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On proximally closed mapping

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Abstract

In this paper, we introduce the concept of proximally closed (or $\delta-$ closed) in the proximity spaces and study some of its properties.

Keywords: proximity space, proximally continuous mapping,δ-continuous, proximally isomorphic, $\delta-$ homeomorphism.

1. Introduction

As early as [3] sketched the concepts of proximity spaces in his "theory of enchainment". However, his idea received no further development at that time. In the early 1950's, Efremovič, [1] a Russian mathematician, rediscovered the subject and gave the definition of a proximity space. An analysis of proximity spaces was carried out by [4]. Many research papers were written about approximation spaces until the mathematician Radoslav Dimitriyevic came in 2010 to collect the important of these researches in his book" Proximity and uniform spaces".
In the first section, we will review the most important definitions, proposition and properties that we will need in the next section. If X is a non-empty set, then a binary relation δ on the P(X) is called a proximity if it satisfies $B_1 - B_5$ on the Boolean algebra $(X, \varnothing, \cap, \cup, -)$. The pair (X, δ) is called proximity space.

Definition 1.1. *A relation δ on the family $P(X)$ of all subsets of a set X is called (Efremovic) a proximity on X if δ satisfies the following conditions:*

(B_1) *If $A\delta B$, then $B\delta A$,*

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(B_2) $A\delta(B\cup C)$ *if and only if* $A\delta B$ *or* $A\delta C$,
(B_3) $X\overline{\delta}\varnothing$,
(B_4) $\{x\}\ \delta\{x\}$, $\forall x\in X$,
(B_5) *If* $A\ \overline{\delta}\ B$, *then* $\exists\ E\in P(x)\ni A\overline{\delta}\ E$ *and* $E^c\overline{\delta}B$. *the pair* (X,δ) *is called a proximity space.*

Definition 1.2. δ_D *is called discrete proximity on* X*, if we define* $A\ \delta_D B$ *if and only if* $A\bigcap B\neq\varnothing$.

Proposition 1.3. *[2] Let* $(X,\ \delta)$ *be a proximity space.Then:*

(i) *If* $A\ \delta\ B$ *and* $B\subseteq C$*, then* $A\delta C$
(ii) *If* $A\ \overline{\delta}\ \ B$ *and* $C\subseteq B$*, then* $A\overline{\delta}_C$
(iii) *If* $\exists x\in X\ \ni A\delta\{x\}$ *and* $\{x\}\delta B$, *then* $A\delta B$;
(iv) *If* $A\bigcap B\neq\varnothing$, *then* δB
(v) $A\overline{\delta}\varnothing$, $\forall\ A\subseteq X$;
(vi) *If* $A\delta B$ *then* $A\neq\varnothing$ *and* $B\neq\varnothing$.

Definition 1.4. *[2] Let* $(X,\ \delta)$ *be a proximity space. We say that the sets* $A,B\subset X$ *are in the relation*$\ll$ *and write* $A\ll B$ *if* $A\overline{\delta}(X-B)$ *when* $A\ll B$, *we call* B *a proximity or* $\delta-$*neighbrhood of* A.

Theorem 1.5. *[2] If* $(X,\ \delta)$ *be a proximity space. Then the relation satisfies the following properties:*

(a) $X\ll X$;
(b) *If* $A\ll B$, *then* $A\subset B$;
(c) *If* $A\subset B\ll C\subset D$ *implies* $A\ll B$.
(d) *If* $A\ll B$, *then* $C\subset X$ *s.t* $A\ll C\ll B$.

Definition 1.6. *If* δ_1*and* δ_2 *are two proximity on the set* X*, then we define* $\delta_1>\delta_2$ *if and only if* $A\delta_1 B$ *then* $A\delta_2 B$. *In this case we say that* δ_1*is finer than* δ_2 *or* δ_2*is coarser than* δ_1.

Definition 1.7. *Let* $(X,\ \delta)$ *be a proximity space. A subset* $F\subseteq X$ *is defined to be closed if and only if* $x\delta F\to x\in F$.

Theorem 1.8. *[2] If* (X,δ)*be a proximity space, then the family* $\tau_\delta=\{G\subseteq X: X-G \text{ is closed}\}$ *a topology on the set* X.

Theorem 1.9. *[2] If* $(X,\ \delta)$ *be a proximity space and* $\tau=\tau_\delta$, *then*τ_δ*-closure* $cl^\delta(A)$ *of a set* A *is given by* $cl^\delta(A)=\{x\in X:\{x\}\delta A\}$.

Proposition 1.10. *[2]For the subsets* A *and* B *of the proximity space* $(X,\ \delta)$ *we have that* $A\delta B$ *if and only if* $cl^\delta(A)\,\delta\ cl^\delta(B)$.

Theorem 1.11. *[2] Let* $(X,\ \delta)$*be a proximity space and* $\varnothing\neq Y\subset X$*. For sets* $A,B\subset Y$, *let* $A\delta_Y B$ *if and only if* $A\delta B$, *then* $(Y,\ \delta_Y)$ *is a proximity space.*

Definition 1.12. *[2]Let* $(X,\ \delta)$ *be a proximity space and* $\varnothing\neq Y\subset X$. *The proximity relation* δ_Y *defined in the (Theorem 1.11) on the subset* Y *of the set* X *is called the restriction on* Y *of the proximity* δ *and it is denoted by* $\delta|_Y$. *The order pair* $(Y,\ \delta|_Y)$*is called the proximity subspace of* $(X,\ \delta)$.

Corollary 1.13. *Let* $(X,\ \delta_X)$ *be a proximity spaces on and let* $\varnothing \neq Y \subset X$. *If a mapping* $f: X \to Y$ *is the canonical inclusion, then* $f^{-1}(\delta) = \delta|_Y$.

Definition 1.14. *[2] Let* $(X,\ \delta_X)$ *and* $(Y,\ \delta_Y)$ *be two proximity spaces. A mapping* $f: X \to Y$ *is said to be* δ*-continuous if* $A\delta_X B$ *implies* $f(A)\,\delta_Y f(B)$.

Proposition 1.15. *[2] Let* $f: X \to Y$ *be a mapping from a set* Y *into a proximity space* $(X,\ \delta_Y)$. *A proximity relation* $\delta_X = f^{-1}(\delta_Y)$ *is the coarsest proximity on* Y *for which* f *is a* δ*- continuous mapping.*

Theorem 1.16. *Let* $(X,\ \tau)$ *be a normal topological space and* $A\overline{\delta}B$ *if and only if* $cl(A) \cap cl(B) = \varnothing$. *Then,* δ *is a proximity relation on* X.

Proposition 1.17. *[2] Let* $f: X \to Y$ *be a mapping, when* $(Y,\ \delta_Y)$ *is a proximity space an let us define relation on the power set* $P(X)$ *of the set* X *in the following way* $A\delta B$ *if and only if* $f(A)\delta^*\ f(B)$, *then* δ^* *is proximity relation on* X.

Proposition 1.18. *Let* $\{\delta^i, i \in I\}$ *be a non-empty family of proximities on* X *and* $\delta\ = \sup\ \{\delta^i, i \in I\}$, *then* $f(\delta)\ = \sup\ \{f(\delta^i), i \in I\}$, *when* $f:\ (X,\ \delta_X) \to\ (Y,\ \delta_Y)$.

Proof . *Let* $F\delta^*\ E$ *, where* $F = f(A),\ E = f(B)$ *and* $\delta^*\ = \sup\ \left\{f(\delta^i), i \in I\right\}$. *For any compositions* $\{F_j, j \in J_m\}$ *and* $\{E_k, k \in J_n\}$ *of sets* F *and* E *respectively. Then there exists some indices* $j \in J_m$ *and* $k \in J_n$ *s.t* $F_j f(\delta^i) E_k$ *and* $f^{-1}(F_j)\delta^i f^{-1}(E_j)\ \forall i \in I$, *but* $F f(\delta)\ E$ *implies to* $A\delta B$, *indeed, let* $\left\{\acute{F}_j, j \in \acute{J}_r\right\}$ *and* $\left\{\acute{E}_k, k \in \acute{J}_s\right\}$ *be the decompositions of the sets* A *and* B *respectively.*
Then $\{F \cap f\left(\acute{F}_j\right), j \in \acute{J}_r\}$ *and* $\{E \cap f\left(\acute{E}_k\right), k \in \acute{J}_s\}$ *are the decompositions of the sets* F *and* E, *so that* $f^{-1}(\ F \cap f\left(\acute{F}_j\right))\overset{i}{\delta} f^{-1}(\ E \cap f\left(\acute{E}_k\right))\ \forall i \in I$. *Since* $f^{-1}(\ F \cap f\left(\acute{F}_j\right)) \subseteq \acute{F}_j$ *and* $f^{-1}(\ E \cap f\left(\acute{E}_k\right)) \subseteq \acute{E}_k$, *then by proposition* $\acute{F}_j \delta^i \acute{E}_k \forall i \in I$.
Conversely, if $F f(\delta) E$, *then* $A\delta B$. *Thus for every two decompositions* $\left\{f^{-1}(F_j), j \in J_m\right\}$ *and* $\{f^{-1}(E_k), k \in J_n\}$ *of the sets* A *and* B *respectively, there exists indices* $j \in J_m$ *and* $k \in J_n$ *s.t* $f^{-1}(F_j)\delta^i f^{-1}(E_k)\ \forall i \in I$. *Therefore* $F_j f^{-1}(\delta^i)\, E_k \forall i \in I$, *so that* $F\delta^*\ E$. □

2. Proximally closed mapping

Definition 2.1. *Let* $(X,\ \delta_X)$ *and* $(Y,\ \delta_Y)$ *be two proximity spaces. A mapping* $f: X \to Y$ *is said to be proximally closed (*δ*-closed) if* $F\delta_Y E$ *implies* $A\delta_X B$, *where* $F = f(A), E = f(B)$ *and* $A \neq \varnothing \neq B$.
Equivalently, $f: X \to Y$ *is said to be* δ*-closed if* $A\overline{\delta}_X B$ *implies* $F\overline{\delta}_Y E$.

Proposition 2.2. *Let* $(X,\ \delta_X)$ *and* $(Y,\ \delta_Y)$ *be two proximity spaces and* $f: X \to Y$ *is* δ*-closed mapping, then* f *it is a closed with respect to the topologies* τ_{δ_X} *and* τ_{δ_Y}.
Proof . *Let* $y \in cl^\delta(f(A))$ *, then* $\{y\}\ \delta_Y\{\ f(A)\}$. *But* $y = f(x)$ *and* $\{f(x)\}\delta_Y\{\ f(A)\}$ *, if there isn't* $x \in X$ *s.t* $f(x) = y$ *then* $\{f(\varnothing)\}\delta_Y\{\ f(A)\}$, *by (Definition 2.1)* $\varnothing\delta_X A$ *which is contradiction. Therefore,* $\{x\}\delta_X A$ *and* $x \in cl^\delta(A)$ *implies to* $f(x) \in f(cl^\delta(A)$, *this prove that* $cl^\delta(f(A))\ \subseteq f(cl^\delta(A))$. *Hence the mapping* f *is closed.* □

Corollary 2.3. *Let* $(X,\ \delta_X)$ *and* $(Y,\ \delta_Y)$ *be two proximity spaces and* $A \subseteq X$. *If* $f(A)$ *is dense in* Y *and* $f: X \to Y$ *is* δ*-closed mapping, then* f *is a surjective.*

Corollary 2.4. *Let $(X,\ \delta_X)$, $(Z,\ \delta_Z)$ and $(Y,\ \delta_Y)$ be proximity spaces and the maps $f: X \to Y$ and $g: Y \to Z$ are δ-closed, then $f \circ g: X \to Z$ is also δ-closed.*

Corollary 2.5. *If δ_1 and δ_2 are two proximity on the set X, then the identity mapping $i: (X,\ \delta_1) \to (X,\ \delta_2)$ is a δ-closed if and only if $\delta_2 > \delta_1$.*
Proof . *Let $F\delta_2 E$, since i is δ-closed, then by definition of identity mapping $F\delta_1 E$. Hence $\delta_2 > \delta_1$.* □

Corollary 2.6. *Let $(X,\ \delta_X)$ and $(Y,\ \delta_Y)$ be two proximity spaces. A mapping $f: X \to Y$ is δ-closed if and only if $A \ll B$ implies that $f(A) \ll f(B)$.*
Proof . *Suppose that f is δ-closed and $A \ll B$, then $A\overline{\delta}_X X - B$, by Definition 2.1 $f(A)\,\overline{\delta}_Y f(X - B)$. But $Y - f(B) \subseteq f(X - B)$, then $f(A)\,\overline{\delta}_Y Y - f(B)$, this prove that $f(A) \ll f(B)$.* □

Corollary 2.7. *Let $(X,\ \delta_X)$ and $(Y,\ \delta_Y)$ be two proximity spaces. A bijective mapping $f: X \to Y$ is δ-closed if and only if $f^{-1}: Y \to X$ is a δ-continuous mapping.*
Proof .
$\Rightarrow$ *Let $F\delta_Y E$. Since $f: X \to Y$ is δ-closed, then $A\delta_X B$ where $F = f(A)$ and $E = f(B)$, also f is injective then $A = f^{-1}(F)$ and $B = f^{-1}(E)$. Thus $f^{-1}(F)\ \delta_X B f^{-1}(E)$.*
$\Leftarrow$*Let $F\delta_Y E$. Since $f^{-1}: Y \to X$ is a δ-continuous mapping, then $f^{-1}(F)\ \delta_X f^{-1}(E)$, also f is a bijective, then $\exists\ A$ and B in X s.t $F = f(A)$ and $E = f(B)$. Thus $A\delta_X B$.* □

Corollary 2.8. *Let $(X,\ \delta_X)$ and $(Y,\ \delta_Y)$ be two proximity spaces and $f: (X,\ \delta_X) \to (Y,\ \delta_Y)$ is δ-closed mapping, then $f|_{X_0}: \left(X_0,\ \delta_X|_{X_0}\right) \to (Y,\ \delta_Y)$ is δ-closed where $\varnothing \neq X_0 \subset X$.*
Proof . *By (Definition 2.1 and Theorem 1.11)* □

Proposition 2.9. *Let $(X,\ \delta_X)$ and $(Y,\ \delta_Y)$ be two proximity spaces and Y is discrete space, then every mapping $f: X \to Y$ is δ-closed.*
Proof . *Let $F\delta_Y E$, by (Definition 1.12) $f(A) \bigcap f(B) \neq \varnothing$ where $F = f(A)$ and $E = f(A)$, thus $A \bigcap B \neq \varnothing$ which implies to $A\delta_X B$. Hence f is δ-closed.* □

Proposition 2.10. *Let $(X,\ \delta_X)$ and $(Y,\ \delta_Y)$ be two proximity spaces and Y is normal space with respect to τ_{δ_Y}, then every closed mapping $f: X \to Y$ with respect to τ_{δ_X} and τ_{δ_Y} is also δ-closed.*
Proof . *Let $F\delta_Y E$, by (Theorem 1.16) $cl^\delta f(A) \bigcap cl^\delta f(B) \neq \varnothing$ where $F = f(A)$ and $E = f(A)$. Since f is closed mapping $[cl^\delta f(A) \subseteq f(cl^\delta(A))]$, then $f(cl^\delta A) \bigcap f(cl^\delta B) \neq \varnothing$, that is $\exists\ x \in X$ s. t $f(x) \in f(cl^\delta A)$ and $f(x) \in f(cl^\delta B)$, this implies $x \in\ cl^\delta A \bigcap cl^\delta B$ and then $cl^\delta A \bigcap cl^\delta B\ \neq \varnothing$, therefore, by (Proposition 1.10) $cl^\delta A\ \delta_X\ cl^\delta B$ and $A\delta_X B$. Hence f is δ-closed .* □

Proposition 2.11. *Let $f: X \to Y$ be a mapping from a proximity space $(X,\ \delta_X)$ into a set Y. A proximity relation $\delta_Y = f(\delta_X)$ is the coarsest proximity on Y for which f is a δ-closed mapping.*
Proof . *Show that δ_Y is proximity relation on Y is clear. Now, let $\acute{\delta_Y}$ be any proximity on Y and assume that f is δ-closed mapping with respect to this proximity. Therefore, if $F\acute{\delta_Y}E$ then $A\delta_X B$ where , $F = f(A)$ and $E = f(B)$, this show that $F\acute{\delta_Y}E$ is equivalent to $Ff(\delta_X)E$. Hence $\delta_Y = f(\delta_X) < \acute{\delta_Y}$.* □

Proposition 2.12. *Let δ_Y be a proximity relation on set Y, $\{\delta_X^i, i \in I\}$ is a non-empty family of proximities on X and $\delta_X = \sup\ \{\delta_X^i, i \in I\}$. A mapping $f:\ (X,\ \delta_X) \to\ (Y,\ \delta_Y)$ is a δ-closed if and only if $f:\ (X,\ \delta_X^i) \to\ (Y,\ \delta_Y)$ is a δ-closed mapping $\forall i \in I$.*
Proof . *By (Proposition 2.11) a mapping $f:\ (X,\ \delta_X) \to\ (Y,\ \delta_Y)$ is a δ-closed if and only if $f(\delta_X) <\ \delta_Y$, but the mapping $f:\ (X,\ \delta_X^i) \to\ (Y,\ \delta_Y)$ is δ-closed if and only if $f(\delta_X^i) <\ \delta_Y$. By (Proposition 1.17) the assertion follows.* □

Proposition 2.13. *Let f be a bijective mapping from a proximity space $(X,\ \delta_X)$ into a proximity space $(Y,\ \delta_Y)$. The coarsest proximity δ_Y which may be assigned to Y in order that f be δ-closeis defined by $F\overline{\delta}_Y E$ if and only if $\exists\ C \subset X$ s.t $A\overline{\delta}_X (X - C)$ and $f(C) \subset Y - E$, where $F = f(A)$ and $E = f(B)$.*

Proof . *First, let us prove that δ_Y is proximity on the set Y.*

(i) *Suppose that $F\overline{\delta}_Y E$ and $C \subset X$ s.t $A\overline{\delta}_X (X - C)$ and $f(C) \subset Y - E$ is hold. Let $D = X - A$, then $A = X - D$, we have $E \subset Y - f(C)$ implies $f^{-1}(E) \subset f^{-1}(Y - f(C))$ and $B \subset X - C$, thus $(X - C)\overline{\delta}_X(X - D)$ and $B\overline{\delta}_X A$, this implies $E\overline{\delta}_Y F$, so axiom $B_1$is hold.*

(ii) *Let $(F \cup E)\,\overline{\delta}_Y G$, then $\exists D \subset X$ s.t $(A \cup B)\,\overline{\delta}_X (X - D)$, where $F = f(A)$, $E = f(B)$ and $f(D) \subset Y - G$ Therefore, by proximity of δ_X, $A\overline{\delta}_X (X - D)$ and $B\overline{\delta}_X (X - D)$ this implies $f(A)\overline{\delta}_Y f(X - D)$ and $f(B)\overline{\delta}_Y f(X - D)$ that is $f(A)\,\overline{\delta}_Y Y - f(D)$ and $f(B)\,\overline{\delta}_Y Y - f(D)$, this show that $F\overline{\delta}_Y G$ and $E\overline{\delta}_Y G$.*
Conversely, if $F\overline{\delta}_Y G$ and $E\overline{\delta}_Y G$, then $\exists D_1$and D_2in Xs.t $A\overline{\delta}_X (X - D_1)$ and $B\overline{\delta}_X (X - D_2)$, but $f(D_1) \subset Y - G$ and $f(D_2) \subset (Y - G)$, thus $f(A)\overline{\delta}_Y G$ and $f(B)\overline{\delta}_Y G$, this show $F\overline{\delta}_Y G$ and $E\overline{\delta}_Y G$ so axiom $B_2$is hold.

(iii) *If $F = \varnothing$, then for $C = \varnothing$ and $F = f(A) = f(\varnothing)$, we have $A\overline{\delta}_X X$, $f(\varnothing) \subseteq Y - Y$, so axiom $B_2$is hold.*

(iv) *We must prove (if $F\overline{\delta}_Y E$ then $F \cap E = \varnothing$)which is equivalent $\{y\}\,\delta_Y\,\{y\}$ for every $y \in Y$.*
Let $F\overline{\delta}_Y E$ then $\exists C \subset X$ s.t $A\overline{\delta}_X X - C$ and $f(C) \subset Y - E$ thus $A \cap (X - C) = \varnothing$, since f is bijective $f(A \cap (X - C)) = \varnothing$, therefore $f(A) \cap (Y - f(C)) = \varnothing$ and $f(A) \cap f(B) = \varnothing$, so axiom $B_4$is hold.

(v) *Let $F\overline{\delta}_Y E$ then $\exists C \subset X$ s.t $A\overline{\delta}_X X - C$ and $f(C) \subset Y - E$, by proximity of δ_X, there exist $D \subset X$ s.t $A\overline{\delta}_X D$ and $X - D\overline{\delta}_X X - C$, thus*
$f(A)\,\overline{\delta}_Y f(D)$ and $F\overline{\delta}_Y G$, where $G = f(D)$. Since $X - D\overline{\delta}_X X - C$ and $B \subset X - C$, then $X - f^{-1}(G)\overline{\delta}_X B$ this implies to $f(X - f^{-1}(G))\overline{\delta}_X f(B)$ that is $Y - G\overline{\delta}_X E$. So axiom $B_5$is hold.
Now, we will prove that $f:\ (X,\ \delta_X) \to\ (Y,\ \delta_Y)$ is a δ-closed mapping.
Suppose that $A\overline{\delta}_X B$, by proposition (Theorem 1.5 (d)) $X \ll X - B$ and $\exists C \subset X$ s. t $X \ll C \ll X - B$, that is $A\overline{\delta}_X X - C$ and $C\overline{\delta}_X B$, by (Proposition 1.3(iv)), $C \cap B = \varnothing$ and $C \subset X - B$ which implies to $f(C) \subset f(X - B) = Y - E$,by assumption $F\overline{\delta}_Y E$. Thus f is a δ-closed.

To prove δ_Y is coarsest proximity, let $\delta_1$be any proximity on Y s.t $f:\ (X,\ \delta_X) \to\ (Y,\ \delta_Y)$ is a δ-close. If $F\overline{\delta}_Y E$, then $\exists C \subset X$ s.t $A\overline{\delta}_X X - C$ and $f(C) \subset Y - E$. Since f is a δ-closed, then $F\delta_1 Y - f(C)$. But $E \subset Y - f(C)$, then $F\overline{\delta}_1 E$. Thus $\delta_Y < \delta_1$. □

Definition 2.14. *Let $(X,\ \delta_X)$ and $(Y,\ \delta_Y)$ be two proximity spaces. A mapping $f: X \to Y$ is said to be proximally open (δ-open) if $F\delta_Y Y - E$ implies $A\delta_X X - B$, where $F = f(A)$ and $E = f(B)$. Equivalently, $f: X \to Y$ is said to be δ-open if $A\overline{\delta}_X X - B$ implies $F\overline{\delta}_Y Y - E$.*

Proposition 2.15. *Let $(X,\ \delta_X)$ and $(Y,\ \delta_Y)$ be two proximity spaces and $f: X \to Y$ is δ-open mapping, then f it is an open with respect to the topologies $\tau_{\delta_X}$and τ_{δ_Y}.*

Proof . *Let $y \in f(int(A)) = f\left(X - cl^{\delta}(X - A)\right)$, then $y = f(x)$ and $x \in X - cl^{\delta}(X - A)$, if not then implies contradiction. Therefore, $x \notin cl^{\delta}(X - A)$ that is $\{x\}\,\overline{\delta}_X X - A$, and since f is δ-open, then $f(x)\overline{\delta}_Y Y - f(A)$ i.e $f(x) \notin cl^{\delta}(Y - f(A))$. Thus $f(x) \in Y - cl^{\delta}(Y - f(A))$ which implies $f(x) \in intf(A)$, this prove that $f(int(A)) \subseteq intf(A)$. Hence the mapping f is open.* □

Corollary 2.16. *Let $(X,\ \delta_X)$ and $(Y,\ \delta_Y)$ be two proximity spaces and $f: (X,\ \delta_X) \to (Y,\ \delta_Y)$ is a bijective mapping, then $f$is δ-closed if and only if $f$is δ-open.*

Proof . *Let $F\delta_Y Y-E$, then $A\delta_X B$ where $F=f(A)$ and $Y-E=f(B)$, but f is injective, thus $E=f(X-B)$, so that $f$is δ-open.*
Conversely, let $F\delta_Y E$, then $A\delta_X B$ where $F=f(A)$ and $Y-E=f(X-B)$, but f is a bijective, thus $E=f(B)$, so that f is δ-closed. □

Definition 2.17. *Let $(X,\ \delta_X)$ and $(Y,\ \delta_Y)$ be two proximity spaces. A bijective mapping $f : X\to Y$ is said to be proximally isomorphic or $\sigma-$homeomorphism if it is δ-closed (orδ-open) and $\sigma-$continuous.*

Proposition 2.18. *Let f be a bijective mapping from a proximity space $(X,\ \delta_X)$ into a proximity space $(Y,\ \delta_Y)$. Then $\delta_Y=f(\delta_X)$ if and only if the mapping $h=f|_{X_0}:\left(X_0,\ \delta_X|_{X_0}\right)\to(Y,\ f(\delta_X))$ is a $\delta-$homeomorphism.*
Proof .
$\Rightarrow$ Let $\delta_Y=f(\delta_X)$, then by (Proposition 2.10) $f:\ (X,\ \delta_X)\to\ (Y,\ f(\delta_X))$ is δ-close. But, by (Corollary 2.8) $h=f|_{X_0}:\left(X_0,\ \delta_X|_{X_0}\right)\to(Y,\ f(\delta_X))$ is δ-close. Previously, $f^{-1}oh:\left(X_0,\ \delta_X|_{X_0}\right)\to(X,\ \delta_X)$ is a canonical inclusion, then by (Corollary 1.13), it follows that $\delta_X|_{X_0}=f^{-1}(f^{-1}oh\,(\delta_X))=h^{-1}(f(\delta_X)$. Thus, by (Proposition 1.15) $h:(X_0,\ h^{-1}(f(\delta_X).\ )\to(Y,\ f(\delta_X))$ is δ-continuous, so that h is a $\sigma-$homeomorphism.
$\Leftarrow$ suppose that that $h=f|_{X_0}:\left(X_0,\ \delta_X|_{X_0}\right)\to(Y,\ f(\delta_X))$ is a $\delta-$ homeomorphism, then the identical mapping hoh^{-1} is $\delta-$ homeomorphism from$\left(X_0,\ \delta_X|_{X_0}\right)$ on to $\left(\ f(X_0),\ \ f(\delta_X)|_{X_0}\right)$. By corollary 66 we obtain $\delta_Y=f(\delta_X)$. □

Corollary 2.19. *Let f be a bijective mapping from a proximity space $(X,\ \delta_X)$ into a proximity space $(Y,\ \delta_Y)$. Then $\delta_Y=f(\delta_X)$ if and only if the mapping f is a $\sigma-$homeomorphism.*
Proof . *Straightway, since it is special case of proposition.* □

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