Some of the sufficient conditions to get the G-Bi-shadowing action

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Abstract
The aim of this paper is introduced some examples of a G-bi-shadowing actions on the metric G-space, by study a sufficient conditions of actions to be G-bi-shadowing. We show the G-λ-Contraction actions, G-(λ, L)-Contraction action, and G-Hardy-Rogers contraction action are G-bi-shadowing by proved some theorems.

Keywords: G-space, G-Bi-Shadowing, Sufficient Conditions, G-Contraction.

1. Introduction

The concept of shadowing is of great importance in studying and understanding dynamical systems because it often accounts for the accuracy of a computer simulation of the system being used. Work on it began to be developed by many researchers in recent years as an important link for dynamical systems with stability and chaos. The map with has shadowing property is assumed to have a true orbit fairly close to each pseudo orbit of this map. The researcher who gave the concept of shadowing is [15], see more [13 1 2].

Some researchers have evolved the shadowing into the bi-shadowing by assuming that the true orbit be on another maps under specific conditions. The researcher who gave the concept of bi-shadowing is [9]. Later in [3 4] the researchers studied many relations between the bi-shadowing and other concept.


In this paper we presented the concepts of Gs-λ-contraction, Gs-(λ, L)-contraction, and Gs-Hardy-Rogers contraction, then studied the actions satisfies it is G-bi-shadowing.

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2. Preliminaries

Let $G$ be a group, $X$ be a Hausdorff topological space and $\phi$ be a map. Then the triple $(G, X, \phi)$ is called topological transformation group.

**Definition 2.1.** \cite{12}

The map $\phi: G \times X \to X$ which satisfies:

1. $\phi(g, .)$ is a homeomorphism of $X$ for any $g \in G$,
2. $\phi(e, x) = x$ for all $x \in X$ where $e$ is the identity of the group $G$,
3. $\phi(g_1, \phi(g_2, x)) = \phi(g_1g_2, x)$, for all $g_1, g_2 \in G$, $x \in X$.

is called an action of a group $G$ on $X$. And a space $X$ with an action $\phi$ of $G$ is called a $G$-space.

**Definition 2.2.** \cite{12, 10}

The group $G$ is called generated by $S$ if $\langle S \rangle = G$.

The group $G$ is called finitely generated if the generating set $S$ is finite.

The generating set $S$ is called symmetric if for any $s \in S$ then $s^{-1} \in S$.

In this paper we suppose that $G$ is a finitely generated group, $X$ is a metric $G$-space with metric $d$, and $\phi: G \times X \to X$ be an action. And we fix finite symmetric generating set $s$ of $G$.

**Remark 2.3.** For $x \in X$ and $n \in \mathbb{N}$, we have:

1. By Definition 2.1 the image of $x$ by $\phi$ is $\phi(s, x)$ for $s \in S$.
2. We denote to the inverse image of $\phi$ by $\phi^{-1}$, and the inverse image of $x$ by $\phi$ is $\phi^{-1}(s, x) = \phi(s^{-1}, x)$, for $s \in S$.
3. The $n$-iterate of $x$ by $\phi$ is

$$
\underbrace{\phi(s, \ldots , \phi(s, x))}_{n\text{-iterate}} = \phi \left( \underbrace{s \ldots s}_{n\text{-items}}, x \right) = \phi(ns, x), \quad \text{for } s \in S,
$$

so we denote to $n$ iterate of $\phi$ by $\phi^n$ and $\phi^n(s, .) = \phi(ns, .)$, for $s \in S$.
4. The $n$ inverse iterate of $x$ by $\phi$ is

$$
\underbrace{\phi(s^{-1}, \ldots , \phi(s^{-1}, x))}_{n\text{-iterate}} = \phi \left( \underbrace{s^{-1} \ldots s^{-1}}_{n\text{-items}}, x \right) = \phi(ns^{-1}, x), \quad \text{for } s \in S,
$$

so we denote to $n$ inverse iterate of $\phi$ by $\phi^{-n}$ and $\phi^{-n}(s, .) = \phi(ns^{-1}, .)$, for $s \in S$.

**Definition 2.4.** \cite{8} For $x \in X$, the sequence $O(x) = \{\phi(g, x) \in X \mid g \in G\}$ which generated by $x$ and a group $G$ is called $G$-orbit of $x$ for $\phi$ in $X$.

**Definition 2.5.** \cite{17} A sequence $x = \{x_g \in X \mid g \in G\}$ is called $G_0$-orbit for $\phi$ if satisfying

$$
x_{sg} = \phi(s, x_g) \quad \text{for } s \in S \text{ and } g \in G. \quad (2.1)
$$
Remark 2.6.
1. We rewrite a $G$-orbit $x = \{x_g \in X \mid g \in G\}$ in Definition 2.5 as a sequence associated with a subset of integer numbers, then a sequence $x$ is **became** $x = \{x_n \in X \mid n \in I \subseteq \mathbb{Z}\}$, when the length of an interval $I \subseteq \mathbb{Z}$ depends on the members of a group $G$. We can reformulated the Condition (2.1) as follows:

$$x_{n+1} = \phi(s, x_n) \text{ for } n \in \mathbb{Z}, s \in S.$$ 

2. Note that a sequence $x$ can be finite or infinite.

**Definition 2.7.** [13] For $\delta > 0$, a sequence $y = \{y_g \in X \mid g \in G\}$ is called $G_\delta$-pseudo orbit for $\phi$ if satisfying

$$d(y_n, \phi(s, y_g)) \leq \delta, \text{ for } s \in S \text{ and } g \in G.$$ (2.2)

**Remark 2.8.**
1. As in Remark 2.6, we can reformulated $y = \{y_n \in X \mid n \in I \subseteq \mathbb{Z}\}$ and the condition (2.2) as follows $d(y_{n+1}, \phi(s, y_n)) \leq \delta, \text{ for } n \in \mathbb{Z}, s \in S$.

2. Note that a sequence $y$ can be finite or infinite.

3. Let $O(\phi)$, and $O(\phi, \delta)$ be denote the sets of all (finite or infinite) $G_\delta$-orbits for $\phi$, and $G_\delta$-pseudo orbits.

**Definition 2.9.** Let $\phi : G \times X \to X$ and $\psi : G \times X \to X$ be an actions. The $G_\delta$-distance between $\phi$ and $\psi$ is given by:

$$d_0(\phi, \psi) = \sup \{d(\phi(s, x), \psi(s, x)) \mid x \in X \} \text{ for } s \in S.$$ 

**Definition 2.10.** Let $a, b > 0$, the action $\phi$ is called $G_\delta$-bi-shadowing with a and $b$ on $X$ (for short we denoted by $G_\delta-$bi-shadowing) if there exists $0 < \delta \leq b$ such that for any $G_\delta$-pseudo orbit (finite or infinite) $y = \{y_n \in X \mid n \in I \subseteq \mathbb{Z}\} \in O(\phi, X, \delta)$ and any action $\psi : G \times X \to X$ satisfying $d_0(\phi, \psi) \leq -\delta$ then there exists a $G_\delta$-orbit $x = \{x_n \in X \mid n \in I \subseteq \mathbb{Z}\} \in O(\psi, X)$ such that:

$$d_0(x, y_n) \leq a(\delta + d_0(\phi, \psi)) \leq ab, \text{ for all } n \text{ as define in } y.$$ 

We will introduced the concepts of $G_\delta$-$\lambda$-contraction, $G_\delta(\lambda, L)$-contraction, and $G_\delta$-Hardy-Rogers contraction.

**Definition 2.11.** The action $\phi$ is called $G_\delta$-$\lambda$-contraction if there exists $0 < \lambda < 1$ such that

$$d(\phi(s, x), \phi(s, y)) \leq \lambda d(x, y) \text{ for all } x, y \in X, \text{ and } s \in S.$$ 

**Definition 2.12.** The action $\phi$ is called $G_\delta(\lambda, L)$-contraction, if there exists constants $0 \leq \lambda < 1$ and $L \geq 0$ such that

$$d(\phi(s, x), \phi(s, y)) \leq \lambda d(x, y) + L d(\phi(s, x), \phi(s, y)), \text{ for all } x, y \in X, \text{ and } s \in S.$$ 

**Definition 2.13.** The action $\phi$ is called $G_\delta$-Hardy-Rogers contraction action [11] if there exist nonnegative constants $\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5$ with $0 < \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + \lambda_5 < 1$ such that

$$d(\phi(s, x), \phi(s, y)) \leq \lambda_1 d(x, y) + \lambda_2 d(x, \phi(s, x)) + \lambda_3 d(y, \phi(s, y)) + \lambda_4 d(x, \phi(s, y)) + \lambda_5 d(y, \phi(s, x)),$$

for all $x, y \in X, \text{ and } s \in S$.

**Remark 2.14.**
1. A $G_\delta$-$\lambda$-contraction actions are special case of a $G_\delta(\lambda, L)$-contraction actions such that every $G_\delta$-$\lambda$-contraction action is $G_\delta(\lambda, L)$-contraction with $L = 0$.

2. A $G_\delta$-$\lambda$-contraction actions are special case of a $G_\delta$-Hardy-Rogers contraction actions such that every $G_\delta$-$\lambda$-contraction action is $G_\delta$-Hardy-Rogers contraction with $\lambda_2 = \lambda_3 = \lambda_4 = \lambda_5 = 0$. 


3. Main Theorems

In this section, we proved some theorems that showed the actions which $G_\delta$-\lambda-contraction, $G_\delta$-(\lambda,L)-contraction, and $G_\delta$-Hardy-Rogers contraction respectively are $G$-bi-shadowing.

**Theorem 3.1.** Let $\phi$ be a $G_\delta$-\lambda-contraction action, then $\phi$ is $G_\delta$-(\alpha,\beta)-bi-shadowing given by:

$$a = \frac{2}{1 - \lambda} \text{ and } \beta = (1 - \lambda) \quad (3.1)$$

**Proof.** Fix $\delta < \frac{(1-\lambda)}{2}$. Let $\{y_g \equiv y_n | g \in G, n \in \mathbb{Z}\}$ be a given $G_\delta$-\delta-pseudo orbit for $\phi$, and let $\psi : G \times X \rightarrow X$ be any action such that $d_0(\phi, \psi) \leq \frac{(1-\lambda)}{2}$.

It follows that $\delta + d_0(\phi, \psi) \leq \frac{(1-\lambda)}{2} + \frac{(1-\lambda)}{2} = (1 - \lambda) = \beta$.

Fix $m \in \mathbb{Z}$, consider a $G_\delta$-orbit $\{x_g \equiv x_n | g \in G, n \in \mathbb{Z}\}$ for $\psi$ and satisfying

$$d(\psi, x_m) \leq \frac{1}{1 - \lambda} (\delta + d_0(\phi, \psi)) \quad (3.2)$$

By induction, for $n = m + 1$ then:

$$d(x_{m+1}, y_{m+1}) = d(\psi(x_m), y_{m+1})$$

$$\leq d(\psi(x_m), \phi(x_m)) + d(\phi(x_m), y_{m+1}) + d(\phi(y_m), y_{m+1})$$

$$\leq d_0(\phi, \psi) + \lambda d(x_m, y_m) + \delta = (\delta + d_0(\phi, \psi)) + d(x_m, y_m).$$

For $n = m + 2$ then:

$$d(x_{m+2}, y_{m+2}) = d(\psi(x_{m+1}), y_{m+2})$$

$$\leq d(\psi(x_{m+1}), \phi(x_{m+1})) + d(\phi(x_{m+1}), \phi(x_{m+1})) + d(\phi(y_{m+1}), y_{m+2})$$

$$\leq d_0(\phi, \psi) + \lambda d(x_{m+1}, y_{m+1}) + \delta$$

$$\leq d_0(\phi, \psi) + \lambda (\delta + d_0(\phi, \psi)) + \lambda d(x_m, y_m)$$

$$= (1 + \lambda) (\delta + d_0(\phi, \psi)) + \lambda^2 d(x_m, y_m)$$

For $n = m + 3$ then:

$$d(x_{m+3}, y_{m+3}) = d(\psi(x_{m+2}), y_{m+3})$$

$$\leq d(\psi(x_{m+2}), \phi(x_{m+2})) + d(\phi(x_{m+2}), \phi(x_{m+2})) + d(\phi(y_{m+2}), y_{m+3})$$

$$\leq d_0(\phi, \psi) + \lambda d(x_{m+2}, y_{m+2}) + \delta$$

$$\leq d_0(\phi, \psi) + \lambda [(1 + \lambda) (\delta + d_0(\phi, \psi)) + \lambda^2 d(x_m, y_m)] + \delta$$

$$\leq (\delta + d_0(\phi, \psi)) + \lambda [(\delta + d_0(\phi, \psi)) + \lambda (\delta + d_0(\phi, \psi)) + \lambda^2 d(x_m, y_m)]$$

$$\leq (\delta + d_0(\phi, \psi)) + \lambda (\delta + d_0(\phi, \psi)) + \lambda^2 (\delta + d_0(\phi, \psi)) + \lambda^3 d(x_m, y_m)$$

$$\leq (1 + \lambda + \lambda^2) (\delta + d_0(\phi, \psi)) + \lambda^3 d(x_m, y_m).$$

In general, we obtain

$$d(x_{m+n}, y_{m+n}) \leq (1 + \lambda + \lambda^2 + \ldots + \lambda^{n-1}) (\delta + d_0(\phi, \psi)) + \lambda^n d(x_m, y_m)$$

$$\leq \frac{1}{1 - \lambda} (\delta + d_0(\phi, \psi)) + d(x_m, y_m).$$
Since \( \lambda < 1 \) and using the conditions \((3.1)\) and \((3.2)\) we have
\[
d_{L}(x_{m+n}, y_{m+n}) \leq \frac{2}{1 - \lambda} (\delta + d_{0}(\phi, \psi)) = (\delta + d_{0}(\phi, \psi)).
\]
Since \( m \in \mathbb{Z} \) is arbitrary, then \( d_{L}(x_{m+n}, y_{m+n}) \leq (\delta + d_{0}(\phi, \psi)) \) for \( n \in \mathbb{Z} \). \( \Box \)

Now, we generalize the result in Theorem \(3.1\) to \( \mathbb{G}(\lambda, L) \)-contraction actions.

**Theorem 3.2.** Let \( \phi \) be a \( \mathbb{G}_{\mathbb{G}}(\lambda, L) \)-contraction action for \( 0 < \lambda < 1, \ L \geq 0 \), and \( \lambda + L < 1 \). And let an action \( \phi \) satisfy the conditions below:

i) For any \( \mathbb{G}_{\mathbb{G}}\)-\( \delta \)-pseudo orbit \( \{y_{g} \equiv y_{n}|g \in \mathbb{G}, \ n \in \mathbb{Z}\} \) of \( \phi \) with \( \delta < \frac{(1-\lambda-L)}{2} \), then the series \( S = \sum_{n=0}^{\infty} d_{L}(\phi(s, y_{n}), y_{n}) \) is convergent.

ii) For every action \( \psi : \mathbb{G} \times \mathbb{X} \to \mathbb{X} \) satisfying \( d_{0}(\phi, \psi) \leq \frac{(1-\lambda-L)}{2} \), the following inequality is satisfied:
\[
L \ S \leq \delta + d_{0}(\phi, \psi).
\]

Then the action \( \phi \) is \( \mathbb{G}_{\mathbb{G}}(\lambda, b) \)-bi-shadowing given by
\[
a = \frac{2}{1 - \lambda - L} \text{ and } b = (1 - \lambda - L).
\]

**Proof.** Fix \( \delta < \frac{(1-\lambda-L)}{2} \). Let \( \{y_{g} \equiv y_{n}|g \in \mathbb{G}, \ n \in \mathbb{Z}\} \) be a given \( \mathbb{G}_{\mathbb{G}}\)-\( \delta \)-pseudo orbit for \( \phi \), and let \( \psi : \mathbb{G} \times \mathbb{X} \to \mathbb{X} \) be any action such that \( d_{0}(\phi, \psi) \leq \frac{(1-\lambda-L)}{2} \). It follows that
\[
\delta + d_{0}(\phi, \psi) \leq \frac{(1 - \lambda - L)}{2} + \frac{(1 - \lambda - L)}{2} = (1 - \lambda - L) = b.
\]
Consider a \( \mathbb{G}_{\mathbb{G}}\)-orbit \( \{x_{g} \equiv x_{n}|g \in \mathbb{G}, \ n \in \mathbb{Z}\} \) for \( \psi \) satisfying Definition 2.10, and we use \((3.3)\) to choose \( m \in \mathbb{Z} \) such that \( x_{m} \) with the following property:
\[
d_{L}(x_{m}, y_{m}) + \frac{L \ S}{1 - \lambda - L} \leq \frac{\delta + d_{0}(\phi, \psi)}{1 - \lambda - L}.
\]

By induction, for \( n = m + 1 \):
\[
d_{L}(x_{m+1}, y_{m+1}) = d_{L}(\psi(s, x_{m}), y_{m+1})
\leq d_{L}(\psi(s, x_{m}), \phi(s, x_{m}))) + d_{L}(\phi(s, x_{m}), \phi(s, y_{m})) + d_{L}(\phi(s, y_{m}), y_{m+1})
\leq d_{0}(\phi, \psi) + \lambda d_{L}(x_{m}, y_{m}) + L d_{L}(x_{m}, \phi(s, y_{m}))) + \delta
\leq (\delta + d_{0}(\phi, \psi)) + \lambda d_{L}(x_{m}, y_{m}) + L [d_{L}(x_{m}, y_{m}) + (y_{m}, \phi(s,y_{m}))]
\leq (\delta + d_{0}(\phi, \psi)) + (\lambda + L) d_{L}(x_{m}, y_{m}) + L d_{L}(y_{m}, \phi(s,y_{m}))).
\]

For \( n = m + 2 \):
\[
\begin{align*}
d_{L}(x_{m+2} , y_{m+2}) & = d_{L}(\psi(s, x_{m+1}) , y_{m+2}) \\
& \leq d_{L}(\psi(s, x_{m+1}), \phi(s, x_{m+1}))) + d_{L}(\phi(s, x_{m+1}), \phi(s, y_{m+1})) + d_{L}(\phi(s, y_{m+1}), y_{m+2}) \\
& \leq d_{0}(\phi, \psi) + \lambda d_{L}(x_{m+1}, y_{m+1}) + L d_{L}(x_{m+1}, \phi(s,y_{m+1}))) + \delta \\
& \leq (\delta + d_{0}(\phi, \psi)) + \lambda d_{L}(x_{m+1}, y_{m+1}) + L [d_{L}(x_{m+1}, y_{m+1}) + (y_{m+1}, \phi(s,y_{m+1}))]
\leq (\delta + d_{0}(\phi, \psi)) + (\lambda + L) d_{L}(x_{m+1}, y_{m+1}) + L d_{L}(y_{m+1}, \phi(s,y_{m+1}))) \\
& \leq (\delta + d_{0}(\phi, \psi)) + (\lambda + L) ([\delta + d_{0}(\phi, \psi)) + (\lambda + L) d_{L}(x_{m}, y_{m}) + L d_{L}(y_{m}, \phi(s,y_{m}))) \\
& + L d_{L}(y_{m+1}, \phi(s,y_{m+1}))) \\
& \leq (1 + (\lambda + L)) \ (\delta + d_{0}(\phi, \psi)) + (\lambda + L)^2 \ d_{L}(x_{m}, y_{m}) + L \ (\lambda + L) \ d_{L}(y_{m}, \phi(s,y_{m}))) \\
& + L \ d_{L}(y_{m+1}, \phi(s,y_{m+1}))).
\end{align*}
\]
For \( n = m + 3 \):

\[
\begin{align*}
&
\dd (x_{m+3}, y_{m+3}) = \dd (\psi(s, x_{m+2}), y_{m+3}) \\
&\leq \dd (\psi(s, x_{m+2}), \phi(s, x_{m+2})) + \dd (\phi(s, x_{m+2}), \phi(s, y_{m+2})) + \dd (\phi(s, y_{m+2}), y_{m+3}) \\
&\leq (\delta + \dd_0 (\phi, \psi)) + (\lambda + L) \dd (x_{m+2}, y_{m+2}) + L \dd (y_{m+2}, \phi(s, y_{m+2})) \\
&\leq (\delta + \dd_0 (\phi, \psi)) + (\lambda + L) [(1 + (\lambda + L)) (\delta + \dd_0 (\phi, \psi)) + (\lambda + L)^2 \dd (x_m, y_m) \\
&+ L (\lambda + L) \dd (y_m, \phi(s, y_m)) + L (\lambda + L) \dd (y_{m+1}, \phi(s, y_{m+1})) + L \dd (y_{m+2}, \phi(s, y_{m+2})) \\
&\leq (1 + (\lambda + L) + (\lambda + L)^2) (\delta + \dd_0 (\phi, \psi)) + (\lambda + L)^3 \dd (x_m, y_m) + L (\lambda + L)^2 \dd (y_m, \phi(s, y_m)) \\
&+ L (\lambda + L) \dd (y_{m+1}, \phi(s, y_{m+1})) + L \dd (y_{m+2}, \phi(s, y_{m+2})).
\end{align*}
\]

In general, we obtain

\[
\dd (x_{m+n}, y_{m+n}) \leq (\delta + \dd_0 (\phi, \psi)) \sum_{k=0}^{n-1} (\lambda + L)^k + (\lambda + L)^n \dd (x_m, y_m) \\
+ \sum_{k=0}^{n-1} L (\lambda + L)^{n-k-1} \dd (y_{m+k}, \phi(s, x_{m+k}))
\]

Note, if we write

\[
\sum_{k=0}^{n-1} L (\lambda + L)^{n-k-1} \dd (y_{m+k}, \phi(s, x_{m+k})) = \sum_{k=0}^{n-1} a_k \ b_{n-k-1}
\]

\[a_n = \dd (y_{m+k}, \phi(s, x_{m+k})), \text{ and } b_n = L (\lambda + L)^n \text{ for } n \geq 0, \text{ and if}
\]

\[c_n = \sum_{k=0}^{n-1} a_k \ b_{n-k-1}
\]

then Theorem 8.46 in [7] implies that the series \( \sum_{k=0}^\infty c_n \) is convergent and

\[
\sum_{k=0}^\infty c_n = \frac{L S}{1 - \lambda - L}.
\]

Therefore, by the conditions (3.4) and (3.5) and since \( \lambda + L < 1 \) we have

\[
\dd (x_{m+n}, y_{m+n}) \leq (\delta + \dd_0 (\phi, \psi)) \sum_{n=0}^\infty (\lambda + L)^n + (\lambda + L)^n \dd (x_m, y_m) + \sum_{n=0}^\infty c_n \\
\leq \frac{\delta + \dd_0 (\phi, \psi)}{1 - \lambda - L} + (x_m, y_m) + \frac{L S}{1 - \lambda - L} \leq \frac{\delta + \dd_0 (\phi, \psi)}{1 - \lambda - L} + \frac{\delta + \dd_0 (\phi, \psi)}{1 - \lambda - L} \\
= \frac{2 (\delta + \dd_0 (\phi, \psi))}{1 - \lambda - L} = a (\delta + \dd_0 (\phi, \psi)).
\]

Since \( m \in \mathbb{Z} \) is arbitrary, then \( \dd (x_{m+n}, y_{m+n}) \leq a (\delta + \dd_0 (\phi, \psi)) \) for \( n \in \mathbb{Z} \). \( \square \)
Theorem 3.3. Let \( \phi \) be a \( G_s \)-Hardy-Rogers contraction action as in Definition 2.13, then it is \( G_s \)-\((\lambda, L)\)-contraction such that
\[
\lambda = \frac{\lambda_1 + \lambda_2 + \lambda_4}{1 - \lambda_3 - \lambda_4} \quad \text{and} \quad L = \frac{\lambda_2 + \lambda_3 + \lambda_4 + \lambda_5}{1 - \lambda_3 - \lambda_4} \tag{3.6}
\]

Proof. 
\[
ed (\phi(s, x), \phi(s, y)) \\
\leq \lambda_1 ed (x, y) + \lambda_2 ed (x, \phi (s, x)) + \lambda_3 ed (y, \phi (s, y)) + \lambda_4 ed (x, \phi (s, y)) + \lambda_5 ed (y, \phi (s, x)) \\
\leq \lambda_1 ed (x, y) + \lambda_2 \left[ ed (x, y) + (y, \phi (s, x)) \right] + \lambda_3 \left[ ed (y, \phi (s, x)) + (\phi (s, x), \phi (s, y)) \right] \\
+ \lambda_4 \left[ ed (x, y) + ed (y, \phi (s, x)) + ed (\phi (s, x), \phi (s, y)) \right] + \lambda_5 ed (y, \phi (s, x)) \\
\leq (\lambda_1 + \lambda_2 + \lambda_4) ed (x, y) + (\lambda_2 + \lambda_3 + \lambda_4 + \lambda_5) ed (y, \phi (s, x)) + (\lambda_3 + \lambda_4) ed (\phi (s, x), \phi (s, y)) ,
\]

It follows that
\[
ed (\phi (s, x), \phi (s, y)) \leq \frac{\lambda_1 + \lambda_2 + \lambda_4}{1 - \lambda_3 - \lambda_4} ed (x, y) + \frac{\lambda_2 + \lambda_3 + \lambda_4 + \lambda_5}{1 - \lambda_3 - \lambda_4} ed (y, \phi (s, x))
\]

Take
\[
\lambda = \frac{\lambda_1 + \lambda_2 + \lambda_4}{1 - \lambda_3 - \lambda_4} \quad \text{and} \quad L = \frac{\lambda_2 + \lambda_3 + \lambda_4 + \lambda_5}{1 - \lambda_3 - \lambda_4} ,
\]

Then by assumption \( \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + \lambda_5 < 1 \) we have \( L \geq 0 \) and \( \lambda < 1 \).
This shows that \( \phi \) is \( G_s \)-\((\lambda, L)\)-contraction. \( \square \)

Theorem 3.4. Let \( \phi \) be \( G_s \)-Hardy-Rogers contraction action satisfying (i) and (ii) of Theorem 3.2.
Then the action \( \phi \) is \( G_s \)-(a, b)-bi-shadowing provided that \( \lambda_1 + 2 \lambda_2 + 2 \lambda_3 + 3 \lambda_4 + \lambda_5 < 1 \) such that
\[
a = 2 \frac{1 - \lambda_3 - \lambda_4}{1 - \lambda_1 - 2 \lambda_2 - 2 \lambda_3 - 3 \lambda_4 - \lambda_5} \quad \text{and} \quad b = 1 - \frac{1 - \lambda_1 - 2 \lambda_2 - 2 \lambda_3 - 3 \lambda_4 - \lambda_5}{1 - \lambda_3 - \lambda_4} \tag{3.7}
\]

Proof. Let \( \phi \) be a \( G_s \)-Hardy-Rogers contraction action and satisfying (i) and (ii) of Theorem 3.2. By Theorem 3.3 a \( G_s \)-Hardy-Rogers contraction action is \( G_s \)-\((\lambda, L)\)-contraction then \( \phi \) is \( G_s \)-bi-shadowing provided that
\[
\lambda + L = \frac{\lambda_1 + 2 \lambda_2 + \lambda_3 + 2 \lambda_4 + \lambda_5}{1 - \lambda_3 - \lambda_4} < 1 ,
\]
that is \( \lambda_1 + 2 \lambda_2 + 2 \lambda_3 + 3 \lambda_4 + \lambda_5 < 1 \).
Moreover, we can find the values of \( a \) and \( b \) in (3.7) substituting \( \lambda \) and \( L \) which given by (3.6) in (3.4). \( \square \)

4. conclusion

The \( G - \lambda \)-Contraction actions, \( G - (\lambda, L) \)-Contraction action with some conditions, and
\( G \)-Hardy-Rogers contraction action with some conditions are \( G \)-bi-shadowing.
References