigma \in \mathcal{C} : \sum_{n=1}^{\infty} c_n(\sigma) \le r\Big\},$$

the set of those σ whose support has cardinality at most s, and let

$$\Phi = \Big\{\phi = (\phi_n) \in l^0 : 0 < \phi_1 \le \phi_n \le \phi_{n+1} \text{ and } (n+1)\phi_n \ge n\phi_{n+1}\Big\},$$

(see [20]). For $\phi \in \Phi$, we define the following sequence space which were further studied in [27].

$$m(\phi) = \Big\{x = (x_n) \in l^0 : \|x\|_{m(\phi)} = \sup_{r \ge 1} \sup_{\sigma \in \mathcal{C}_r} \Big(\frac{1}{\phi_r}\sum_{n \in \sigma} |x_n|\Big) < \infty\Big\}.$$

Remark 2.1. (1) *The space $m(\phi)$ is a BK space with its respective norm.*
(2) *If $\phi_n = 1(n = 1, 2, \dots,)$, then $m(\phi) = l_1$ and if $\phi_n = n(n = 1, 2, \dots,)$, then $m(\phi) = l_\infty$.*
(3) *$l_1 \subseteq m(\phi) \subseteq l_\infty$ for all $\phi \in \Phi$.*

Theorem 2.2. *[20] Let Q be a bounded subset of $m(\phi)$. Then*

$$\mathcal{X}_{m(\phi)}(Q) = \lim_{k\to\infty} \sup_{x\in Q} \Big(\sup_{r>k} \sup_{\tau\in\mathcal{C}_r} \frac{1}{\phi_r} \sum_{n\in\tau} |x_n| \Big).$$

Assume that $\beta = (\beta_i)_{i=1}^{\infty}$ is a sequence with positive terms which is nonincreasing. The space c_0^β consists of all sequences (x_i) such that the sequence $(\beta_i x_i)$ converges to zero. The norm in the space c_0^β is defined by the formula

$$\|x\|_{c_0^\beta} = \|(x_i)\|_{c_0^\beta} = \sup\{\beta_i |x_i| : i = 1, 2, \dots,\}.$$

It is trivial that c_0^β forms a linear space over the field of real (or complex) numbers.
Based on the similar approach we introduce a new tempered sequence space $m^\beta(\phi)$. The set $\Re$ consists of all real sequences $x = (x_n)_{i=1}^{\infty}$ such that
$\sup_{r\geq 1} \sup_{\sigma\in\mathcal{C}_r} \Big(\frac{1}{\phi_r} \sum_{n\in\sigma} \beta_n |x_n| \Big) < \infty$. Clearly $\Re$ forms a linear space over the field of real numbers and it becomes a Banach space if we normed it by norm

$$\|x\|_{m^\beta(\phi)} = \sup_{r\geq 1} \sup_{\sigma\in\mathcal{C}_r} \Big(\frac{1}{\phi_r} \sum_{n\in\sigma} \beta_n |x_n| \Big). \tag{2.1}$$

Proposition 2.3. *The spaces $m^\beta(\phi)$ and $m(\phi)$ are isometric.*
Proof . *We consider the mapping $F : m^\beta(\phi) \to m(\phi)$ defined by*

$$F(x) = F\big((x_n)_{n=1}^{\infty}\big) = (\beta_n x_n)_{n=1}^{\infty} = \beta x,$$

where $x = (x_n)_{n=1}^{\infty}$ and $(\beta_n x_n)_{n=1}^{\infty} = \beta x$ belong to $m(\phi)$.
Let us fix $y = (y_n)_{n=1}^{\infty}, z = (z_n)_{n=1}^{\infty} \in m^\beta(\phi)$ so we have

$$\begin{aligned}\|F(y) - F(z)\|_{m(\phi)} &= \|(\beta_n y_n)_{n=1}^{\infty} - (\beta_n z_n)_{n=1}^{\infty}\|_{m(\phi)} \\ &= \|\beta y - \beta z\|_{m(\phi)} \\ &= \sup_{r\geq 1} \sup_{\sigma\in\mathcal{C}_r} \Big(\frac{1}{\phi_r} \sum_{i\in\sigma} |\beta_i y_i - \beta_i z_i| \Big) \\ &= \sup_{r\geq 1} \sup_{\sigma\in\mathcal{C}_r} \Big(\frac{1}{\phi_r} \sum_{i\in\sigma} \beta_i |(y_i - z_i)| \Big).\end{aligned}$$

This means that $\|F(y) - F(z)\|_{m(\phi)} = \|y - z\|_{m^\beta(\phi)}$ and F is an isometry between $m^\beta(\phi)$ and $m(\phi)$.
□

Now, we determine the Hausdorff measure of noncompactness on $m^\beta(\phi)$.
In view of Theorem 2.2 and Proposition 2.3 we have

$$\mathcal{X}_{m^\beta(\phi)}(\mathbb{B}^\beta) = \lim_{k\to\infty} \sup_{x\in\mathbb{B}^\beta} \Big(\sup_{r>k} \sup_{\tau\in\mathcal{C}_r} \frac{1}{\phi_r} \sum_{n\in\tau} \beta_n |x_n| \Big), \tag{2.2}$$

where $\mathbb{B}^\beta \in \mathfrak{M}_{m^\beta(\phi)}$.

3. Existence of solution of infinite system of fractional differential equations in $m^\beta(\phi)$

In this section, we investigate the solvability of the infinite systems of nonlinear fractional integral equations (1.1). We provide an illustrative example to show the effectiveness and applicability of our results.

Consider the following conditions:

(*i*) $u_i(0) = 0$ *and* $\{u_i(\xi)\}_{i=1}^\infty \in m^\beta(\phi)$.

(*ii*) $f = (f_1, f_2, \ldots)$ continuously transforms the set $I \times m^\beta(\phi)$ to $m^\beta(\phi)$. Also, the family of functions $\{(fu)(t)\}_{t\in I}$ is equicontinuous at each point of $m^\beta(\phi)$.

(*iii*) For each $t \in I$ and $u \in m^\beta(\phi)$ there exist non-negative real valued functions $p_i(t)$ *and* $q_i(t)$ on I satisfying the following inequality

$$|f_i(t, u(t))| \leq p_i(t) + q_i(t)|u_i(t)| \ for \ i = 1, 2, \ldots$$

such that $p_i(t)$ are continuous, $\frac{1}{\phi_r}\sum_{i\in\sigma}\beta_i p_i(t)$ is uniformly converges on I and $q_i(t)$ is equibounded in I.

Theorem 3.1. *Let the system (1.1) satisfies the above conditions $(i) - (iii)$, then if $MTH < 1$ it has at least one solution $u(t)$ such that $u(t) = \{u_n(t)\}_{i=1}^\infty \in m^\beta(\phi)$ for all $t \in I = [0, T]$, where $M = \sup_{t,s\in I}|K(t, s)|$, $\sup_{t,\in I,i\in\mathbb{N}} q_i(t) \leq H$.*

Proof . *Suppose that $u(t) = \{u_n(t)\}_{i=1}^\infty$ is a function which satisfies the boundary conditions of the problem (1.1), and each $u_i(t)$ continuous for all $t \in I$.*
Define an operator $\mathcal{F} : C(I, m^\beta(\phi)) \to C(I, m^\beta(\phi))$ as follows

$$(\mathcal{F}u)(t) = \int_0^T K(t, s)f(s, u(s)). \tag{3.1}$$

By condition (ii), $\mathcal{F}$ is well defined on $C(I, m^\beta(\phi))$. Now we claim that $\mathcal{F}$ is bounded in the classical supremum norm $\|u\| = \sup_{t\in I}\|u(t)\|_{m^\beta(\phi)}$.

$$\begin{aligned}
\|(\mathcal{F}u)(t)\|_{m^\beta(\phi)} &= \|\int_o^T K(t, s)f(s, u(s))ds\|_{m^\beta(\phi)} \\
&= \sup_{r\geq1}\sup_{\sigma\in C_r}\Big(\frac{1}{\phi_r}\sum_{i\in\sigma}\beta_i|\int_o^T K(t, s)f_i(s, u(s))ds|\Big) \\
&\leq \sup_{r\geq1}\sup_{\sigma\in C_r}\Big(\frac{1}{\phi_r}\sum_{i\in\sigma}\beta_i\Big[\int_o^T |K(t, s)||f_i(s, u(s))|ds\Big]\Big) \\
&\leq \sup_{r\geq1}\sup_{\sigma\in C_r}\Big(\frac{1}{\phi_r}\sum_{i\in\sigma}\beta_i\Big[\int_o^T \big[|K(t, s)|\big(p_i(s) + \sup_{s\in I,i\in\sigma} q_i(s)|u_i(s)|\big)\big]ds\Big]\Big) \\
&\leq M\int_o^T \Big[\sup_{r\geq1}\sup_{\sigma\in C_r}\big(\frac{1}{\phi_r}\sum_{i\in\sigma}\beta_i p_i(s)\big)\Big]ds + MH\Big(\int_o^T \Big[\sup_{r\geq1}\sup_{\sigma\in C_r}\frac{1}{\phi_r}\big(\sum_{i\in\sigma}\beta_i|u_i(s)|\big)\Big]ds\Big). \\
&\leq M\int_o^T \Big[\sup_{r\geq1}\sup_{\sigma\in C_r}\big(\frac{1}{\phi_r}\sum_{i\in\sigma}\beta_i p_i(s)\big)\Big]ds + MTH\|u\|_{C(I,m^\beta(\phi))}.
\end{aligned}$$

Above inequality can be written as

$$\sup_{t\in I}\|(\mathcal{F}u)(t)\|_{m^\beta(\phi)} \le MT\overline{P} + MTH\|u\|_{C(I,m^\beta(\phi))},$$

where $\sup_{t\in I}\Big(\sup_{r\ge 1}\sup_{\sigma\in C_r}\big(\frac{1}{\phi_r}\sum_{i\in\sigma}\beta_i p_i(t)\big)\Big) = \overline{P}$ *for each* $t \in I$.

Hence

$$\|(\mathcal{F}(u)\|_{C(I,m^\beta(\phi))} \le MT\overline{P} + MTH\|u\|_{C(I,m^\beta(\phi))}.$$

Let d_0 *be the optimal solution of the following inequality*

$$d \le MT\Big(\overline{P} + Hd\Big).$$

Consider the set $B^\beta = B^\beta(u_0, r_0) = \{u(t) \in C(I, m^\beta(\phi)) : \|u\|_{C(I,m^\beta(\phi))} \le d,\ u(0) = 0\ ;\ u(T) = au(\xi)\}$ *which is convex, closed and bounded. Now, we show that* $\mathcal{F}$ *is continuous. To prove this fact, let* ν *be an arbitrary fixed point in* B^β *and let* $\varepsilon > 0$ *be given. By using condition (ii), there exists* $\delta > 0$ *such that if* $u \in B^\beta$ *and* $\|u - \nu\|_{C(I,m^\beta(\phi))} \le \delta$, *then* $\|(fu) - (f\nu)\|_{C(I,m^\beta(\phi))} \le \frac{\varepsilon}{MT}$. *Hence, for each* t *in* $[0, T]$, *we have*

$$\begin{aligned}
\|(\mathcal{F}u)(t) - (\mathcal{F}\nu)(t)\|_{m^\beta(\phi)} &= \sup_{r\ge 1}\sup_{\sigma\in\mathcal{C}_r}\Big(\frac{1}{\phi_r}\sum_{i\in\sigma}\beta_i\Big|\int_0^T K(t,s)f_n(s,u(s))ds - \int_0^T K(t,s)f_i(s,\nu(s))ds\Big|\Big)\\
&\le \sup_{r\ge 1}\sup_{\sigma\in\mathcal{C}_r}\Big(\frac{1}{\phi_r}\sum_{i\in\sigma}\beta_i\Big[\int_0^T \big[|K(t,s)||f_i(s,u(s)) - f_i(s,\nu(s))|\big]ds\Big]\Big)\\
&\le MT\sup_{t\in I}\Big[\sup_{r\ge 1}\sup_{\sigma\in\mathcal{C}_r}\Big(\frac{1}{\phi_r}\sum_{i\in\sigma}\beta_i|f_i(t,u(t)) - f_i(t,\nu(t))|\Big)\Big]\\
&\le MT\Big[\|(fu) - (f\nu)\|_{C(I,m^\beta(\phi))}\Big]\\
&\le \varepsilon.
\end{aligned}$$

Taking the supremum on the left side over all $t \in [0, T]$, *we deduce*

$$\|(\mathcal{F}u) - (\mathcal{F}\nu)\|_{C(I,m^\beta(\phi))} \le \varepsilon.$$

Therefore, we infer that $\mathcal{F}$ *is continuous.*

Next, we prove the continuity of $(\mathcal{F}u)$ *in* I. *Let* $t_0 \in (0, T)$ *and* $\varepsilon > 0$ *be arbitrary. By using the continuity of* $K(t, s)$ *w.r.t* t, *we have* $\delta > 0$ *such that for* $|t - t_0| < \delta$,

$$|K(t,s) - K(t_0,s)| < \frac{\varepsilon}{T\Big(\overline{P} + H\|u\|_{C(I,m^\beta(\phi))}\Big)}.$$

In view of condition (iii) we observe that

$$\begin{aligned}
\|(\mathcal{F}u)(t)-(\mathcal{F}u)(t_0)\|_{m^\beta(\phi)} &= \sup_{r\geq 1}\sup_{\sigma\in\mathcal{C}_r}\Big(\frac{1}{\phi_r}\sum_{i\in\sigma}\beta_i|\int_0^T K(t,s)f_i(s,u(s))ds-\int_0^T K(t_0,s)f_i(s,u(s))ds\Big)| \\
&\leq \sup_{r\geq 1}\sup_{\sigma\in\mathcal{C}_r}\Big(\frac{1}{\phi_r}\sum_{i\in\sigma}\beta_i\Big[\int_0^T\big[|K(t,s)-K(t_0,s)||f_i(s,u)s))|\big]ds\Big]\Big) \\
&\leq \sup_{r\geq 1}\sup_{\sigma\in C_r}\Big(\frac{1}{\phi_r}\sum_{i\in\sigma}\beta_i\Big[\int_o^T\big[|K(t,s)-K(t_0,s)|\big(p_i(s)+\sup_{s\in I,i\in\sigma}q_i(s)|u_i(s)|\big)\big]ds\Big]\Big) \\
&\leq \frac{\varepsilon}{T\Big(\overline{P}+H\|u\|_{C(I,m^\beta(\phi))}\Big)}T\sup_{t\in I}\big[\sup_{r\geq 1}\sup_{\sigma\in C_r}\Big(\frac{1}{\phi_r}\sum_{i\in\sigma}\beta_i p_i(t)\Big)\big] \\
&\quad+\frac{\varepsilon}{T\Big(\overline{P}+H\|u\|_{C(I,m^\beta(\phi))}\Big)}HT\sup_{t\in I}\Big(\big[\sup_{r\geq 1}\sup_{\sigma\in C_r}\Big(\frac{1}{\phi_r}\sum_{i\in\sigma}\beta_i|u_(t)|\Big)\big]\Big) \\
&\leq \frac{\varepsilon}{T\Big(\overline{P}+H\|u\|_{C(I,m^\beta(\phi))}\Big)}T\Big(\overline{P}+H\|u\|_{C(I,m^\beta(\phi))}\Big)<\varepsilon.
\end{aligned}$$

Then $(\mathcal{F}u)$ is continuous for each $t\in I$.
In order to finish the proof, we show that $\mathcal{F}$ is a Meir–Keeler condensing operator with respect to the Hausdorff measure of noncompactness χ on the space $C(I,m^\beta(\phi))$, In view of formula (2.1) and Proposition 1.6, we conclude that the Hausdorff measure of noncompactness for $B^\beta\subset C(I,m^\beta(\phi))$ is defined as

$$\mathcal{X}_{C(I,m^\beta(\phi))}(B^\beta)=\sup_{t\in I}\mathcal{X}_{m^\beta(\phi)}(B^\beta(t)). \tag{3.2}$$

Taking into account condition (ii) and (iii), we get

$$\begin{aligned}
\mathcal{X}_{m^\beta(\phi)}[(\mathcal{F}B^\beta)(t)] &= \lim_{k\to\infty}\sup_{u\in\mathbb{B}^\beta}\Big(\sup_{r>k}\sup_{\tau\in\mathcal{C}_r}\big(\frac{1}{\phi_r}\sum_{i\in\tau}\beta_i|\mathcal{F}u_i(t)|\big)\Big) \\
&\leq \lim_{k\to\infty}\sup_{u\in\mathbb{B}^\beta}\Big(\sup_{r>k}\sup_{\tau\in\mathcal{C}_r}\big(\frac{1}{\phi_r}\sum_{i\in\tau}\beta_i|\int_o^T K(t,s)f_i(s,u(s))ds|\big)\Big) \\
&\leq \lim_{k\to\infty}\sup_{u\in\mathbb{B}^\beta}\Big(\sup_{r>k}\sup_{\tau\in\mathcal{C}_r}\big(\frac{1}{\phi_r}\sum_{i\in\tau}\beta_i\Big[\int_o^T|K(t,s)|\big[p_i(s)+\sup_{s\in I,i\in\sigma}q_i(s)|u_i(s)|\big]ds\Big]\big)\Big) \\
&\leq MH\lim_{k\to\infty}\sup_{u\in\mathbb{B}^\beta}\Big(\int_o^T\Big[\sup_{r>k}\sup_{\tau\in\mathcal{C}_r}\big(\frac{1}{\phi_r}\sum_{i\in\tau}\beta_i|u_i|\big)\Big]ds\Big) \\
&\leq MTH\mathcal{X}_{C(I,m^\beta(\phi))}(B^\beta).
\end{aligned}$$

By (3.2) we can write

$$\mathcal{X}_{C(I,m^\beta(\phi))}(\mathcal{F}B^\beta)\leq MTH\mathcal{X}_{C(I,m^\beta(\phi))}(B^\beta).$$

This implies that

$$\mathcal{X}_{C(I,m^\beta(\phi))}(\mathcal{F}B^\beta)\leq MTH\mathcal{X}_{C(I,m^\beta(\phi))}(B^\beta)<\varepsilon.$$

Then

$$\mathcal{X}_{C(I,m^{\beta}(\phi))}(B^{\beta}) < \frac{1}{MTH}\varepsilon.$$

Let us choose $\delta = \varepsilon(\frac{1}{MTH} - 1)$. *It is easy to see that* $\mathcal{F}$ *is a Meir–Keeler condensing operator on* $B^{\beta} \subset m^{\beta}(\phi)$. *Now, by applying Theorem 1.5, we find that* $\mathcal{F}$ *has a fixed point in* B^{β}, *and thus the infinite system of integral equations (1.1) has at least one solution in* $C(I, m^{\beta}(\phi))$. □

Example 3.2. *Consider the following system of fractional differential equations*

$$\begin{cases} D^{\frac{5}{4}}u_n(t) = \sqrt[n]{t} + \sum_{m=n}^{\infty} \frac{sin(t)u_m(t)}{(1+m^2)(n^2)};\ n = 1,2,\dots \text{ and } t \in [0,T], \\ u_n(0) = 0, u_n(T) = \sqrt[4]{5}u_n(\frac{T}{4}), \end{cases}$$

$u(\frac{T}{4}) = \{u_n(\frac{T}{4})\}_{n=1}^{\infty} \in m^{\beta}(\phi)$.

Taking $\beta_n = \frac{1}{n^2}$, $\xi = \frac{T}{4}$, $a = \sqrt[4]{5}$, $f_n(t,u(t)) = \sqrt[n]{t} + \sum_{m=n}^{\infty} \frac{sin(t)u_m(t)}{(1+m^2)(n^2)}$, $p_n(t) = \sqrt[n]{t}$ *and* $q_n(t) = \frac{1}{n^2}\sum_{m=n}^{\infty} \frac{1}{1+m^2}$.

Above Eq. is a special case of Eq. (1.1). Here kernel $K_1(t,s)$ *and* $K_2(t,s)$ *are given as*

$$K(t,s) = \frac{1}{\Gamma(\frac{5}{4})(\sqrt[4]{T} - \sqrt[4]{\frac{5T}{2}})} \begin{cases} K_1(t,s); & 0 \le t \le \xi, \\ K_2(t,s); & \xi \le t \le T. \end{cases}$$

$$K_1(t,s) = \begin{cases} (t-s)^{\frac{1}{4}}(\sqrt[4]{T} - \sqrt[4]{\frac{5T}{2}}) - t^{\frac{1}{4}}[(T-s)^{\frac{1}{4}} - \sqrt[4]{\frac{5}{2}}(T-2s)^{\frac{1}{4}}]; & 0 \le s \le t, \\ -t^{\frac{1}{4}}[(T-s)^{\frac{1}{4}} - \sqrt[4]{\frac{5}{2}}(T-2s)^{\frac{1}{4}}]; & t \le s \le \xi, \\ -(t(T-s)^{\frac{1}{4}}); & \xi \le s \le T. \end{cases}$$

$$K_2(t,s) = \begin{cases} (t-s)^{\frac{1}{4}}(\sqrt[4]{T} - \sqrt[4]{\frac{5T}{2}}) - t^{\frac{1}{4}}[(T-s)^{\frac{1}{4}} - \sqrt[4]{\frac{5}{2}}(T-2s)^{\frac{1}{4}}]; & 0 \le s \le \xi, \\ (t-s)^{\frac{1}{4}})(T^{\frac{1}{4}} - \sqrt[4]{\frac{5T}{2}}) - (t(T-s))^{\frac{1}{4}}; & \xi \le s \le t, \\ -(t(T-s))^{\frac{1}{4}}; & t \le s \le T. \end{cases}$$

Clearly, the functions $p_n(t)$, $f_n(t,u(t))$ *and* $q_n(t)$ *are continuous and equibounded in* I *respectively. Also, It is easy to prove that* $\frac{1}{\phi_s}\sum_{\kappa\in\sigma}\beta_i p_i(t) = \frac{1}{\phi_s}\sum_{\kappa\in\sigma}\frac{\sqrt[\kappa]{t}}{\kappa^2}$ *converges uniformly to* $\frac{T\pi^2}{6}$ *in* I *for any sequence* $\phi \in \Phi$ *and* $q_n(t)$ *is equibounded by* $\frac{\pi^2}{6} = H$.

On the other hand, we have

$$|f_n(t,u(t))| = |\sqrt[n]{t} + \sum_{m=n}^{\infty} \frac{\sin(t)u_m(t)}{(1+m^2)(n^2)}| \le \sqrt[n]{t} + \frac{1}{n^2}\sum_{m=n}^{\infty}\frac{1}{(1+m^2)}|u_m(t)| = p_n(t) + q_n(t)|u_m(t)|,$$

for $n = 1,2,\dots$.

Now, we show that $f(t,u(t)) \in m^{\beta}(\phi)$. *For this aim assume that* $t \in [0,T]$ *is arbitrary and* $u \in m^{\beta}(\phi)$

then for any sequence $\phi \in \Phi$ we have

$$\begin{aligned}
\sup_{r\geq 1}\sup_{\sigma\in\mathcal{C}_r}\Big(\frac{1}{\phi_r}\sum_{\kappa\in\sigma}\beta_\kappa|f_i(s,u(t))|\Big) &= \sup_{r\geq 1}\sup_{\sigma\in\mathcal{C}_r}\Big(\frac{1}{\phi_r}\sum_{\kappa\in\sigma}\beta_\kappa|\sqrt[\kappa]{t}+\sum_{\kappa\geq m}^{\infty}\frac{sin(t)u_m(t)}{(1+m^2)(\kappa^2)}|\Big)\\
&\leq \sup_{r\geq 1}\sup_{\sigma\in\mathcal{C}_r}\Big(\frac{1}{\phi_r}\sum_{\kappa\in\sigma}\beta_\kappa\big(\sqrt[\kappa]{t}+\sum_{m=n}^{\infty}\frac{|u_m(t)|}{(1+m^2)(n^2)}\big)\Big)\\
&\leq \frac{T\pi^2}{6}+H\sup_{r\geq 1}\sup_{\sigma\in\mathcal{C}_r}\Big(\frac{1}{\phi_r}\sum_{\kappa\in\sigma}\beta_\kappa|u_\kappa(t)|\Big)\\
&= \frac{T\pi^2}{6}+H\|u(t)\|_{m^\beta(\phi)}.
\end{aligned}$$

Next, we show that the family of functions $\{(f(u)(t)\}_{t\in I}$ is equicontinuous at each point of $m^{\beta}(\phi)$. Let $t \in I,\ \nu \in m^{\beta}(\phi)$ be arbitrarily fixed, take any $\varepsilon > 0$,

$$\begin{aligned}
\|(fu)(t)-(f\nu)(t)\|_{m^\beta(\phi)} &= \sup_{r\geq 1}\sup_{\sigma\in\mathcal{C}_r}\Big(\frac{1}{\phi_r}\sum_{\kappa\in\sigma}\beta_\kappa|\sqrt[n]{t}+\sum_{\kappa\geq m}^{\infty}\frac{sin(t)u_m(t)}{(1+m^2)(\kappa^2)}-\sqrt[n]{t}-\sum_{\kappa\geq m}^{\infty}\frac{sin(t)\nu_m(t)}{(1+m^2)(\kappa^2)}|\Big)\\
&= \sup_{r\geq 1}\sup_{\sigma\in\mathcal{C}_r}\Big(\frac{1}{\phi_r}\sum_{\kappa\in\sigma}\beta_\kappa|\sum_{\kappa\geq m}^{\infty}\frac{sin(t)u_m(t)}{(1+m^2)(\kappa^2)}-\sum_{\kappa\geq m}^{\infty}\frac{sin(t)\nu_m(t)}{(1+m^2)(\kappa^2)}|\Big)\\
&= \sup_{r\geq 1}\sup_{\sigma\in\mathcal{C}_r}\Big(\frac{1}{\phi_r}\sum_{\kappa\in\sigma}\beta_\kappa\big(\sum_{\kappa\geq m}^{\infty}\frac{sin(t)}{(1+m^2)(\kappa^2)}\big)|u_\kappa(t)-\nu_\kappa(t)|\Big)\\
&\leq \sup_{r\geq 1}\sup_{\sigma\in\mathcal{C}_r}\Big(\frac{1}{\phi_r}\sum_{\kappa\in\sigma}\beta_\kappa\big(\sum_{m=n}^{\infty}\frac{1}{(1+m^2)(n^2)}\big)|u_\kappa(t)-\nu_\kappa(t)|\Big)\\
&\leq \frac{\pi^2}{6}\sup_{r\geq 1}\sup_{\sigma\in\mathcal{C}_r}\Big(\frac{1}{\phi_r}\sum_{\kappa\in\sigma}\beta_\kappa|u_\kappa(t)-\nu_\kappa(t)|\Big)\\
&= \frac{\pi^2}{6}\|u(t)-\nu(t)\|_{m^\beta(\phi)}<\varepsilon,
\end{aligned}$$

where $\|u(t) - \nu(t)\|_{m^{\beta}(\phi)} < \delta = \varepsilon\frac{6}{\pi^2}$. Therefore, by Theorem (3.1), the system (1.1) has at least one solution in $C(I, m^{\beta}(\phi))$

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