Int. J. Nonlinear Anal. Appl. 13 (2022) 1, 1737-1745
ISSN: 2008-6822 (electronic)
http://dx.doi.org/10.22075/ijnaa.2022.5788

Generalized Euler and Runge-Kutta methods for solving classes of fractional ordinary differential equations

Mohammed S. Mechee[a,*], Sameeah H. Aidi[b]

[a] *Information Technology Research and Development Center (ITRDC), UOK, Najaf, Iraq*
[b] *Department of Mathematics, Faculty of Education - Ibn Haithem, Baghdad University, Iraq*

(Communicated by Madjid Eshaghi Gordji)

Abstract

A third-order fractional ordinary differential equation (FrODE) is very important in the mathematical modelling of physical problems. Generally, the third-order ODE is solved by converting the differential equation to a system of first-order ODEs. However, it is a lot more efficient in terms of accuracy, a number of function evaluations as well as computational time if the problem can be solved directly using numerical methods. In this paper, we are focused on the derivation of the direct numerical methods which are one, two and three-stage methods for solving third-order FrODEs. The RKD methods with two- and three stages for solving third-order ODEs are adapted for solving special third-order FrDEs. Numerical examples have been evaluated to show the effectiveness of the new methods compared with the analytical method. Numerical experiments are carried out to verify the accuracy and efficiency of the proposed methods. Applications of proposed methods are also presented which yield impressive results for the proposed and modified methods. The numerical solutions of the test problems using proposed methods agree well with the analytical solutions. From the numerical results obtained using proposed methods, we can conclude that the proposed methods in which derived or modified in this paper are very efficient.

Keywords: RK, RKD, RKM, Ordinary, Third-order, DEs, ODEs, PDEs, FrDEs.

*Corresponding author

Email addresses: mohammeds.abed@uokufa.edu.iq (Mohammed S. Mechee), sameeahaidi9@gmail.com (Sameeah H. Aidi)

Received: May 2021 *Accepted:* March 2021

1. Introduction

Differential equation (DE) is one of the most important branches of applied mathematics and its applications in the fields of science and engineering. Most of mathematical modelling in science and engineering involving different types of DEs. Also, DEs have significate rule in the various fields of applied mathematics such as physics, engineering, biology, medicine, chemistry and economic. The mathematical models of the real problems in applied science and engineering are modelled by using the tools of DEs especially, fractional differential equations (FrDEs). In the 20^{th} century, important research in fractional calculus was published in the engineering and science literature. Progress of fractional calculus is reported in various applications in the field of integral equations, fluid mechanics, viscoelastic models, biological models, and electrochemistry. Many classical or modern analytical and numerical methods for solving DEs have been studied during long time [10]. However, Finding the solutions of different types of DEs, analytically or numerically, had been challenged the ingenuity of mathematicians. At present, several powerful classical and modern numerical and analytical methods are be available to use for scientists and engineers. The literature review of different modern methods for finding the solutions of mathematical models which contains DEs are listed as follows: [39] developed second-, fourth-, sixth- and eighth-orders finite-difference methods for solving IVPs while [33]-[38] developed a second-order method for solving IVPs, [1] solved boundary value problems (BVPs) using the technique of non-polynomial spline and [21] developed new integrator for solving ODEs of seventh-order. The analytical methods for solving DEs are not always able to solve all types of DEs directly or indirectly. This propose make us to study the derivation of direct numerical methods. Many researchers like: [29]-[26] have derived one-step numerical methods for solving IVPs of ODEs of different orders while other authors derived multistep numerical methods for solving this problem [3]-[34]

Undoubtedly, fractional calculus is an efficient mathematical tool to solve various problems in mathematics, engineering, and sciences. Recently, the tool of fractional calculus has been used to analyze the nonlinear dynamics of different problems [37]-[6]. Mostly, the analytical solutions cannot be obtained for fractional differential equations, so that there is a need of semi analytical and numerical methods to understand the effects of the solutions to the nonlinear problems [32]. In the recent decades, different approximated methods have been implemented to solve the linear as well as the nonlinear dynamical systems, such as the Adomian decomposition method (ADM) [43], variational iteration method (VIM) [40], Homotopy perturbation method (HPM) [41], Homotopy perturbation method in association with the Laplace transform method [42], Homotopy analysis method (HAM) [9], Sumudu transform method and Homotopy analysis transform method (HATM) [7]. Also, Odibat and Momani developed the new method with the connection of fractional Euler method and modified trapezoidal rule by using the generalized Taylor series expansion [31]. Lastly [28] and [27] solved FRODEs used collection and least square methods while [5] studied the numerical solutions of first-order FrODEs using developed Euler method and they derived 2-stage fractional Runge-Kutta (FRK) method.

In this work, Euler and Runge-Kutta methods have been derived or developed to be consistent with solving third-order FrDEs. Firstly, one-stage Euler method and two-stage RKD method for solving third-order FrDEs have been derived using the generalized Taylor series expansion. Secondly, we developed the two- and three-stage RKD methods for solving third-order ODEs to be consistent with solving third-order FrDEs. Afterwards, we applied the proposed numerical methods on different cases of fractional differential equations implementations.

2. Preliminary

2.1. A Class of Quasi Linear Third Order Fractional ODEs

Consider the following quasi linear third-order fractional differential equation:

$$D^{3\alpha}u(t) = \Phi(t, u(t)); \qquad t > 0, 0 < \alpha \leq l \tag{2.1}$$

with the condition

$$u(0) = \beta_0; u^{\alpha}(0) = \beta_1, u^{2\alpha}(0) = \beta_2. \tag{2.2}$$

2.2. RKD and RKT Methods for Solving Third-Order ODEs

Many researchers used to solve the ODEs of order more than order two by converting the ODE into a system of first-order equations with the dimension equal to this order. However, [44] and [29] derived direct numerical methods for solving special third-order ODE. The general form of RKD and RKT methods with s-stage for solving special third-order ODEs can be written as

$$u_{n+1} = u_n + hu_n^{'} + \frac{h^2}{2}u_n^{''} + h^3 \sum_{i=1}^{s} b_i k_i, \tag{2.3}$$

$$u_{n+1}^{'} = u_n^{'} + hu_n^{''} + h^2 \sum_{i=1}^{s} b_i^{'} k_i, \tag{2.4}$$

$$u_{n+1}^{''} = u_n^{''} + h \sum_{i=1}^{s} b_i^{''} k_i, \tag{2.5}$$

where,

$$k_1 = f(x_n, y_n), \tag{2.6}$$

$$k_i = f\left(x_n + c_i h, y_n + hc_i y_n^{'} + \frac{h^2}{2}c_i^2 y_n^{''} + h^3 \sum_{j=1}^{i-1} a_{ij} k_j\right), \tag{2.7}$$

for $i = 2, 3, \ldots, s$.

The parameters of the numerical method are c_i, a_{ij}, b_i, $b_i^{'}$, and $b_i^{''}$ for $i = 1, 2, \ldots, s$ and, $j = 1, 2, \ldots, s$ are assumed to be real. If $a_{ij} = 0$ for $i \leq j$, it is an explicit method and implicit otherwise. The two-stage, third-order and three-stages, fourth-order RKD methods which can be expressed in the Table 1 and Table 2.

3. Analysis of Proposed Methods for Solving Third-order FrDEs

In this section, we have constructed two numerical methods and modify another two numerical methods for solving third-order FrODEs which belong to class of quasi linear in Equation (2.1) with initial conditions in Equation (2.2).

3.1. Derivation of Generalized Euler Method

The generalized Taylor expansion of $u(t+h)$ is

$$u(t+h) = u(t) + \frac{h^{\alpha}}{\Gamma(\alpha+1)} D^{\alpha}u(t) + \frac{h^{2\alpha}}{\Gamma(2\alpha+1)} D^{2\alpha}u(t) + \frac{h^{3\alpha}}{\Gamma(3\alpha+1)} D^{3\alpha}u(t) + \ldots\ldots \tag{3.1}$$

For the very small step size, we neglect the higher terms involving $D^{4\alpha}u(t)$ in Equation (3.1) and substituting the value of $D^{3\alpha}y(t)$ from Equation (2.1), we obtain the following formula:

$$u_{n+1} = u_n + \frac{h^{\alpha}}{\Gamma(\alpha+1)} u_n^{\alpha} + \frac{h^{2\alpha}}{\Gamma(\alpha+2)} u_n^{2\alpha} + \frac{h^{3\alpha}}{\Gamma(\alpha+3)} \Phi(t_n, u_n(t_n)). \tag{3.2}$$

By derivation Equation (3.2) once and twice, we obtain the following:

$$u_{n+1}^{\alpha} = u_n^{\alpha} + \frac{h^{\alpha}}{\Gamma(\alpha+1)} u_n^{2\alpha} + \frac{h^{2\alpha}}{\Gamma(2\alpha+1)} \Phi(t_n, u_n(t_n)), \tag{3.3}$$

and

$$u_{n+1}^{2\alpha} = u_n^{2\alpha} + \frac{h^{\alpha}}{\Gamma(\alpha+1)} \Phi(t, u_n(t)). \tag{3.4}$$

The formulas in equations (3.2)-(3.4) can used to generate convergent sequence for solving the the Equation (2.1) with initial conditions (3.1).

3.2. Derivation of Two-Stage RKD Method

Using the chain rule, we obtain the following:

$$D^{4\alpha}u(t) = D^{\alpha}(D^{3\alpha}u(t)) = D^{\alpha}(\Phi(t,u(t))) = D_t^{\alpha}\Phi(t,u(t)) + \Phi(t,u(t)) D_u^{\alpha}\Phi(t,u(t)) \tag{3.5}$$

For the very small step size, we neglect the higher terms involving $D^{5\alpha}u(t)$ in Equation (3.1) and substituting the value of $D^{3\alpha}u(t)$ from Equation (2.1). By substituting Equation (3.5) in Taylor expansion serious $u(t+h)$ in Equation (3.1), we obtain the following formulas

$$\begin{aligned} u_{n+1} &= u_n + \frac{h^{\alpha}}{\Gamma(\alpha+1)} u_n^{\alpha} + \frac{h^{2\alpha}}{\Gamma(2\alpha+1)} u_n^{2\alpha} + \frac{h^{3\alpha}}{\Gamma(3\alpha+1)} \Phi(t_n, u_n(t_n)) \\ &+ \frac{h^{4\alpha}}{\Gamma(4\alpha+1)} (D_t^{\alpha}\Phi(t,u(t)) + \Phi(t,u(t)) D_u^{\alpha}\Phi(t,u(t))), \qquad (3.6) \\ &= u_n + \frac{h^{\alpha}}{\Gamma(\alpha+1)} u_n^{\alpha} + \frac{h^{2\alpha}}{\Gamma(2\alpha+1)} u_n^{2\alpha} + \frac{h^{3\alpha}}{2\Gamma(3\alpha+1)} \Phi(t_n, u_n(t_n)) \\ &+ \frac{h^{3\alpha}}{2\Gamma(3\alpha+1)} \Phi(t + \frac{2h^{3\alpha}\Gamma(3\alpha+1)}{\Gamma(4\alpha+1)}, u(t) + \frac{2h^{3\alpha}\Gamma(3\alpha+1)}{\Gamma(4\alpha+1)} \Phi(t_n, u_n(t_n))). \qquad (3.7) \end{aligned}$$

By derivation Equation (3.7) once and twice, we obtain the following:

$$u_{n+1}^{\alpha} = u_n^{\alpha} + \frac{h^{\alpha}}{\Gamma(\alpha+1)} u_n^{2\alpha} + \frac{h^{2\alpha}}{\Gamma(\alpha+2)} \Phi(t_n, u_n(t_n)), \tag{3.8}$$

$$u_{n+1}^{2\alpha} = u_n^{2\alpha} + \frac{h^{\alpha}}{\Gamma(\alpha+1)} \Phi(t_n, u_n(t_n)). \tag{3.9}$$

Hence, the formulas in equations (3.7)-(3.9) used to generate convergent sequence for solving the the Equation (2.1) with initial conditions (2.2).

3.3. Developed RKD Method for Solving 3^{th}-Order FrODEs

We improve equations (2.3)-(2.7) for solving 3^{th}-order ODEs to be suitable for solving 3^{th}-order FrODEs, we suppose the following formula for numerical solutions of the Equation (2.1) with initial conditions in Equation (2.2):

$$u_{n+1}(t_n) = u_n(t_n) + \frac{h^{\alpha}}{\Gamma(\alpha+1)} u_n^{\alpha}(t_n) + \frac{h^{2\alpha}}{\Gamma(2\alpha+1)} u_n^{2\alpha}(t_n) + h^{3\alpha} \sum_{i=1}^{s} b_i k_i. \tag{3.10}$$

$$u_{n+1}^{\alpha}(t_n) = u_n^{\alpha}(t_n) + \frac{h^{\alpha}}{\Gamma(\alpha+1)} u_n^{2\alpha}(t_n) + h^{2\alpha} \sum_{i=1}^{s} b_i' k_i, \tag{3.11}$$

$$u_{n+1}^{2\alpha}(t_n) = u_n^{2\alpha}(t_n) + h^{\alpha} \sum_{i=1}^{s} b_i'' k_i. \tag{3.12}$$

$$k1 = f(t_n, u_n(t_n)), \tag{3.13}$$

$$k2 = f(t_n + hc_2, u_n(t_n) + ha_{21}k1), \tag{3.14}$$

end

$$k3 = f(t_n + hc_3, u_n(t_n) + ha_{31}k1 + ha_{32}k2). \tag{3.15}$$

Table 1: The Butcher Tableau RKD3 Method of Third-Order

0		
$\frac{2}{3}$	$\frac{11}{200}$	
	$\frac{1}{8}$	$\frac{1}{24}$
	$\frac{1}{4}$	$\frac{1}{4}$
	$\frac{1}{4}$	$\frac{3}{4}$

4. Implementations

In this section, we prove the efficiency of the proposed method by some numerical examples

Example 4.1. *Consider the following third-order FrODE*

$$D^{3\alpha} y(t) = a^{3\alpha} y(t); \qquad 0 < \alpha < 1 \tag{4.1}$$

with initial conditions $y(0) = 1$; $D^{\alpha} y(0) = a^{\alpha}$; $D^{2\alpha} y(0) = a^{2\alpha}$.

The exact solution is $y(t) = e^{at}$

Example 4.2. *Consider the following third-order FrODE*

$$D^{3\alpha} y(t) = a^{3\alpha} sin(at + \frac{3\Pi}{2}\alpha); \qquad 0 < \alpha < 1 \tag{4.2}$$

with initial conditions $y(0) = 0$; $D^{\alpha} y(0) = a^{\alpha} sin(\frac{\Pi\alpha}{2})$; $D^{2\alpha} y(0) = a^{2\alpha} sin(\frac{\Pi\alpha}{2})$.

The exact solution is $y(t) = sin(at)$

Table 2: The Butcher Tableau RKD4 Method of Fourth-Order

0			
$\frac{1}{2}$	$\frac{1}{48}$		
1	$\frac{1}{12}$	$\frac{1}{12}$	
	$\frac{1}{12}$	$\frac{1}{12}$	0
	$\frac{1}{6}$	$\frac{1}{3}$	0
	$\frac{1}{6}$	$\frac{2}{3}$	$\frac{1}{6}$

5. Discussion of the Numerical Results

To discuss the numerical results which shown in Figure 1 for the implementations of this paper, we have a comparisons on approximated solutions for the improved and proposed methods versus analytical solutions for N=number of grids in the interval of definition of differential equation=50 and $\alpha = 0.96$ for solving Example 4.1 for five cases. In case (1) a=-2 using modified Euler method, case (2) a=-2 using the proposed two-stage method, case (3) a=-2 using modified two-stage RKD Method, case (4) a=-2 using modified three-stage RKD method and case (5) a=-1.5 using proposed two-stage method while case (6) for solving Example 4.2 using modified two-stage RKD method. The main contribution of this paper is the establishment of direct methods for solving third-order FrODEs, which are the derivation of new explicit method or develop explicit RKD method for solving third-order FrODEs. The method of RKD and Euler methods are modified to be a suitable method for directly solving some third order FrODEs. The proposed technique of this direct or development methods require less computational work in addition to great features such as fast and effective computation. The numerical solutions are compared with exact solutions to establish the validity of the method. The numerical results of the methods show that the methods are applicable to FrODEs and have a good agreement with exact solutions. The new methods provide encouraging results and efficiency. Also, we studied the developed Euler method and proposed two-stage and three-stages methods by using various cases of examples of third-order FrODEs to compare the efficiency of the developed and proposed methods with analytical solutions. The numerical results in Figure 1 indicate that the new numerical methods showed good agreement with the exact solutions and these methods provide encouraging results. To adapt the RKD methods for solving FrODEs. The adaption for RKD methods have been studied and implemented. These methods are more cost effective in terms of computation time than other existing methods. Furthermore, the function evaluations of the modified RKD methods are few. Overall, the implementations of the numerical methods show that the new methods are agree well with analytical solutions and require less function evaluations.

6. Conclusion

In this paper, we established direct numerical methods for solving third-order FrODEs. RKD methods have been improved to be consistent for solving FrODEs. Various examples of third-order FrODEs prove the efficiency of the proposed methods. The numerical results in Figures 1 indicate that the new numerical methods showed good agreement with exact solutions. The new methods provide

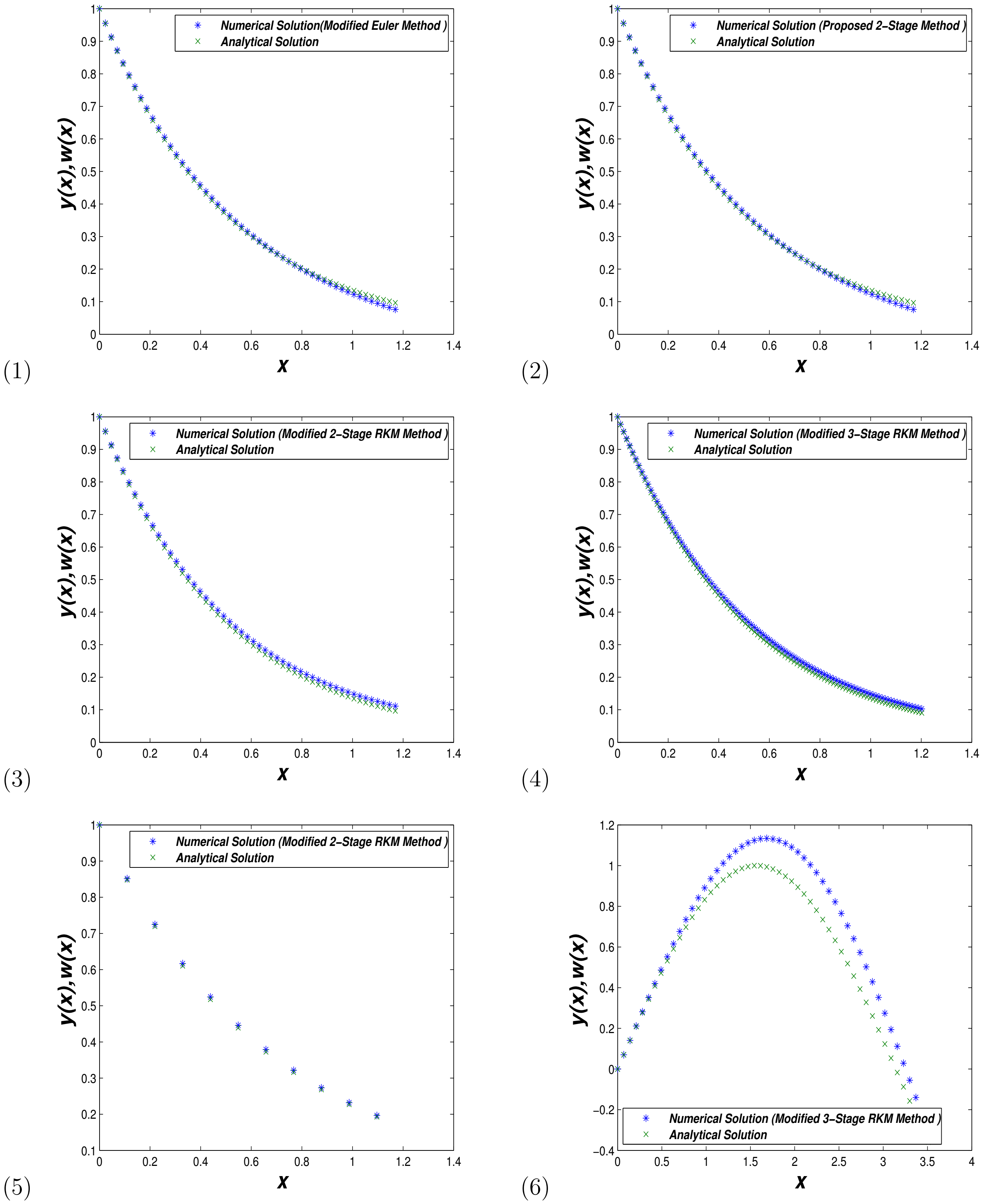

Figure 1: Comparisons on Approximated Solutions versus Analytical Solutions for N=50 and $\alpha = 0.96$ for Solving Example 4.1 Using (1) Modified Euler Method and a=-2 (2) Proposed two-stage method and a=-2 (3) Modified two-stage RKD Method and a=-2 (4) Modified three-stage RKD Method and a=-2 and (5) Proposed two-stage method while (6) for Solving Example 4.2 Using Modified two-stage RKD Method

encouraging results and are efficient. The main contribution of this paper is the establishment of direct methods for solving third-order FrODEs. The derivation of new one-stage explicit method and two-stages, three-stage explicit RKD methods with constant step sizes for solving third-order FrODEs have been introduced. The implementation of the numerical methods show that the new methods agree well with analytical solutions and require less function evaluations. Modified Euler method and modified RKD method are efficient because they are direct methods; hence, we save considerable computational times. These methods are more cost effective in terms of computation time than other existing methods. The proposed technique of these direct methods require less computational work in addition to great features such as fast and effective computation. The numerical solutions are compared with known exact solutions to establish the validity of the method. The numerical results of the methods show that the methods are applicable to FrODEs.

References

[1] G. Akram and S.S. Siddiqi, *Solution of sixth order boundary value problems using non- polynomial spline technique*, Appl. Math. Comput. 181(1) (2006) 708–720.

[2] R. Allogmany and F. Ismail, *many and F. Ismail, Direct solution of $u'' = f(t, u, u')$ using three point block method of order eight with applications*, J. King Saud Univ. Sci. 33(2) (2021) 101337.

[3] R. Allogmany, F. Ismail and Z.B. Ibrahim, *Implicit two-point block method with third and fourth derivatives for solving general second order odes*, Math. Stat. 7(4) (2019) 116–123.

[4] R. Allogmany, F. Ismail, Z. A. Majid and Z. B. Ibrahim, *Implicit two-point block method for solving fourth-order initial value problem directly with application*, Math. Prob. Engin. 2020 (2020).

[5] M.S. Arshad, D. Baleanu, M.B. Riaz and M. Abbas, *A novel 2-stage fractional Runge-Kutta method for a time-fractional logistic growth model*, Discrete Dyn. Nature Soc. 2020 (2020).

[6] S. Arshad, A. M. Siddiqui, A. Sohail, K. Maqbool and Z. Li, *Comparison of optimal homotopy analysis method and fractional homotopy analysis transform method for the dynamical analysis of fractional order optical solitons*, Adv. Mech. Engin. 9(3) (2017) 1687814017692946.

[7] S. Arshad, A. Sohail and K. Maqbool, *Nonlinear shallow water waves: A fractional order approach*, Alexandria Engin. J. 55(1) (2016) 525–532.

[8] A. Boutayeb and E. Twizell, *Numerical methods for the solution of special sixth-order boundary-value problems*, Int. J. Comput. Math. 45(3-4) (1992) 207–223.

[9] D. Das, P. Ray, R. Bera and P. Sarkar, *Solution of nonlinear fractional differential equation (nfde) by homotopy analysis method*, Int. J. Sci. Res. Educ. 3(3) (2015) 3084.

[10] S.J. Farlow, *Partial Differential Equations for Scientists and Engineers*, Courier Dover Publications, 2012.

[11] M.S. Mechee, *Direct Integrators of Runge-Kutta Type for Sspecial Third-Order Differential Equations with Their Applications*, Thesis, ISM, University of Malaya, 2014.

[12] M.S. Mechee, *Generalized RK integrators for solving class of sixth-order ordinary differential equations*, J. Interdiscip. Math. 22(8) (2019) 1457–1461.

[13] M.S. Mechee, G.A. Al-Juaifri and A.K. Joohy, *Modified homotopy perturbation method for solving generalized linear complex differential equations*, Appl. Math. Sci. 11(51) (2017) 2527–2540.

[14] M.S. Mechee, O.I. Al-Shaher and G.A. Al-Juaifri, *Haar wavelet technique for solving fractional differential equations with an application*, J. Al-Qadisiyah Comput. Sci. Math. 11(1) (2019) 70.

[15] M.S. Mechee, Z.M. Hussain and H.R. Mohammed, *On the reliability and stability of direct explicit Runge-Kutta integrators*, Global J. Pure Appl. Math. 12(4) (2016) 3959–3975.

[16] M. Mechee, F. Ismail, Z. Hussain and Z. Siri, *Direct numerical methods for solving a class of third-order partial differential equations*, Appl. Math. Comput. 247 (2014) 663–674.

[17] M.S. Mechee, F. Ismail, N. Senu and Z. Siri, *A third-order direct integrators of Runge-Kutta type for special third-order ordinary and delay differential equations*, J. Appl. Sci. 2(6) (2014).

[18] M.S. Mechee, F. Ismail, N. Senu and Z. Siri, *Directly solving special second order delay differential equations using Runge-Kutta-Nystrom method*, Math. Prob. Engin. 2013 (2013)

[19] M.S. Mechee, F. Ismail, Z. Siri and N. Senu, *A four stage sixth-order RKD method for directly solving special third order ordinary differential equations*, Life Sci. J. 11(3) (2014).

[20] M.S. Mechee and M. Kadhim, *Direct explicit integrators of rk type for solving special fourth-order ordinary differential equations with an application*, Global J. Pure Appl. Math. 12(6) (2016) 4687–4715.

[21] M.S. Mechee and M.A. Kadhim, *Explicit direct integrators of rk type for solving special fifth-order ordinary differential equations*, Amer. J. Appl. Sci. 13 (2016) 1452–1460.

[22] M.S. Mechee and J.K. Mshachal, *Derivation of direct explicit integrators of rk type for solving class of seventh-order ordinary differential equations*, Karbala Int. J. Modern Sci. 5(3) (2019) 8.

[23] M.S. Mechee and J.K. Mshachal, *Derivation of embedded explicit rk type methods for directly solving class of seventh-order ordinary differential equations*, J. Interdiscip. Math. 22(8) (2019) 1451–1456.

[24] M.S. Mechee and K.B. Mussa, *Generalization of RKM integrators for solving a class of eighth-order ordinary differential equations with applications*, Adv. Math. Model. Appl. 5(1) (2020) 111–120.

[25] M. S. Mechee and Y. Rajihy, *Generalized RK integrators for solving ordinary differential equations: A survey and comparison study*, Global J. Pure Appl. Math. 13(7) (2017) 2923–2949.

[26] M.S. Mechee and N. Senu, *A new numerical multistep method for solution of second order of ordinary differential equations*, Asian Trans. Sci. Technol. 2(2) (2012) 18–22.

[27] M.S. Mechee and N. Senu, *Numerical study of fractional differential equations of Lane-Emden type by method of collocation*, Appl. Math. 3(8) (2012) 851.

[28] M.S. Mechee and N. Senu, *Numerical study of fractional differential equations of Lane-Emden type by the least square method*, Int. J. Diff. Equ. Appl. 11(3) (2012) 157–168.

[29] M.S. Mechee, N. Senu, F. Ismail, B. Nikouravan and Z. Siri, *A three-stage fifth-order Runge-Kutta method for directly solving special third-order differential equation with application to thin film ow problem*, Math. Prob. Engin. 2013 (2013).

[30] M.S. Mechee, H.M. Wali and K.B. Mussa, *Developed rkm method for solving ninth-order ordinary differential equations with applications*, J. Phys. Conf. Ser. 1664(1) IOP Publishing, 2020, p. 012102.

[31] Z.M. Odibat and S. Momani, *An algorithm for the numerical solution of differential equations of fractional order*, J. Appl. Math. Inf. 26(1-2) (2008) 15–27.

[32] A. Sohail, S. Arshad, and Z. Ehsan, *Numerical analysis of plasma kdv equation: time- fractional approach*, Int. J. Appl. Comput. Math. 3(1) (2017) 13251336.

[33] J. Toomre, J.-P. Zahn, J. Latour and E. Spiegel, *Stellar convection theory. ii-single- mode study of the second convection zone in an a-type star*, Astrophysical J. 207 (1976) 545–563.

[34] M.Y. Turki, *Second Derivative Block Methods for Solving First and Higher Order Ordinary Differential Equations*, PhD Thesis, UPM, Putra University, 2018.

[35] M.Y. Turki, F. Ismail, N. Senu and Z. Bibi, *Two and three point implicit second derivative block methods for solving first order ordinary differential equations*, ASM Sci. J. 12 (2019) 10–23.

[36] M.Y. Turki, F. Ismail, N. Senu and Z.B. Ibrahim, *Second derivative multistep method for solving first-order ordinary differential equations*, AIP Conf. Proc. 1739(1) (2016) 020054.

[37] M.Y. Turki, F. Ismail, N. Senu and Z.B. Ibrahim, *Direct integrator of block type methods with additional derivative for general third order initial value problems*, Adv. Mech. Engin. 12(10) (2020) 1687814020966188.

[38] E. Twizell, *Numerical methods for sixth-order boundary value problems*, Numerical Math. Singapore 1988, Springer, 1988, 495–506.

[39] E. Twizell and A. Boutayeb, *Numerical methods for the solution of special and general sixth-order boundary-value problems, with applications to benard layer eigenvalue problems*, Proc. R. Soc. Lond. A 431(1883) (1990) 433–450.

[40] S.-Q. Wang, Y.-J. Yang and H.K. Jassim, *Local fractional function decom- position method for solving inhomogeneous wave equations with local fractional derivative*, Abstr. Appl. Anal. 2014 (2014) 1–7.

[41] A.-M. Wazwaz, *New (3+ 1)-dimensional nonlinear evolution equation: multiple soliton solutions*, Central European J. Engin. 4(4) (2014) 352–356.

[42] Y. Xiao-Jun, H. Srivastava and C. Cattani, *Local fractional homotopy perturbation method for solving fractal partial differential equations arising in mathematical physics*, Romanian Rep. Phys. 67(4) (2015) 752–761.

[43] X.-J. Yang, J. Tenreiro Machado and H. Srivastava, *A new numerical technique for solving the local fractional diffusion equation: Two-dimensional extended differential transform approach*, Appl. Math. Comput. 274(C) (2016) 143–151.

[44] X. You and Z. Chen, *Direct integrators of Runge-Kutta type for special third-order ordinary differential equations*, Appl. Numerical Math. 74 (2013) 128–150.