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Near fixed point, near fixed interval circle and their equivalence classes in a $b-$interval metric space

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Abstract

We introduce a novel distance structure called a b-interval metric space to generalize and extend metric interval space. Also, we demonstrate that the collection of open balls, which forms a basis of a b-interval metric space, generates a T_0-topology on it. Further, we define topological notions like an open ball, closed ball, b-convergence, b-Cauchy sequence and completeness of the space on a b-interval metric space to create an environment for the survival of a near fixed point and a unique equivalence class of near fixed point. Towards the end, we introduce notions of interval circle, fixed interval circle, its equivalence class and established the existence of a near fixed interval circle and its equivalence interval C-class of near fixed interval circle to study the geometric properties of non-unique equivalence C-classes of nearly fixed interval circles.

Keywords: Continuity, $b-$convergence, completeness, fixed interval circle, null set, $\mathcal{T}_0-$topology.
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1. Introduction

In 1922, Banach established the acclaimed Banach contraction principle exploiting metric spaces. Subsequently, this principle was generalized in different frameworks. Recently, Wu [20] familiarized metric interval spaces exploiting the null set to study near fixed points. It is interesting to mention that metric interval space is not a conventional metric space and all the closed and bounded intervals on the collection of real numbers may not be a vector space, as the additive inverse of each of its elements may not exist in it. Acknowledging the work of Wu [20], we familiarize a novel distance structure called a $b-$interval metric space and study its topology. In spite of the fact that the metric

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space may not essentially be a vector space, we equip a $b-$metric to the collection of closed and bounded intervals in the set of real numbers utilizing a null set. Besides, we introduce notions like $b-$convergence, $b-$Cauchy sequence, completeness and demonstrate that the collection of open balls, which forms a basis of a $b-$interval metric space, generates a $\mathcal{T}_0$ topology on it. We illustrate by means of examples that conventional Banach contraction principle, Theorem 5 of Sehgal [17] and consequently results in, [3], [6], [8], [9]-[13], [15], [16], [18]-[19], and so on , may not be proved in a $b-$interval metric space. Thereby, we infer that the celebrated fixed point conclusions may not be proved in a conventional way in a novel $b-$interval metric space which demonstrates the prominence of $b-$interval metric space over celebrated distance structures. We also include examples to demonstrate that a $b-$interval metric space is neither a $b-$metric space (see, Bhaktin [1], Czerwik [5]) nor a metric interval space [20]. Towards the end, we introduce notions of interval circle, near fixed interval circle, its equivalence class and establish the existence of near fixed interval circle and its equivalence $\mathcal{C}-$class to study the geometric properties of non-unique equivalence $\mathcal{C}-$classes of near fixed interval circles. These near fixed points and near fixed interval circles results promote further examinations and applications in metric fixed point theory, which has been utilized in solving numerous real-life problems.

2. Preliminaries

The addition and scalar multiplication on the set $\mathcal{U}$ of closed and bounded intervals in $\mathbb{R}$ is defined as:

$$[\mathfrak{p},\mathfrak{q}]\oplus[\mathfrak{r},\mathfrak{s}]=[\mathfrak{p}+\mathfrak{r},\mathfrak{q}+\mathfrak{s}],\text{ and}$$

$$\mathfrak{k}[\mathfrak{p},\mathfrak{q}]=\begin{cases}[\mathfrak{k}\mathfrak{p},\mathfrak{k}\mathfrak{q}], & \mathfrak{k}\geq 0\\ [\mathfrak{k}\mathfrak{q},\mathfrak{k}\mathfrak{p}], & \mathfrak{k}<0\end{cases},\ [\mathfrak{p},\mathfrak{q}],[\mathfrak{r},\mathfrak{s}]\in\mathcal{U}.$$

$[0,0]\in\mathcal{U}$ is zero element of $\mathcal{U}$. For any $[\mathfrak{p},\mathfrak{q}]\in\mathcal{U}$, $[\mathfrak{p},\mathfrak{q}]\ominus[\mathfrak{p},\mathfrak{q}]=[\mathfrak{p},\mathfrak{q}]\oplus[-\mathfrak{q},-\mathfrak{p}]=[\mathfrak{p}-\mathfrak{q},\mathfrak{q}-\mathfrak{p}]$, that is, $\mathcal{U}$ is not a vector space in a conventional way, under the operations of addition and scalar multiplication defined above, since the additive inverse of each of its non-degenerated closed interval may not exist.

Now, the null set is defined as:

$$\begin{aligned}\mathcal{N}&=\{[\mathfrak{p},\mathfrak{q}]\ominus[\mathfrak{p},\mathfrak{q}]:\ [\mathfrak{p},\mathfrak{q}]\in\mathcal{U}\}\\&=\{[-\mathfrak{a},\mathfrak{a}]:\ \mathfrak{a}\geq 0\}\\&=\{\mathfrak{a}[-1,1]:\ \mathfrak{a}\geq 0\},\end{aligned}$$

that is, $\mathcal{N}$ is generated by $[-1,1]$.

Remark 2.1. *[20]*

1. *In general,* $(\mathfrak{a}+\mathfrak{b})[\mathfrak{p},\mathfrak{q}]\neq\mathfrak{a}[\mathfrak{p},\mathfrak{q}]+\mathfrak{b}[\mathfrak{p},\mathfrak{q}]$.
2. *If* $\mathfrak{a},\mathfrak{b}\geq 0$, $(\mathfrak{a}+\mathfrak{b})[\mathfrak{p},\mathfrak{q}]=\mathfrak{a}[\mathfrak{p},\mathfrak{q}]+\mathfrak{b}[\mathfrak{p},\mathfrak{q}]$.
3. *If* $\mathfrak{a},\mathfrak{b}\leq 0$, $(\mathfrak{a}+\mathfrak{b})[\mathfrak{p},\mathfrak{q}]=\mathfrak{a}[\mathfrak{p},\mathfrak{q}]+\mathfrak{b}[\mathfrak{p},\mathfrak{q}]$, $\forall\mathfrak{a},\mathfrak{b}\in\mathbb{R}$.
4. $[\mathfrak{p},\mathfrak{q}]\overset{\mathcal{N}}{=}[\mathfrak{r},\mathfrak{s}]$ *if and only if there exist* $\mathfrak{n}_1,\mathfrak{n}_2\in\mathcal{N}$ *such that*

$$[\mathfrak{p},\mathfrak{q}]+\mathfrak{n}_1=[\mathfrak{r},\mathfrak{s}]+\mathfrak{n}_2.$$

Clearly, $[\mathfrak{p},\mathfrak{q}] = [\mathfrak{r},\mathfrak{s}] \Longrightarrow [\mathfrak{p},\mathfrak{q}] + \mathfrak{n}_1 = [\mathfrak{r},\mathfrak{s}] + \mathfrak{n}_2, \mathfrak{n}_1 = \mathfrak{n}_2 = [0,0] \Longrightarrow [\mathfrak{p},\mathfrak{q}] \overset{\mathcal{N}}{=} [\mathfrak{r},\mathfrak{s}]$. *However, the converse may not essentially be true. Exploiting the binary relation* $\overset{\mathcal{N}}{=}$, *for any* $[\mathfrak{p},\mathfrak{q}] \in \mathcal{U}$, *we define*

$$\big\langle[\mathfrak{p},\mathfrak{q}]\big\rangle = \{[\mathfrak{r},\mathfrak{s}] \in \mathcal{U} : [\mathfrak{p},\mathfrak{q}] \overset{\mathcal{N}}{=} [\mathfrak{r},\mathfrak{s}]\}. \tag{2.1}$$

The family of all classes $\big\langle[\mathfrak{p},\mathfrak{q}]\big\rangle$ *for* $[\mathfrak{p},\mathfrak{q}] \in \mathcal{U}$ *is symbolized by* $\big\langle\mathcal{U}\big\rangle$.

The binary relation $\overset{\mathcal{N}}{=}$ is an equivalence relation [20]. Noticeably, class (2.1) constitutes the equivalence class and the family $\big\langle\mathcal{U}\big\rangle$ of all the classes (2.1) is said to be the quotient set of $\mathcal{U}$. It is significant to mention that a quotient set $\big\langle\mathcal{U}\big\rangle$ is also not a conventional vector space. Moreover, $[\mathfrak{r},\mathfrak{s}] \in \big\langle[\mathfrak{p},\mathfrak{q}]\big\rangle \Longrightarrow \big\langle[\mathfrak{p},\mathfrak{q}]\big\rangle = \big\langle[\mathfrak{r},\mathfrak{s}]\big\rangle$. Consequently, the family of equivalence classes constitutes a partition of the entire collection of closed and bounded intervals $\mathcal{U}$ in $\mathbb{R}$.

Definition 2.2. *[20] Let* $\mathcal{U}$ *be the set of closed and bounded intervals and* $\mathcal{N}$ *be the null set. A metric interval space is a pair* $(\mathcal{U}, d)$, *on a non-empty set* $\mathcal{U}$ *if and only if a map* $d : \mathcal{U} \times \mathcal{U} \longrightarrow \mathbb{R}^+$ *satisfies the subsequent conditions:*

1. $d([\mathfrak{p},\mathfrak{q}],[\mathfrak{r},\mathfrak{s}]) = 0$ *if and only if* $[\mathfrak{p},\mathfrak{q}] \overset{\mathcal{N}}{=} [\mathfrak{r},\mathfrak{s}]$;
2. $d([\mathfrak{p},\mathfrak{q}],[\mathfrak{r},\mathfrak{s}]) = d([\mathfrak{r},\mathfrak{s}],[\mathfrak{p},\mathfrak{q}])$;
3. $d([\mathfrak{p},\mathfrak{q}],[\mathfrak{r},\mathfrak{s}]) \leq d([\mathfrak{p},\mathfrak{q}],[\mathfrak{t},\mathfrak{u}]) + d([\mathfrak{t},\mathfrak{u}],[\mathfrak{r},\mathfrak{s}])$, $[\mathfrak{p},\mathfrak{q}],[\mathfrak{r},\mathfrak{s}],[\mathfrak{t},\mathfrak{u}] \in \mathcal{U}$.

Remark 2.3. *In this paper, we slightly modify the name of this notion and call it interval metric space instead of metric interval space in the next section.*

Definition 2.4. *[20]* $d : \mathcal{U} \times \mathcal{U} \longrightarrow \mathbb{R}^+$ *is said to satisfy null equalities, if for* $\mathfrak{n}_1, \mathfrak{n}_2 \in \mathcal{N}$ *and* $[\mathfrak{p},\mathfrak{q}],[\mathfrak{r},\mathfrak{s}] \in \mathcal{U}$, *the subsequent conditions holds:*

1. $d([\mathfrak{p},\mathfrak{q}] \oplus \mathfrak{n}_1, [\mathfrak{r},\mathfrak{s}] + \mathfrak{n}_2) = d([\mathfrak{p},\mathfrak{q}],[\mathfrak{r},\mathfrak{s}])$;
2. $d([\mathfrak{p},\mathfrak{q}] \oplus \mathfrak{n}_1, [\mathfrak{r},\mathfrak{s}]) = d([\mathfrak{p},\mathfrak{q}],[\mathfrak{r},\mathfrak{s}])$;
3. $d([\mathfrak{p},\mathfrak{q}], [\mathfrak{r},\mathfrak{s}] \oplus \mathfrak{n}_2) = d([\mathfrak{p},\mathfrak{q}],[\mathfrak{r},\mathfrak{s}])$.

Definition 2.5. *[20] Let* $\mathcal{M} : \mathcal{U} \to \mathcal{U}$ *be a function. A point* $[\mathfrak{p},\mathfrak{q}] \in \mathcal{U}$ *is known as a near fixed point of* $\mathcal{U}$ *if and only if* $\mathcal{M}([\mathfrak{p},\mathfrak{q}]) \overset{\mathcal{N}}{=} [\mathfrak{p},\mathfrak{q}]$.

3. Main results

First, we present formally the novel distance structure as a $b-$interval metric space.

Definition 3.1. *Let* $\mathcal{U}$ *be the set of closed and bounded intervals and* $\mathcal{N}$ *be the null set. A* $b-$*interval metric on a nonempty set* $\mathcal{U}$ *with* $s \geq 1$, *is a map* $d_b : \mathcal{U} \times \mathcal{U} \longrightarrow \mathbb{R}^+$ *satisfying:*

(d_{b1}) $d_b([\mathfrak{p},\mathfrak{q}],[\mathfrak{r},\mathfrak{s}]) = 0$ *if and only if* $[\mathfrak{p},\mathfrak{q}] \overset{\mathcal{N}}{=} [\mathfrak{r},\mathfrak{s}]$;
(d_{b2}) $d_b([\mathfrak{p},\mathfrak{q}],[\mathfrak{r},\mathfrak{s}]) = d_b([\mathfrak{r},\mathfrak{s}],[\mathfrak{p},\mathfrak{q}])$;
(d_{b3}) $d_b([\mathfrak{p},\mathfrak{q}],[\mathfrak{r},\mathfrak{s}]) \leq \boldsymbol{s}\big[d_b([\mathfrak{p},\mathfrak{q}],[\mathfrak{t},\mathfrak{u}]) + d_b([\mathfrak{t},\mathfrak{u}],[\mathfrak{r},\mathfrak{s}])\big]$, $[\mathfrak{p},\mathfrak{q}]$, $[\mathfrak{r},\mathfrak{s}]$, $[\mathfrak{t},\mathfrak{u}] \in \mathcal{U}$.

A pair $(\mathcal{U}, d_b)$ *is known as a* $b-$*interval metric space.*
A $b-$*interval metric space reduces to an interval metric space [20], for* $\boldsymbol{s} = 1$.

Following, Wu [20], we introduce null equalities in a $b-$interval metric space.

Definition 3.2. *$d_b : \mathcal{U} \times \mathcal{U} \longrightarrow \mathbb{R}^+$ satisfies $b-$null equalities, if for $\mathfrak{n}_1, \mathfrak{n}_2 \in \mathcal{N}$, the null set, $\boldsymbol{s} \geq 1$, and $[\mathfrak{p}, \mathfrak{q}], [\mathfrak{r}, \mathfrak{s}] \in \mathcal{U}$, the subsequent conditions holds:*

1. $d_b([\mathfrak{p}, \mathfrak{q}] \oplus \mathfrak{n}_1, [\mathfrak{r}, \mathfrak{s}] + \mathfrak{n}_2) = d_b([\mathfrak{p}, \mathfrak{q}], [\mathfrak{r}, \mathfrak{s}])$;
2. $d_b([\mathfrak{p}, \mathfrak{q}] \oplus \mathfrak{n}_1, [\mathfrak{r}, \mathfrak{s}]) = d_b([\mathfrak{p}, \mathfrak{q}], [\mathfrak{r}, \mathfrak{s}])$;
3. $d_b([\mathfrak{p}, \mathfrak{q}], [\mathfrak{r}, \mathfrak{s}] \oplus \mathfrak{n}_2) = d_b([\mathfrak{p}, \mathfrak{q}], [\mathfrak{r}, \mathfrak{s}])$.

Example 3.3. *Let $\mathcal{U}$ be the set of closed and bounded intervals and $\mathcal{N}$ be the null set. Let d_b : $\mathcal{U} \times \mathcal{U} \rightarrow \mathbb{R}^+$ be defined as:*

$$d_b([\mathfrak{p}, \mathfrak{q}], [\mathfrak{r}, \mathfrak{s}]) = (\mathfrak{p} + \mathfrak{q} - \mathfrak{r} - \mathfrak{s})^2. \tag{3.1}$$

We assert that $(\mathcal{U}, d_b)$ is a $b-$interval metric space and $\boldsymbol{s} = 2$.

(d_{b1}) Let $[\mathfrak{p}, \mathfrak{q}], [\mathfrak{r}, \mathfrak{s}] \in \mathcal{U}$, $\mathfrak{p} \leq \mathfrak{q}$, $\mathfrak{r} \leq \mathfrak{s}$. Now,

$$\begin{aligned}
&d_b([\mathfrak{p}, \mathfrak{q}], [\mathfrak{r}, \mathfrak{s}]) = 0,\\
&\Longrightarrow (\mathfrak{p} + \mathfrak{q} - \mathfrak{r} - \mathfrak{s})^2 = 0,\\
&\Longrightarrow \mathfrak{p} + \mathfrak{q} = \mathfrak{r} + \mathfrak{s}, \text{ which is possible if and only if } \mathfrak{q} \geq \mathfrak{r},\\
&\text{that is, } \mathfrak{p} + \mathfrak{r} - \mathfrak{s} = 2\mathfrak{r} - \mathfrak{q}.
\end{aligned}$$

Since, $\mathfrak{p} \leq \mathfrak{q}, \mathfrak{r} \leq \mathfrak{s}$, and $\mathfrak{q} \geq \mathfrak{r}$, $\mathfrak{p} + \mathfrak{r} - \mathfrak{s} \leq \mathfrak{q} + \mathfrak{s} - \mathfrak{r}$ and $2\mathfrak{r} - \mathfrak{q} \leq \mathfrak{q} + \mathfrak{s} - \mathfrak{r}$, we have two identical intervals $[\mathfrak{p} + \mathfrak{r} - \mathfrak{s}, \mathfrak{q} + \mathfrak{s} - \mathfrak{r}]$ and $[2\mathfrak{r} - \mathfrak{q}, \mathfrak{q} + \mathfrak{s} - \mathfrak{r}]$. These intervals may be written as $[\mathfrak{p} + \mathfrak{r} - \mathfrak{s}, \mathfrak{q} + \mathfrak{s} - \mathfrak{r}] = [\mathfrak{p}, \mathfrak{q}] \oplus [\mathfrak{r} - \mathfrak{s}, \mathfrak{s} - \mathfrak{r}]$ and $[2\mathfrak{r} - \mathfrak{q}, \mathfrak{q} + \mathfrak{s} - \mathfrak{r}] = [\mathfrak{r}, \mathfrak{s}] \oplus [\mathfrak{r} - \mathfrak{q}, \mathfrak{q} - \mathfrak{r}]$.
Suppose, $\mathfrak{n}_1 = [\mathfrak{r} - \mathfrak{s}, \mathfrak{s} - \mathfrak{r}]$ and $\mathfrak{n}_2 = [\mathfrak{r} - \mathfrak{q}, \mathfrak{q} - \mathfrak{r}]$, $\mathfrak{n}_1, \mathfrak{n}_2 \in \mathcal{N}$.
Now, we have $[\mathfrak{p}, \mathfrak{q}] \oplus \mathfrak{n}_1 = [\mathfrak{r}, \mathfrak{s}] \oplus \mathfrak{n}_2$.
Hence, $[\mathfrak{p}, \mathfrak{q}] \overset{\mathcal{N}}{=} [r, s]$.
Conversely, suppose that $[\mathfrak{p}, \mathfrak{q}] \overset{\mathcal{N}}{=} [r, s]$, then $[\mathfrak{p}, \mathfrak{q}] \oplus \mathfrak{n}_1 = [\mathfrak{r}, \mathfrak{s}] \oplus \mathfrak{n}_2$, $\mathfrak{n}_1, \mathfrak{n}_2 \in \mathcal{N}$, where, $\mathfrak{n}_1 = [-(\mathfrak{s} - \mathfrak{r}), \mathfrak{s} - \mathfrak{r}]$ and $\mathfrak{n}_2 = [-(\mathfrak{q} - \mathfrak{r}), \mathfrak{q} - \mathfrak{r}]$.
It is easy to verify that, $d_b([\mathfrak{p}, \mathfrak{q}] \oplus \mathfrak{n}_1, [\mathfrak{r}, \mathfrak{s}] \oplus \mathfrak{n}_2) = 0$.

(d_{b2}) Since,

$$\begin{aligned}
d_b([\mathfrak{p}, \mathfrak{q}], [\mathfrak{r}, \mathfrak{s}]) &= (\mathfrak{p} + \mathfrak{q} - \mathfrak{r} - \mathfrak{s})^2\\
&= (\mathfrak{r} + \mathfrak{s} - \mathfrak{p} - \mathfrak{q})^2\\
&= d_b([\mathfrak{r}, \mathfrak{s}], [\mathfrak{p}, \mathfrak{q}]).
\end{aligned}$$

(d_{b3}) For $[\mathfrak{p}, \mathfrak{q}], [\mathfrak{r}, \mathfrak{s}], [\mathfrak{t}, \mathfrak{u}] \in \mathcal{U}$,

$$\begin{aligned}
d_b([\mathfrak{p}, \mathfrak{q}], [\mathfrak{r}, \mathfrak{s}]) &= (\mathfrak{p} + \mathfrak{q} - \mathfrak{r} - \mathfrak{s})^2\\
&= (\mathfrak{p} + \mathfrak{q} - \mathfrak{t} - \mathfrak{u} + \mathfrak{t} + \mathfrak{u} - \mathfrak{r} - \mathfrak{s})^2\\
&\leq 2\big[(\mathfrak{p} + \mathfrak{q} - \mathfrak{t} - \mathfrak{u})^2 + (\mathfrak{t} + \mathfrak{u} - \mathfrak{r} - \mathfrak{s})^2\big]\\
&= 2\big[d_b([\mathfrak{p}, \mathfrak{q}], [\mathfrak{t}, \mathfrak{u}]) + d_b([\mathfrak{t}, \mathfrak{u}], [\mathfrak{r}, \mathfrak{s}])\big].
\end{aligned}$$

Hence, $(\mathcal{U}, d_b)$ is a $b-$interval metric space but d_b is neither a $b-$metric nor an interval metric on $\mathcal{U}$.

Example 3.4. *Let $\mathcal{U}$ be the set of closed and bounded intervals and $\mathcal{N}$ be the null set.*
Let $d_b([\mathfrak{p}, \mathfrak{q}], [\mathfrak{r}, \mathfrak{s}]) = d([\mathfrak{p}, \mathfrak{q}], [\mathfrak{r}, \mathfrak{s}])^p$, $p > 1$ is a real number. One may verify that d_b is a $b-$interval metric and $s = 2^{p-1}$, but d_b is neither an interval metric nor a $b-$metric on $\mathcal{U}$.

Remark 3.5. *It is fascinating to note that for some $\mathfrak{n}_1 = [-\mathfrak{l}, \mathfrak{l}]$ and $\mathfrak{n}_2 = [-\mathfrak{m}, \mathfrak{m}]$, $\mathfrak{l}, \mathfrak{m} \in \mathbb{R}$, $b-$interval metrics d_b defined in Examples 3.3 and 3.4 satisfy $b-$null equalities.*

Example 3.6. *Let* $\mathcal{U} = \{[-1,0],[0,1],[-1,1]\}$, $\mathcal{N} = \{[0,0]\}$ *and*
$d_b([-1,0],[-1,1]) = d_b([-1,1],[-1,0]) = m, m > 2,$
$d_b([-1,0],[0,1]) = d_b([0,1],[-1,1]) = 1,$ *and*
$d_b([-1,0],[-1,0]) = d_b([0,1],[0,1]) = d_b([-1,1],[-1,1]) = 0,$
then one may verify that $(\mathcal{U}, d_b)$ *is a* $b-$*interval metric space but* d_b *is neither an interval metric nor a* $b-$*metric on* $\mathcal{U}$. *For* $m = 2$, $(\mathcal{U}, d_b)$ *is an interval metric space.*
Since, $d_b([-1,0]\oplus[-1,1],[0,1]) = d_b([-2,1],[0,1])$ *is not defined, that is, a* $b-$*interval metric* d_b *does not satisfy null equalities.*

To discuss the topology corresponding to $b-$interval metric with $\mathbf{s} \geq 1$ and the null set $\mathcal{N}$, the open ball centred at $[\mathfrak{p}_\mathfrak{o}, \mathfrak{q}_\mathfrak{o}]$ and radius $\epsilon \in (0,\infty)$ is defined as:
$\mathcal{O}([\mathfrak{p}_\mathfrak{o}, \mathfrak{q}_\mathfrak{o}],\epsilon) = \{[\mathfrak{p},\mathfrak{q}] \in \mathcal{U} : d_b([\mathfrak{p}_\mathfrak{o}, \mathfrak{q}_\mathfrak{o}],[\mathfrak{p},\mathfrak{q}]) < \frac{\epsilon}{\mathbf{s}}\}.$
The closed ball centred at $\mathfrak{u}$ and radius $\epsilon \in (0,\infty)$ is defined as:
$C([\mathfrak{p}_\mathfrak{o}, \mathfrak{q}_\mathfrak{o}],\epsilon) = \{[\mathfrak{p},\mathfrak{q}] \in \mathcal{U} : d_b([\mathfrak{p}_\mathfrak{o}, \mathfrak{q}_\mathfrak{o}],[\mathfrak{p},\mathfrak{q}]) \leq \frac{\epsilon}{\mathbf{s}}\}.$

Lemma 3.7. *Let* $(\mathcal{U}, d_b)$ *be a* $b-$*interval metric space,* $\mathcal{N}$ *be the null set, and* $\mathbf{s} \geq 1$. *Then the collection of all open balls,*
$\mathcal{O}([\mathfrak{p}_\mathfrak{o}, \mathfrak{q}_\mathfrak{o}],\epsilon) = \{[\mathfrak{p},\mathfrak{q}] \in \mathcal{U} : d_b([\mathfrak{p}_\mathfrak{o}, \mathfrak{q}_\mathfrak{o}],[\mathfrak{p},\mathfrak{q}]) < \frac{\epsilon}{s}\}$ *forms a basis of* $\mathcal{U}$.

Proof . Let $[\mathfrak{r}_\mathfrak{o}, \mathfrak{s}_\mathfrak{o}] \in \mathcal{O}([\mathfrak{p}_\mathfrak{o}, \mathfrak{q}_\mathfrak{o}],\epsilon)$, then $d_b([\mathfrak{p}_\mathfrak{o}, \mathfrak{q}_\mathfrak{o}],[\mathfrak{r}_\mathfrak{o}, \mathfrak{s}_\mathfrak{o}]) < \frac{\epsilon}{\mathbf{s}}$. Suppose there exists $\epsilon_1 > 0$ such that $d_b([\mathfrak{p}_\mathfrak{o}, \mathfrak{q}_\mathfrak{o}],[\mathfrak{r}_\mathfrak{o}, \mathfrak{s}_\mathfrak{o}]) + \frac{\epsilon_1}{\mathbf{s}} = \frac{\epsilon}{\mathbf{s}}$.
Again, let $[\mathfrak{r}_1, \mathfrak{s}_1] \in \mathcal{O}([\mathfrak{r}_\mathfrak{o}, \mathfrak{s}_\mathfrak{o}],\epsilon_1)$, so $d_b([\mathfrak{r}_\mathfrak{o}, \mathfrak{s}_\mathfrak{o}],[\mathfrak{r}_1, \mathfrak{s}_1]) < \frac{\epsilon_1}{\mathbf{s}}$. Again, suppose there exists $\epsilon_2 > 0$ such that $d_b([\mathfrak{r}_\mathfrak{o}, \mathfrak{s}_\mathfrak{o}],[\mathfrak{r}_1, \mathfrak{s}_1]) + \frac{\epsilon_2}{\mathbf{s}} = \frac{\epsilon_1}{\mathbf{s}}$.
Now,

$$\begin{aligned} d_b([\mathfrak{p}_\mathfrak{o}, \mathfrak{q}_\mathfrak{o}],[\mathfrak{r}_1, \mathfrak{s}_1]) &\leq \mathbf{s}\big[d_b([\mathfrak{p}_\mathfrak{o}, \mathfrak{q}_\mathfrak{o}],[\mathfrak{r}_\mathfrak{o}, \mathfrak{s}_\mathfrak{o}]) + d_b([\mathfrak{r}_\mathfrak{o}, \mathfrak{s}_\mathfrak{o}],[\mathfrak{r}_1, \mathfrak{s}_1])\big] \\ &= \mathbf{s}\Big[\frac{\epsilon}{\mathbf{s}} - \frac{\epsilon_1}{\mathbf{s}} + \frac{\epsilon_1}{\mathbf{s}} - \frac{\epsilon_2}{\mathbf{s}}\Big] \\ &= \epsilon - \epsilon_2. \end{aligned}$$

Hence, $\mathcal{O}([\mathfrak{r}_\mathfrak{o}, \mathfrak{s}_\mathfrak{o}],\epsilon_1) \subseteq \mathcal{O}([\mathfrak{p}_\mathfrak{o}, \mathfrak{q}_\mathfrak{o}],\epsilon)$. □

Theorem 3.8. *If* $(\mathcal{U}, d_b)$ *is a* $b-$*interval metric space,* $\mathcal{N}$ *is the null set,* $\mathbf{s} \geq 1$, τ_b *is a topology generated by the open ball* $\mathcal{O}([\mathfrak{p}_\mathfrak{o}, \mathfrak{q}_\mathfrak{o}],\epsilon)$, *then* $(\mathcal{U}, \tau_b)$ *is a* $\mathcal{T}_0-$*space.*

Proof . Let $(\mathcal{U}, d_b)$ be a $b-$interval metric space and $[\mathfrak{p},\mathfrak{q}]$, $[\mathfrak{r},\mathfrak{s}] \in \mathcal{U}$ are two distinct points.
Then $d_b([\mathfrak{p},\mathfrak{q}],[\mathfrak{r},\mathfrak{s}]) > 0$. Firstly, assume that for $\mathfrak{s} > 1$, if we chose $\epsilon > 0$ so that $d_b([\mathfrak{p},\mathfrak{q}],[\mathfrak{r},\mathfrak{s}]) = \frac{\epsilon}{\mathbf{s}}$, then $[\mathfrak{r},\mathfrak{s}] \notin \mathcal{O}([\mathfrak{p},\mathfrak{q}],\epsilon)$.
Next, assume that for $\mathfrak{s} > 1$, if we chose $\epsilon_1 > 0$ such that $d_b([\mathfrak{p},\mathfrak{q}],[\mathfrak{r},\mathfrak{s}]) = \frac{\epsilon}{\mathbf{s}}$, then $[\mathfrak{p},\mathfrak{q}] \in \mathcal{O}([\mathfrak{p},\mathfrak{q}],\epsilon_1)$ and $[\mathfrak{r},\mathfrak{s}] \notin \mathcal{O}([\mathfrak{p},\mathfrak{q}],\epsilon_1)$.
So proceeding as above, one may easily find an open ball so that $[\mathfrak{p},\mathfrak{q}] \in \mathcal{O}([\mathfrak{p},\mathfrak{q}],\epsilon_1)$ and $[\mathfrak{r},\mathfrak{s}] \notin \mathcal{O}([\mathfrak{p},\mathfrak{q}],\epsilon_1)$,
that is, for two distinct points $[\mathfrak{p},\mathfrak{q}]$, $[\mathfrak{r},\mathfrak{s}] \in \mathcal{U}$, there exists a ball including the point $[\mathfrak{p},\mathfrak{q}]$ but not including the other point $[\mathfrak{r},\mathfrak{s}]$. Thus, $(\mathcal{U}, d_b)$ is a $\mathcal{T}_0-$space. □

Now, we discuss $b-$convergence, completeness, and $b-$Cauchy sequence in the $b-$interval metric space.

Definition 3.9. *Let* $(\mathcal{U}, d_b)$ *be a* $b-$*interval metric space,* $\mathbf{s} \geq 1$ *and* $\mathcal{N}$ *be the null set. The sequence* $\{[\mathfrak{p}_\mathfrak{n}, \mathfrak{q}_\mathfrak{n}]\}_{n=1}^{\infty}$ *in* $\mathcal{U}$ *is said to be* $b-$*convergent if and only if* $\lim_{n\to\infty} d_b([\mathfrak{p}_\mathfrak{n}, \mathfrak{q}_\mathfrak{n}],[\mathfrak{p},\mathfrak{q}]) = 0$, $[\mathfrak{p},\mathfrak{q}] \in \mathcal{U}$. *The element* $[\mathfrak{p},\mathfrak{q}]$ *is known as a* $b-$*limit of the sequence* $\{[\mathfrak{p}_\mathfrak{n}, \mathfrak{q}_\mathfrak{n}]\}_{n=1}^{\infty}$.

If there exists $[\mathfrak{p},\mathfrak{q}],\ [\mathfrak{r},\mathfrak{s}]\in\mathcal{U}$ so that $\lim_{n\to\infty} d_b([\mathfrak{p}_\mathfrak{n},\mathfrak{q}_\mathfrak{n}],[\mathfrak{p},\mathfrak{q}]) = \lim_{n\to\infty} d_b([\mathfrak{p}_\mathfrak{n},\mathfrak{q}_\mathfrak{n}],[\mathfrak{r},\mathfrak{s}]) = 0$, then

$$d([\mathfrak{p},\mathfrak{q}],[\mathfrak{r},\mathfrak{s}]) \le s\Big[d([\mathfrak{p},\mathfrak{q}],[\mathfrak{p}_\mathfrak{n},\mathfrak{q}_\mathfrak{n}]) + d([\mathfrak{p}_\mathfrak{n},\mathfrak{q}_\mathfrak{n}],[\mathfrak{r},\mathfrak{s}])\Big] \to 0,\ \text{as } \mathfrak{n}\to\infty, \tag{3.2}$$

so, by Definition 3.1, $[\mathfrak{p},\mathfrak{q}] \overset{\mathcal{N}}{=} [\mathfrak{r},\mathfrak{s}]$, that is, $[\mathfrak{r},\mathfrak{s}]\in\big\langle[\mathfrak{p},\mathfrak{q}]\big\rangle$.

Proposition 3.10. *Let $\{[\mathfrak{p}_\mathfrak{n},\mathfrak{q}_\mathfrak{n}]\}_{\mathfrak{n}=1}^{\infty}$ be a sequence in a $b-$interval metric space $(\mathcal{U},d_b)$, $\boldsymbol{s}\ge 1$, and $\mathcal{N}$ be the null set, satisfying $\lim_{n\to\infty} d_b([\mathfrak{p}_\mathfrak{n},\mathfrak{q}_\mathfrak{n}],[\mathfrak{p},\mathfrak{q}]) = 0$. Then,*

$$\lim_{n\to\infty} d_b([\mathfrak{p}_\mathfrak{n},\mathfrak{q}_\mathfrak{n}],[\mathfrak{r},\mathfrak{s}]) = 0, \quad \textit{for any}[\mathfrak{r},\mathfrak{s}]\in\big\langle[\mathfrak{p},\mathfrak{q}]\big\rangle.$$

Proof . For $[\mathfrak{r},\mathfrak{s}]\in\big\langle[\mathfrak{p},\mathfrak{q}]\big\rangle$, we have $[\mathfrak{p},\mathfrak{q}]\oplus\mathfrak{n}_1 = [\mathfrak{r},\mathfrak{s}]\oplus\mathfrak{n}_2$, for some $\mathfrak{n}_1,\mathfrak{n}_2\in\mathcal{N}$. Exploiting the null equality, we attain

$$0\le d([\mathfrak{p}_\mathfrak{n},\mathfrak{q}_\mathfrak{n}],[\mathfrak{r},\mathfrak{s}]) = d([\mathfrak{p}_\mathfrak{n},\mathfrak{q}_\mathfrak{n}],[\mathfrak{r},\mathfrak{s}]\oplus\mathfrak{n}_2) = d([\mathfrak{p}_\mathfrak{n},\mathfrak{q}_\mathfrak{n}],[\mathfrak{p},\mathfrak{q}]\oplus\mathfrak{n}_1) = d([\mathfrak{p}_\mathfrak{n},\mathfrak{q}_\mathfrak{n}],[\mathfrak{p},\mathfrak{q}]),$$

which tends to 0 as $\mathfrak{n}\to\infty$. Hence, $\lim_{n\to\infty} d([\mathfrak{p}_\mathfrak{n},\mathfrak{q}_\mathfrak{n}],[\mathfrak{r},\mathfrak{s}]) = 0$. □

Definition 3.11. *Let $(\mathcal{U},d_b)$ be a $b-$interval metric space, $\mathcal{N}$ be the null set and $\boldsymbol{s}\ge 1$. If $\{[\mathfrak{p}_\mathfrak{n},\mathfrak{q}_\mathfrak{n}]\}_{\mathfrak{n}=1}^{\infty}$ is a sequence in $\mathcal{U}$ satisfying $\lim_{n\to\infty} d_b([\mathfrak{p}_\mathfrak{n},\mathfrak{q}_\mathfrak{n}],\big\langle[\mathfrak{p},\mathfrak{q}]\big\rangle) = 0,\ [\mathfrak{p},\mathfrak{q}]\in\mathcal{U}$ or $\lim_{n\to\infty}[\mathfrak{p}_\mathfrak{n},\mathfrak{q}_\mathfrak{n}] = \big\langle[\mathfrak{p},\mathfrak{q}]\big\rangle$, then the equivalence class $\big\langle[p,q]\big\rangle$ is known as a $b-$class limit of the sequence $\{[\mathfrak{p}_\mathfrak{n},\mathfrak{q}_\mathfrak{n}]_{\mathfrak{n}=1}^{\infty}\}$.*

Proposition 3.12. *The $b-$class limit in the $b-$interval metric space $(\mathcal{U},d_b)$, $\boldsymbol{s}\ge 1$, and the null set $\mathcal{N}$ is unique.*

Proof . Let, if possible, the sequence $\{[\mathfrak{p}_\mathfrak{n},\mathfrak{q}_\mathfrak{n}]_{\mathfrak{n}=1}^{\infty}\}$ be $b-$convergent to two distinct $b-$class limits $\big\langle[\mathfrak{p},\mathfrak{q}]\big\rangle$ and $\big\langle[\mathfrak{r},\mathfrak{s}]\big\rangle$. Consequently, $lim_{n\to\infty}d([\mathfrak{p}_\mathfrak{n},\mathfrak{q}_\mathfrak{n}],\big\langle[\mathfrak{p},\mathfrak{q}]\big\rangle) = 0$ and $\lim_{n\to\infty} d([\mathfrak{p}_\mathfrak{n},\mathfrak{q}_\mathfrak{n}],\big\langle[\mathfrak{r},\mathfrak{s}]\big\rangle) = 0$, that is, $d(\big\langle[\mathfrak{p},\mathfrak{q}]\big\rangle,\big\langle[\mathfrak{r},\mathfrak{s}]\big\rangle) = 0$ by referring to (3.2). Therefore, we obtain $[\mathfrak{r},\mathfrak{s}]\in\big\langle[\mathfrak{p},\mathfrak{q}]\big\rangle$, that is, $\big\langle[\mathfrak{r},\mathfrak{s}]\big\rangle = \big\langle[\mathfrak{p},\mathfrak{q}]\big\rangle$. This completes the proof. □

Definition 3.13. *Let $(\mathcal{U},d_b)$ be a $b-$interval metric space, $\mathcal{N}$ be the null set, and $\boldsymbol{s}\ge 1$.*

1. *A sequence $\{[\mathfrak{p}_\mathfrak{n},\mathfrak{q}_\mathfrak{n}]\}_{n=1}^{\infty}$ in a $b-$interval metric space $(\mathcal{U},d_b)$ is known as a $b-$Cauchy sequence if for given $\epsilon>0$, there exists numbers $\mathfrak{n},\mathfrak{m},N\in\mathbb{N}$ so that $d_b([\mathfrak{p}_\mathfrak{n},\mathfrak{q}_\mathfrak{n}],\ [\mathfrak{p}_\mathfrak{m},\mathfrak{q}_\mathfrak{m}])<\epsilon$, $\mathfrak{n}>N$, and $\mathfrak{m}>N$.*
   Equivalently, $\{[\mathfrak{p}_\mathfrak{n},\mathfrak{q}_\mathfrak{n}]\}_{n=1}^{\infty}$ in a topological $b-$interval space $(\mathcal{U},\mathcal{T}_b)$ is known as a $b-$Cauchy sequence if and only if, for given $\epsilon>0$, there exists numbers $\mathfrak{n},\mathfrak{m},N\in\mathbb{N}$ so that $[\mathfrak{p}_\mathfrak{n},\mathfrak{q}_\mathfrak{n}],\ [\mathfrak{p}_\mathfrak{m},\mathfrak{q}_\mathfrak{m}]\in\mathcal{O}([\mathfrak{p}_\mathfrak{o},\mathfrak{q}_\mathfrak{o}],\epsilon)$, $\mathfrak{n}>N$ and $\mathfrak{m}>N$.
2. *Let $\mathcal{V}\subseteq\mathcal{U}$, then $(\mathcal{V},d_b)$ is a complete subspace of $(\mathcal{U},d_b)$ if and only if each $b-$Cauchy sequence in $(\mathcal{V},d_b)$ is $b-$convergent in $(\mathcal{V},d_b)$.*

Proposition 3.14. *Every $b-$convergent sequence in a $b-$interval metric space $(\mathcal{U},d_b)$, $\boldsymbol{s}\ge 1$, and the null set $\mathcal{N}$ is a $b-$Cauchy sequence.*

Proof . If $\{[\mathfrak{p}_\mathfrak{n},\mathfrak{q}_\mathfrak{n}]\}_{n=1}^{\infty}$ is a $b-$convergent sequence, then, for any $\epsilon>0$, $d_b([\mathfrak{p}_\mathfrak{n},\mathfrak{q}_\mathfrak{n}],[\mathfrak{p},\mathfrak{q}])<\frac{\epsilon}{2\mathbf{s}}$, for $n>N$. Therefore for $\mathfrak{m},\mathfrak{n}>N$,

$$\begin{aligned} d_b([\mathfrak{p}_\mathfrak{n},\mathfrak{q}_\mathfrak{n}],[\mathfrak{p}_\mathfrak{m},\mathfrak{q}_\mathfrak{m}]) &\le \mathbf{s}\big[d_b([\mathfrak{p}_\mathfrak{n},\mathfrak{q}_\mathfrak{n}],[\mathfrak{p},\mathfrak{q}]) + d_b([\mathfrak{p},\mathfrak{q}],[\mathfrak{p}_\mathfrak{m},\mathfrak{q}_\mathfrak{m}])\big] \\ &< \mathbf{s}[\frac{\epsilon}{2\mathbf{s}}+\frac{\epsilon}{2\mathbf{s}}] \\ &= \epsilon, \end{aligned}$$

that is, $\{[\mathfrak{p}_\mathfrak{n},\mathfrak{q}_\mathfrak{n}]\}_{n=1}^{\infty}$ is a $b-$Cauchy sequence. □

Following example demonstrate that the opposite of above result may not essentially be true, that is, every $b-$Cauchy sequence may not be a $b-$convergent in a $b-$interval metric space.

Example 3.15. *Let* $\mathcal{U} = \{[\mathfrak{p},\mathfrak{q}] : 0 < \mathfrak{p},\mathfrak{q} < 1\}$. *Define a* $b-$*interval metric,* $d_b : \mathcal{U}\times\mathcal{U} \to \mathbb{R}^+$ *as* $d_b([\mathfrak{p},\mathfrak{q}],[\mathfrak{r},\mathfrak{s}]) = |\mathfrak{p}+\mathfrak{q}-\mathfrak{r}-\mathfrak{s}|^3$, *with* $\boldsymbol{s} = 4$ *and null set* $\mathcal{N}$. *Define a sequence* $\{[\mathfrak{p}_\mathfrak{n},\mathfrak{q}_\mathfrak{n}]\} = \{[\frac{1}{n}, 1-\frac{1}{n}]\}$, *which is a* $b-$*Cauchy sequence but not a* $b-$*convergent sequence in a* $b-$*interval metric space.*

Next, we prove the first main result for a $b-$interval metric variant of Banach contraction [2] for determining near fixed points of the function $\mathcal{M}$ and its equivalence classes.

Theorem 3.16. *Let* $(\mathcal{U}, d_b)$ *be a complete* $b-$*interval metric space and* $\mathfrak{s} \geq 1$ *satisfying null equalities. Suppose a self map* $\mathcal{M} : \mathcal{U} \longrightarrow \mathcal{U}$ *satisfies*

$$d_b(\mathcal{M}[\mathfrak{p},\mathfrak{q}],\mathcal{M}[\mathfrak{r},\mathfrak{s}]) \leq \eta d_b([\mathfrak{p},\mathfrak{q}],[\mathfrak{r},\mathfrak{s}]), \quad \eta < \frac{1}{\boldsymbol{s}} \;\; \text{and} \;\; [\mathfrak{p},\mathfrak{q}],[\mathfrak{r},\mathfrak{s}] \in \mathcal{U}. \tag{3.3}$$

Then $\mathcal{M}$ *has a near fixed point* $[\mathfrak{p},\mathfrak{q}] \in \mathcal{U}$. *Further,* $\mathcal{M}$ *has a unique equivalence class of near fixed points* $\langle [p,q]\rangle$, *that is, if* $[\bar{\mathfrak{p}},\bar{\mathfrak{q}}]$ *is a near fixed point of* $\mathcal{M}$, *then* $[\bar{\mathfrak{p}},\bar{\mathfrak{q}}] \in \langle [\mathfrak{p},\mathfrak{q}]\rangle$ *or* $\langle [\mathfrak{p},\mathfrak{q}]\rangle = \langle [\bar{\mathfrak{p}},\bar{\mathfrak{q}}]\rangle$. *Equivalently, if* $[\mathfrak{p},\mathfrak{q}]$ *and* $[\bar{\mathfrak{p}},\bar{\mathfrak{q}}]$ *are the near fixed points of* $\mathcal{M}$, *then* $[\mathfrak{p},\mathfrak{q}] \overset{\mathcal{N}}{=} [\bar{\mathfrak{p}},\bar{\mathfrak{q}}]$.

Proof . Given an initial element $[\mathfrak{p}_\mathfrak{o},\mathfrak{q}_\mathfrak{o}] \in \mathcal{U}$, the iterative sequence $\{[\mathfrak{p}_\mathfrak{n},\mathfrak{q}_\mathfrak{n}]\}_{n=1}^{\infty}$, utilizing the function $\mathcal{M}$, is defined as follows:

$$[\mathfrak{p}_{\mathfrak{n}+1},\mathfrak{q}_{\mathfrak{n}+1}] = \mathcal{M}([\mathfrak{p}_\mathfrak{n},\mathfrak{q}_\mathfrak{n}]) = \mathcal{M}^{n+1}([\mathfrak{p}_\mathfrak{o},\mathfrak{q}_\mathfrak{o}]). \tag{3.4}$$

Now, we assert that $\{[\mathfrak{p}_\mathfrak{n},\mathfrak{q}_\mathfrak{n}]\}$ is a $b-$convergent sequence, converging to a near fixed point of $\mathcal{M}$ in a $b-$interval metric space. Utilizing, (3.3)

$$\begin{aligned}
d_b([\mathfrak{p}_{\mathfrak{n}+1},\mathfrak{q}_{\mathfrak{n}+1}],[\mathfrak{p}_\mathfrak{n},\mathfrak{q}_\mathfrak{n}]) &= d_b(\mathcal{M}[\mathfrak{p}_\mathfrak{n},\mathfrak{q}_\mathfrak{n}],\mathcal{M}[\mathfrak{p}_{\mathfrak{n}-1},\mathfrak{q}_{\mathfrak{n}-1}])\\
&\leq \eta d_b([\mathfrak{p}_\mathfrak{n},\mathfrak{q}_\mathfrak{n}],[\mathfrak{p}_{\mathfrak{n}-1},\mathfrak{q}_{\mathfrak{n}-1}])\\
&= \eta d_b(\mathcal{M}[\mathfrak{p}_{\mathfrak{n}-1},\mathfrak{q}_{\mathfrak{n}-1}],\mathcal{M}[\mathfrak{p}_{\mathfrak{n}-2},\mathfrak{q}_{\mathfrak{n}-2}])\\
&\leq \eta^2 d_b([\mathfrak{p}_{\mathfrak{n}-1},\mathfrak{q}_{\mathfrak{n}-1}],[\mathfrak{p}_{\mathfrak{n}-2},\mathfrak{q}_{\mathfrak{n}-2}])\\
&\quad\vdots\\
&\leq \eta^n d_b([\mathfrak{p}_1,\mathfrak{q}_1],[\mathfrak{p}_\mathfrak{o},\mathfrak{q}_\mathfrak{o}]).
\end{aligned}$$

Next, for $\mathfrak{m} > \mathfrak{n}$, we have

$$\begin{aligned}
d_b([\mathfrak{p}_\mathfrak{n},\mathfrak{q}_\mathfrak{n}],[\mathfrak{p}_\mathfrak{m},\mathfrak{q}_\mathfrak{m}]) &\leq \mathbf{s}\big[d_b([\mathfrak{p}_\mathfrak{n},\mathfrak{q}_\mathfrak{n}],[\mathfrak{p}_{\mathfrak{n}+1},\mathfrak{q}_{\mathfrak{n}+1}]) + d_b([\mathfrak{p}_{\mathfrak{n}+1},\mathfrak{q}_{\mathfrak{n}+1}],[\mathfrak{p}_\mathfrak{m},\mathfrak{q}_\mathfrak{m}])\big]\\
&\leq \mathbf{s} d_b([\mathfrak{p}_\mathfrak{n},\mathfrak{q}_\mathfrak{n}],[\mathfrak{p}_{\mathfrak{n}+1},\mathfrak{q}_{\mathfrak{n}+1}]) + \mathbf{s}^2\big[d_b([\mathfrak{p}_{\mathfrak{n}+1},\mathfrak{q}_{\mathfrak{n}+1}],[\mathfrak{p}_{\mathfrak{n}+2},\mathfrak{q}_{\mathfrak{n}+2}]) + d_b([\mathfrak{p}_{\mathfrak{n}+2},\mathfrak{q}_{\mathfrak{n}+2}],[\mathfrak{p}_\mathfrak{m},\mathfrak{q}_\mathfrak{m}])\big]\\
&\quad\vdots\\
&\leq \mathbf{s} d_b([\mathfrak{p}_\mathfrak{n},\mathfrak{q}_\mathfrak{n}],[\mathfrak{p}_{\mathfrak{n}+1},\mathfrak{q}_{\mathfrak{n}+1}]) + \mathbf{s}^2 d_b([\mathfrak{p}_{\mathfrak{n}+1},\mathfrak{q}_{\mathfrak{n}+1}],[\mathfrak{p}_{\mathfrak{n}+2},\mathfrak{q}_{\mathfrak{n}+2}])\\
&+ \mathbf{s}^3 d_b([\mathfrak{p}_{\mathfrak{n}+2},\mathfrak{q}_{\mathfrak{n}+2}],[\mathfrak{p}_{\mathfrak{n}+3},\mathfrak{q}_{\mathfrak{n}+3}]) + \cdots + \mathbf{s}^{m-n} d_b([\mathfrak{p}_{\mathfrak{m}-1},\mathfrak{q}_{\mathfrak{m}-1}],[\mathfrak{p}_\mathfrak{m},\mathfrak{q}_\mathfrak{m}])\\
&\leq \mathbf{s}\eta^\mathfrak{n} d_b([\mathfrak{p}_1,\mathfrak{q}_1],[\mathfrak{p}_\mathfrak{o},\mathfrak{q}_\mathfrak{o}]) + \mathbf{s}^2\eta^{\mathfrak{n}+1} d_b([\mathfrak{p}_1,\mathfrak{q}_1],[\mathfrak{p}_\mathfrak{o},\mathfrak{q}_\mathfrak{o}]) + \dots\\
&\;+ \mathbf{s}^{m-n}\eta^{\mathfrak{m}-1} d_b([\mathfrak{p}_1,\mathfrak{q}_1],[\mathfrak{p}_\mathfrak{o},\mathfrak{q}_\mathfrak{o}])\\
&= \mathbf{s}\eta^n[1 + \mathbf{s}\eta + (\mathbf{s}\eta)^2 + \cdots + (\mathbf{s}\eta)^{m-n-1}] d_b([\mathfrak{p}_1,\mathfrak{q}_1],[\mathfrak{p}_\mathfrak{o},\mathfrak{q}_\mathfrak{o}])\\
&= \mathbf{s}\eta^n\Big(\frac{1-(\mathbf{s}\eta)^{\mathfrak{m}-\mathfrak{n}}}{1-\mathbf{s}\eta}\Big) d_b([\mathfrak{p}_1,\mathfrak{q}_1],[\mathfrak{p}_\mathfrak{o},\mathfrak{q}_\mathfrak{o}]) \to 0, \quad \text{as} \;\; \mathfrak{n} \to \infty.
\end{aligned}$$

Therefore, the sequence $\{[\mathfrak{p}_\mathfrak{n},\mathfrak{q}_\mathfrak{n}]\}_{\mathfrak{n}=1}^{\infty}$ is a $b-$Cauchy sequence in $\mathcal{U}$. As $(\mathcal{U}, d_b)$ is a complete $b-$interval metric space, we have $[\mathfrak{p},\mathfrak{q}] \in \mathcal{U}$ so that $d_b([\mathfrak{p}_\mathfrak{n},\mathfrak{q}_\mathfrak{n}],[\mathfrak{p},\mathfrak{q}]) \to 0$, that is, $[\mathfrak{p}_\mathfrak{n},\mathfrak{q}_\mathfrak{n}] \to [\mathfrak{p},\mathfrak{q}]$, that is,

$[\mathfrak{p},\mathfrak{q}] \in \langle[\mathfrak{p},\mathfrak{q}]\rangle$.
Now, we establish that any $[\bar{\mathfrak{p}},\bar{\mathfrak{q}}] \in \langle[\mathfrak{p},\mathfrak{q}]\rangle$ is a near fixed point of $\mathcal{M}$. Since, d_b satisfies null equalities, $[\bar{\mathfrak{p}},\bar{\mathfrak{q}}] \oplus \mathfrak{n}_1 = [\mathfrak{p},\mathfrak{q}] \oplus \mathfrak{n}_2$, for some $\mathfrak{n}_1, \mathfrak{n}_2 \in \mathcal{N}$.
Now,

$$\begin{aligned} d_b(\mathcal{M}[\bar{\mathfrak{p}},\bar{\mathfrak{q}}],[\bar{\mathfrak{p}},\bar{\mathfrak{q}}]) &= d_b(\mathcal{M}[\bar{\mathfrak{p}},\bar{\mathfrak{q}}],[\bar{\mathfrak{p}},\bar{\mathfrak{q}}]\oplus\mathfrak{n}_1) \\ &\le \mathbf{s}\big[d_b(\mathcal{M}[\bar{\mathfrak{p}},\bar{\mathfrak{q}}],[\mathfrak{p}_\mathfrak{n},\mathfrak{q}_\mathfrak{n}]) + d_b([\mathfrak{p}_\mathfrak{n},\mathfrak{q}_\mathfrak{n}],[\bar{\mathfrak{p}},\bar{\mathfrak{q}}]\oplus\mathfrak{n}_1)\big] \\ &= \mathbf{s}\big[d_b(\mathcal{M}[\bar{\mathfrak{p}},\bar{\mathfrak{q}}],\mathcal{M}[\mathfrak{p}_{\mathfrak{n}-1},\mathfrak{q}_{\mathfrak{n}-1}]) + d_b([\mathfrak{p}_\mathfrak{n},\mathfrak{q}_\mathfrak{n}],[\bar{\mathfrak{p}},\bar{\mathfrak{q}}]\oplus\mathfrak{n}_1)\big] \\ &\le \mathbf{s}\big[\eta d_b([\bar{\mathfrak{p}},\bar{\mathfrak{q}}],[\mathfrak{p}_\mathfrak{n},\mathfrak{q}_\mathfrak{n}]) + d_b([\mathfrak{p}_\mathfrak{n},\mathfrak{q}_\mathfrak{n}],[\mathfrak{p},\mathfrak{q}]\oplus\mathfrak{n}_2)\big] \\ &= \mathbf{s}\big[\eta d_b([\bar{\mathfrak{p}},\bar{\mathfrak{q}}]\oplus\mathfrak{n}_1,[\mathfrak{p}_\mathfrak{n},\mathfrak{q}_\mathfrak{n}]) + d_b([\mathfrak{p}_\mathfrak{n},\mathfrak{q}_\mathfrak{n}],[\mathfrak{p},\mathfrak{q}]\oplus\mathfrak{n}_2)\big] \\ &= \mathbf{s}\big[\eta d_b([\mathfrak{p},\mathfrak{q}]\oplus\mathfrak{n}_2,[\mathfrak{p}_\mathfrak{n},\mathfrak{q}_\mathfrak{n}]) + d_b([\mathfrak{p}_\mathfrak{n},\mathfrak{q}_\mathfrak{n}],[\mathfrak{p},\mathfrak{q}]\oplus\mathfrak{n}_2)\big] \\ &= \mathbf{s}\big[\eta d_b([\mathfrak{p},\mathfrak{q}],[\mathfrak{p}_\mathfrak{n},\mathfrak{q}_\mathfrak{n}]) + d_b([\mathfrak{p}_\mathfrak{n},\mathfrak{q}_\mathfrak{n}],[\mathfrak{p},\mathfrak{q}])\big] \\ &\to 0, \quad \text{as} \quad \mathfrak{n}\to\infty, \end{aligned}$$

that is, $\mathcal{M}[\bar{\mathfrak{p}},\bar{\mathfrak{q}}] \overset{\mathcal{N}}{=} [\bar{\mathfrak{p}},\bar{\mathfrak{q}}]$, for any $[\bar{\mathfrak{p}},\bar{\mathfrak{q}}] \in \langle[\mathfrak{p},\mathfrak{q}]\rangle$.
Suppose $[\mathfrak{p},\mathfrak{q}]$ and $[\mathfrak{r},\mathfrak{s}]$ are two distinct near fixed points of $\mathcal{M}$ with $[\mathfrak{p},\mathfrak{q}] \in \langle[\mathfrak{p},\mathfrak{q}]\rangle$ and $[\mathfrak{r},\mathfrak{s}] \notin \langle[\mathfrak{p},\mathfrak{q}]\rangle$, however $[\mathfrak{r},\mathfrak{s}]$ belongs to some different equivalence class. So, $\mathcal{M}[\mathfrak{p},\mathfrak{q}] \overset{\mathcal{N}}{=} [\mathfrak{p},\mathfrak{q}]$ and $\mathcal{M}[\mathfrak{r},\mathfrak{s}] \overset{\mathcal{N}}{=} [\mathfrak{r},\mathfrak{s}]$. Then $\mathcal{M}[\mathfrak{p},\mathfrak{q}]\oplus\mathfrak{n}_1 = [\mathfrak{p},\mathfrak{q}]\oplus\mathfrak{n}_2$ and $\mathcal{M}[\mathfrak{r},\mathfrak{s}]\oplus\mathfrak{n}_3 = [\mathfrak{r},\mathfrak{s}]\oplus\mathfrak{n}_4$, for some $\mathfrak{n}_1,\mathfrak{n}_2,\mathfrak{n}_3,\mathfrak{n}_4 \in \mathcal{N}$.
Now,

$$\begin{aligned} d_b([\mathfrak{p},\mathfrak{q}],[\mathfrak{r},\mathfrak{s}]) &= d_b([\mathfrak{p},\mathfrak{q}]\oplus\mathfrak{n}_2,[\mathfrak{r},\mathfrak{s}]\oplus\mathfrak{n}_4) \\ &= d_b(\mathcal{M}[\mathfrak{p},\mathfrak{q}]\oplus\mathfrak{n}_1,\mathcal{M}[\mathfrak{r},\mathfrak{s}]\oplus\mathfrak{n}_3) \\ &= d_b(\mathcal{M}[\mathfrak{p},\mathfrak{q}],\mathcal{M}[\mathfrak{r},\mathfrak{s}]) \\ &\le \eta d_b([\mathfrak{p},\mathfrak{q}],[\mathfrak{r},\mathfrak{s}]), \end{aligned}$$

a contradiction. Hence, $[\mathfrak{r},\mathfrak{s}] \in \langle[\mathfrak{p},\mathfrak{q}]\rangle$, concluding thereby that $\langle[\mathfrak{p},\mathfrak{q}]\rangle$ is a unique equivalence class of near fixed points of a self map $\mathcal{M}$. □

Next, we contribute an explanatory example to validate the efficiency and strength of our novel $b-$interval metric in creating an environment for the survival of near fixed points as well as a unique equivalence class of near fixed points.

Example 3.17. *Let $\mathcal{U}$ be the set of closed and bounded intervals and $\mathcal{N}$ be the null set. Define a $b-$interval metric, $d_b : \mathcal{U}\times\mathcal{U} \to \mathbb{R}^+$ on $\mathcal{U}$ as $d_b([\mathfrak{p},\mathfrak{q}],[\mathfrak{r},\mathfrak{s}]) = (\mathfrak{p}+\mathfrak{q}-\mathfrak{r}-\mathfrak{s})^2$. Then $(\mathcal{U}, d_b)$ is a complete $b-$interval metric space and $\mathbf{s} = 2$. Now, define a map $\mathcal{M} : \mathcal{U}\to\mathcal{U}$ as $\mathcal{M}[\mathfrak{p},\mathfrak{q}] = [-1+\frac{3}{5}\mathfrak{p}, 1+\frac{3}{5}\mathfrak{q}]$.*
Observe that, for $[\mathfrak{p},\mathfrak{q}],[\mathfrak{r},\mathfrak{s}] \in \mathcal{U}$,

$$\begin{aligned} d_b(\mathcal{M}[\mathfrak{p},\mathfrak{q}],\mathcal{M}[\mathfrak{r},\mathfrak{s}]) &= d_b\Big([-1+\frac{3}{5}\mathfrak{p}, 1+\frac{3}{5}\mathfrak{q}],[-1+\frac{3}{5}\mathfrak{r}, 1+\frac{3}{5}\mathfrak{s}]\Big) \\ &= \big(-1+\frac{3}{5}\mathfrak{p}+1+\frac{3}{5}\mathfrak{q}+1-\frac{3}{4}\mathfrak{r}-1-\frac{3}{5}\mathfrak{s}\big)^2 \\ &= \frac{9}{25}(\mathfrak{p}+\mathfrak{q}-\mathfrak{r}-\mathfrak{s})^2 \\ &\le \frac{9}{25}d_b([\mathfrak{p},\mathfrak{q}],[\mathfrak{r},\mathfrak{s}]), \end{aligned}$$

that is, $\mathcal{M}$ satisfies inequality (3.3) for $\eta = \frac{9}{25} < \frac{1}{\mathbf{s}}$. Hence, $\mathcal{M}$ has a unique equivalence class of near fixed points $\langle[-1,1]\rangle$, and $[-4,4] \overset{\mathcal{N}}{=} [-1,1]$. Noticeably, $\mathcal{M}$ has infinitely many near fixed points.

Remark 3.18. *It is fascinating to note that Example 3.17 can not be covered by near fixed point theorem 1 of Wu [20] and consequently, Theorem 3.16 is a genuine generalization and extension of Theorem 1 of Wu [20] to b−interval metric space.*

Next, we discuss some examples to establish the significant fact that contraction condition (3.3) is an essential prerequisite for the existence of a unique equivalence class of near fixed points. Otherwise, an equivalence class of near fixed points may or may not be unique.

Example 3.19. *Let* $\mathcal{U}$ *be the set of closed and bounded intervals and* $\mathcal{N}$ *be the null set. Define a b−interval metric* $d_b : \mathcal{U}\times\mathcal{U} \to \mathbb{R}^+$ *on* $\mathcal{U}$ *as* $d_b([\mathfrak{p},\mathfrak{q}],[\mathfrak{r},\mathfrak{s}]) = (\mathfrak{p}+\mathfrak{q}-\mathfrak{r}-\mathfrak{s})^2$ *with* $s = 2$*. Then* $(\mathcal{U}, d_b)$ *is a complete b−interval metric space. If* $\mathcal{M} : \mathcal{U} \to \mathcal{U}$ *is defined as* $\mathcal{M}[\mathfrak{p},\mathfrak{q}] = [2,4] - [\mathfrak{p},\mathfrak{q}]$*, then for* $[\mathfrak{p},\mathfrak{q}],\ [\mathfrak{r},\mathfrak{s}] \in \mathcal{U}$*,*

$$\begin{aligned} d_b(\mathcal{M}[\mathfrak{p},\mathfrak{q}],\mathcal{M}[\mathfrak{r},\mathfrak{s}]) &= d_b\Big([2,4]-[\mathfrak{p},\mathfrak{q}],[2,4]-[\mathfrak{r},\mathfrak{s}]\Big)\\ &= (\mathfrak{p}+\mathfrak{q}-\mathfrak{r}-\mathfrak{s})^2\\ &= d_b([\mathfrak{p},\mathfrak{q}],[\mathfrak{r},\mathfrak{s}]), \end{aligned}$$

that is, there does not exist any $\eta \in [0,1)$*, so that inequality (3.3) is satisfied. But* $\mathcal{M}$ *has a near fixed point* $[1,2]$ *and a unique equivalence class of near fixed point* $\big\langle[1,2]\big\rangle$ *in* $\mathcal{U}$*.*
Again, if $\mathcal{M} : \mathcal{U} \to \mathcal{U}$ *is defined as* $\mathcal{M}[\mathfrak{p},\mathfrak{q}] = 2[\mathfrak{p},\mathfrak{q}] - [-2,2]$*, then for* $[\mathfrak{p},\mathfrak{q}],\ [\mathfrak{r},\mathfrak{s}] \in \mathcal{U}$*,*

$$\begin{aligned} d_b(\mathcal{M}[\mathfrak{p},\mathfrak{q}],\mathcal{M}[\mathfrak{r},\mathfrak{s}]) &= d_b\Big(2[\mathfrak{p},\mathfrak{q}]-[-2,2],2[\mathfrak{r},\mathfrak{s}]-[-2,2]\Big)\\ &= 4(\mathfrak{p}+\mathfrak{q}-\mathfrak{r}-\mathfrak{s}])^2\\ &= 4d_b([\mathfrak{p},\mathfrak{q}],[\mathfrak{r},\mathfrak{s}]), \end{aligned}$$

that is, there does not exist any $\eta \in [0,1)$*, so that inequality (3.3) is satisfied. But* $\mathcal{M}$ *has a near fixed point* $[-2,2]$ *and a unique equivalence class of near fixed points* $\big\langle[-2,2]\big\rangle$ *in* $\mathcal{U}$*.*

Example 3.20. *Let* $\mathcal{U}$ *be the set of closed and bounded intervals and* $\mathcal{N}$ *be the null set. Define a b−interval metric* $d_b : \mathcal{U}\times\mathcal{U} \to \mathbb{R}^+$ *on* $\mathcal{U}$ *as* $d_b([\mathfrak{p},\mathfrak{q}],[\mathfrak{r},\mathfrak{s}]) = |\mathfrak{p}+\mathfrak{q}-\mathfrak{r}-\mathfrak{s}|^3$ *with* $s = 4$*. Then* $(\mathcal{U}, d_b)$ *is a complete b−interval metric space. If* $\mathcal{M} : \mathcal{U} \to \mathcal{U}$ *is defined as* $\mathcal{M}[\mathfrak{p},\mathfrak{q}] = [\mathfrak{p},\mathfrak{q}]$*, then for* $[\mathfrak{p},\mathfrak{q}],\ [\mathfrak{r},\mathfrak{s}] \in \mathcal{U}$*,*

$$\begin{aligned} d_b(\mathcal{M}[\mathfrak{p},\mathfrak{q}],\mathcal{M}[\mathfrak{r},\mathfrak{s}]) &= d_b\Big([\mathfrak{p},\mathfrak{q}],[\mathfrak{r},\mathfrak{s}]\Big)\\ &= |\mathfrak{p}+\mathfrak{q}-\mathfrak{r}-\mathfrak{s}|^3\\ &= d_b([\mathfrak{p},\mathfrak{q}],[\mathfrak{r},\mathfrak{s}]), \end{aligned}$$

that is, there does not exist any $\eta \in [0,1)$*, so that inequality (3.3) is satisfied. Here,* $\mathcal{M}$ *have infinitely many near fixed points and infinitely many equivalence classes of near fixed points corresponding to near fixed points in* $\mathcal{U}$*.*

Next, we present an improved $b-$interval metric variant of Theorem 5 of Sehgal [17] which is an extension of Banach [2], Bianchini [3], Edelstein [6], Kannan [8], Rakotch [15], Reich [16], and so on.

Theorem 3.21. *Theorem 3.16 still holds, if (3.2) is replaced by the following:*

$$d_b(\mathcal{M}[\mathfrak{p},\mathfrak{q}],\mathcal{M}[\mathfrak{r},\mathfrak{s}]) \le \eta \max\{d_b([\mathfrak{p},\mathfrak{q}],[\mathfrak{r},\mathfrak{s}]),\ d_b([\mathfrak{p},\mathfrak{q}],\mathcal{M}[\mathfrak{p},\mathfrak{q}]),\ d_b([\mathfrak{r},\mathfrak{s}],\mathcal{M}[\mathfrak{r},\mathfrak{s}])\}, \tag{3.5}$$

$\eta < \frac{1}{s}$ *and* $[\mathfrak{p},\mathfrak{q}],[\mathfrak{r},\mathfrak{s}] \in \mathcal{U}$*.*

Proof . Let the sequence $\{[\mathfrak{p}_\mathfrak{n}, \mathfrak{q}_\mathfrak{n}]\}$ be defined as in Theorem 3.16. Now,

$$\begin{aligned} d_b([\mathfrak{p}_{\mathfrak{n}+1}, \mathfrak{q}_{\mathfrak{n}+1}], [\mathfrak{p}_\mathfrak{n}, \mathfrak{q}_\mathfrak{n}]) &= d_b(\mathcal{M}[\mathfrak{p}_\mathfrak{n}, \mathfrak{q}_\mathfrak{n}], \mathcal{M}[\mathfrak{p}_{\mathfrak{n}-1}, \mathfrak{q}_{\mathfrak{n}-1}]) \\ &\leq \eta \max\{d_b([\mathfrak{p}_\mathfrak{n}, \mathfrak{q}_\mathfrak{n}], [\mathfrak{p}_{\mathfrak{n}-1}, \mathfrak{q}_{\mathfrak{n}-1}]),\ d_b([\mathfrak{p}_\mathfrak{n}, \mathfrak{q}_\mathfrak{n}], \mathcal{M}[\mathfrak{p}_\mathfrak{n}, \mathfrak{q}_\mathfrak{n}]), \\ &\quad d_b([\mathfrak{p}_{\mathfrak{n}-1}, \mathfrak{q}_{\mathfrak{n}-1}], \mathcal{M}[\mathfrak{p}_{\mathfrak{n}-1}, \mathfrak{q}_{\mathfrak{n}-1}])\}, \\ &= \eta \max\{d_b([\mathfrak{p}_\mathfrak{n}, \mathfrak{q}_\mathfrak{n}], [\mathfrak{p}_{\mathfrak{n}-1}, \mathfrak{q}_{\mathfrak{n}-1}]),\ d_b([\mathfrak{p}_\mathfrak{n}, \mathfrak{q}_\mathfrak{n}], [\mathfrak{p}_{\mathfrak{n}+1}, \mathfrak{q}_{\mathfrak{n}+1}]),\ d_b([\mathfrak{p}_{\mathfrak{n}-1}, \mathfrak{q}_{\mathfrak{n}-1}], [\mathfrak{p}_\mathfrak{n}, \mathfrak{q}_\mathfrak{n}])\}, \\ &= \eta \max\{d_b([\mathfrak{p}_\mathfrak{n}, \mathfrak{q}_\mathfrak{n}], [\mathfrak{p}_{\mathfrak{n}+1}, \mathfrak{q}_{\mathfrak{n}+1}]),\ d_b([\mathfrak{p}_{\mathfrak{n}-1}, \mathfrak{q}_{\mathfrak{n}-1}], [\mathfrak{p}_\mathfrak{n}, \mathfrak{q}_\mathfrak{n}])\}. \end{aligned}$$

We discuss two cases:

Case (i) If $\max\{d_b([\mathfrak{p}_\mathfrak{n}, \mathfrak{q}_\mathfrak{n}], [\mathfrak{p}_{\mathfrak{n}+1}, \mathfrak{q}_{\mathfrak{n}+1}]),\ d_b([\mathfrak{p}_{\mathfrak{n}-1}, \mathfrak{q}_{\mathfrak{n}-1}], [\mathfrak{p}_\mathfrak{n}, \mathfrak{q}_\mathfrak{n}])\} = d_b([\mathfrak{p}_\mathfrak{n}, \mathfrak{q}_\mathfrak{n}], [\mathfrak{p}_{\mathfrak{n}+1}, \mathfrak{q}_{\mathfrak{n}+1}])$, then $d_b([\mathfrak{p}_{\mathfrak{n}+1}, \mathfrak{q}_{\mathfrak{n}+1}], [\mathfrak{p}_\mathfrak{n}, \mathfrak{q}_\mathfrak{n}]) \leq \eta d_b([\mathfrak{p}_\mathfrak{n}, \mathfrak{q}_\mathfrak{n}], [\mathfrak{p}_{\mathfrak{n}+1}, \mathfrak{q}_{\mathfrak{n}+1}])$, a contradiction.

Case (ii) If $\max\{d_b([\mathfrak{p}_\mathfrak{n}, \mathfrak{q}_\mathfrak{n}], [\mathfrak{p}_{\mathfrak{n}+1}, \mathfrak{q}_{\mathfrak{n}+1}]),\ d_b([\mathfrak{p}_{\mathfrak{n}-1}, \mathfrak{q}_{\mathfrak{n}-1}], [\mathfrak{p}_\mathfrak{n}, \mathfrak{q}_\mathfrak{n}])\} = d_b([\mathfrak{p}_{\mathfrak{n}-1}, \mathfrak{q}_{\mathfrak{n}-1}], [\mathfrak{p}_\mathfrak{n}, \mathfrak{q}_\mathfrak{n}])$, then $d_b([\mathfrak{p}_{\mathfrak{n}+1}, \mathfrak{q}_{\mathfrak{n}+1}], [\mathfrak{p}_\mathfrak{n}, \mathfrak{q}_\mathfrak{n}]) \leq \eta d_b([\mathfrak{p}_{\mathfrak{n}-1}, \mathfrak{q}_{\mathfrak{n}-1}], [\mathfrak{p}_\mathfrak{n}, \mathfrak{q}_\mathfrak{n}])$.

Thus, the sequence $\{[\mathfrak{p}_\mathfrak{n}, \mathfrak{q}_\mathfrak{n}]\}_{n=1}^{\infty}$ satisfies all the hypotheses of Theorem 3.16. So, following similar steps as in Theorem 3.16, we may deduce that $\mathcal{U}$ has a near fixed point and a unique equivalence class of near fixed points $\langle [p, q] \rangle$. □

The following example is given to illustrate the above theorem and demonstrate the significant fact that Theorem 5 of Sehgal [17] is not valid in the $b-$interval metric space.

Example 3.22. *Let $\mathcal{U}$ be the set of closed and bounded intervals and $\mathcal{N}$ be the null set. Define a $b-$interval metric, $d_b : \mathcal{U} \times \mathcal{U} \to \mathbb{R}^+$ on $\mathcal{U}$ as $d_b([\mathfrak{p}, \mathfrak{q}], [\mathfrak{r}, \mathfrak{s}]) = (\mathfrak{p} + \mathfrak{q} - \mathfrak{r} - \mathfrak{s})^2$. Then $(\mathcal{U}, d_b)$ is a complete $b-$interval metric space and $\boldsymbol{s} = 2$. Now, define a map $\mathcal{M} : \mathcal{U} \to \mathcal{U}$ as $\mathcal{M}[\mathfrak{p}, \mathfrak{q}] = [\frac{3}{10}\mathfrak{p}, \frac{3}{10}\mathfrak{q}]$. Observe that, for $[\mathfrak{p}, \mathfrak{q}], [\mathfrak{r}, \mathfrak{s}] \in \mathcal{U}$,*

$$\begin{aligned} d_b(\mathcal{M}[\mathfrak{p}, \mathfrak{q}], \mathcal{M}[\mathfrak{r}, \mathfrak{s}]) &= d_b\Big([\frac{3}{10}\mathfrak{p}, \frac{3}{10}\mathfrak{q}], [\frac{3}{10}\mathfrak{r}, \frac{3}{10}\mathfrak{s}]\Big) \\ &= \big(\frac{3}{10}\mathfrak{p} + \frac{3}{10}\mathfrak{q} - \frac{3}{10}\mathfrak{r} - \frac{3}{10}\mathfrak{s}\big)^2 \\ &= \frac{9}{100}(\mathfrak{p} + \mathfrak{q} - \mathfrak{r} - \mathfrak{s})^2 \\ &\leq \frac{9}{100} \max\{d_b([\mathfrak{p}, \mathfrak{q}], [\mathfrak{r}, \mathfrak{s}]), d_b(\mathcal{M}[\mathfrak{p}, \mathfrak{q}],[\mathfrak{r}, \mathfrak{s}]), d_b(\mathcal{M}[\mathfrak{p}, \mathfrak{q}],[\mathfrak{r}, \mathfrak{s}]), d_b([\mathfrak{p}, \mathfrak{q}],\mathcal{M}[\mathfrak{r}, \mathfrak{s}])\}, \end{aligned}$$

that is, $\mathcal{M}$ satisfies inequality (3.5) for $\eta = \frac{9}{100} < \frac{1}{s}$. Hence, $\mathcal{M}$ has a unique equivalence class of near fixed points $\langle [-1, 1] \rangle$, a near fixed point $[0, 0]$ and $[-1, 1] \stackrel{\mathcal{N}}{=} [0, 0]$. Noticeably, $\mathcal{M}$ have infinitely many near fixed points.

4. Application

Since the $b-$interval metric space $(\mathcal{U}, d_b)$ discussed here is not a metric space, we can not study the fixed circle introduced by, Taş and Özgür [14] on $b-$interval metric space $(\mathcal{U}, d_b)$ in a traditional way. So, we shall study the geometric properties of the set of non-unique near fixed points of a self map in reference to a so called near fixed interval circle and define its equivalence class as equivalence $\mathcal{C}-$class of interval circles.

Now, we define an interval circle in a $b-$interval metric space as:

Definition 4.1. *Let $\mathcal{U}$ be the set of closed and bounded intervals and $\mathcal{N}$ be the null set. Let a function $\mathcal{M} : \mathcal{U} \to \mathcal{U}$ be defined on a b−interval metric space $(\mathcal{U}, d_b)$ with $\boldsymbol{s} \geq 1$. An interval circle $\mathcal{C}([\mathfrak{p}_\circ, \mathfrak{q}_\circ], \boldsymbol{r})$ having centre at $[\mathfrak{p}_\circ, \mathfrak{q}_\circ]$ and radius $\boldsymbol{r}$ on $(\mathcal{U}, d_b)$ is defined as:*

$$\mathcal{C}([\mathfrak{p}_\circ, \mathfrak{q}_\circ], \boldsymbol{r}) = \{[\mathfrak{p}, \mathfrak{q}] \in \mathcal{U} : d_b([\mathfrak{p}, \mathfrak{q}], [\mathfrak{p}_\circ, \mathfrak{q}_\circ]) = \boldsymbol{r},\ [\mathfrak{p}_\circ, \mathfrak{q}_\circ] \in \mathcal{U}, \boldsymbol{r} \in [0, \infty)\}. \tag{4.1}$$

Remark 4.2. *For $\boldsymbol{s} = 1$, (4.1) is the interval circle in an interval metric space. Noticeably, an interval circle is not necessarily the same as a circle in a Euclidean space.*

Example 4.3. *Let $\mathcal{U}$ be the set of closed and bounded intervals and $\mathcal{N}$ be the null set. Let a b−interval metric $d_b : \mathcal{U} \to \mathcal{U}$ be defined as $d_b([\mathfrak{p}, \mathfrak{q}], [\mathfrak{r}, \mathfrak{s}]) = |\mathfrak{p} + \mathfrak{q} - \mathfrak{r} - \mathfrak{s}|^3$ with $\boldsymbol{s} = 4$. Then an interval circle centred at $[2, 5] \in \mathcal{U}$ having radius $\boldsymbol{r} = 6$ is*

$$\begin{aligned}\mathcal{C}([2, 5], 5) &= \{[\mathfrak{p}, \mathfrak{q}] \in \mathcal{U} : d([\mathfrak{p}, \mathfrak{q}], [2, 5]) = 6\} \\ &= \{\mathfrak{p}, \mathfrak{q}] \in \mathcal{U} : |\mathfrak{p} + \mathfrak{q} - 2 - 5|^3 = 6\} \\ &= \{\mathfrak{p}, \mathfrak{q}] \in \mathcal{U} : |\mathfrak{p} + \mathfrak{q} - 7|^3 = 6\}.\end{aligned}$$

For any $[\mathfrak{r}, \mathfrak{s}] \in \langle [\mathfrak{p}_\circ, \mathfrak{q}_\circ] \rangle$, $\mathcal{C}([\mathfrak{p}_\circ, \mathfrak{q}_\circ], \mathbf{r}) = \mathcal{C}([\mathfrak{r}, \mathfrak{s}], \mathbf{r})$. So, we define a $\mathcal{C}-$class of interval circles having radius $\mathbf{r}$, using a binary relation $\overset{\mathcal{R}}{\approx}$ as:

$$\begin{aligned}&\langle \mathcal{C}([\mathfrak{p}, \mathfrak{q}], \mathbf{r}) \rangle = \{\mathcal{C}([\mathfrak{r}, \mathfrak{s}], \mathbf{r}) : \mathcal{C}([\mathfrak{p}, \mathfrak{q}], \mathbf{r}) \overset{\mathcal{R}}{\approx} \mathcal{C}([\mathfrak{r}, \mathfrak{s}], \mathbf{r}), \\ &\text{if } \mathcal{C}([\mathfrak{p}, \mathfrak{q}], \mathbf{r}) = \mathcal{C}([\mathfrak{r}, \mathfrak{s}], \mathbf{r}) \text{ and } [\mathfrak{p}, \mathfrak{q}] \overset{\mathcal{N}}{=} [\mathfrak{r}, \mathfrak{s}],\ [\mathfrak{p}, \mathfrak{q}], [\mathfrak{r}, \mathfrak{s}] \in \mathcal{U}\}.\end{aligned} \tag{4.2}$$

If $\breve{\mathbf{C}}$ denotes the set of all $\mathcal{C}-$classes of interval circles defined on elements of $\mathcal{U}$.

Proposition 4.4. *The binary relation $\overset{\mathcal{R}}{\approx}$ is an equivalence relation.*

Proof .

1. For $\mathcal{C}([\mathfrak{p}, \mathfrak{q}], \mathbf{r}) \in \breve{\mathbf{C}}$, $\mathcal{C}([\mathfrak{p}, \mathfrak{q}], \mathbf{r}) = \mathcal{C}([\mathfrak{p}, \mathfrak{q}], \mathbf{r})$ for $\mathfrak{n}_1 = \mathfrak{n}_2 = [0, 0]$, $[\mathfrak{p}, \mathfrak{q}] \overset{\mathcal{R}}{\approx} [\mathfrak{p}, \mathfrak{q}]$, $\mathcal{C}([\mathfrak{p}, \mathfrak{q}], \mathbf{r}) \overset{\mathcal{R}}{\approx} \mathcal{C}([\mathfrak{p}, \mathfrak{q}], \mathbf{r})$, which shows the reflexivity.
2. Let $\mathcal{C}([\mathfrak{p}, \mathfrak{q}], \mathbf{r}) \overset{\mathcal{R}}{\approx} \mathcal{C}([\mathfrak{r}, \mathfrak{s}], \mathbf{r})$,
   that is, $\mathcal{C}([\mathfrak{p}, \mathfrak{q}], \mathbf{r}) = \mathcal{C}([\mathfrak{r}, \mathfrak{s}], \mathbf{r})$ and $[\mathfrak{p}, \mathfrak{q}] \oplus \mathfrak{n}_1 = [\mathfrak{r}, \mathfrak{s}] + \mathfrak{n}_2$,
   or $\mathcal{C}([\mathfrak{r}, \mathfrak{s}], \mathbf{r}) = \mathcal{C}([\mathfrak{p}, \mathfrak{q}], \mathbf{r})$ and $[\mathfrak{r}, \mathfrak{s}] \oplus \mathfrak{n}_2 = [\mathfrak{p}, \mathfrak{q}] + \mathfrak{n}_1$,
   $\Longrightarrow \mathcal{C}([\mathfrak{r}, \mathfrak{s}], \mathbf{r}) \overset{\mathcal{R}}{\approx} \mathcal{C}([\mathfrak{p}, \mathfrak{q}], \mathbf{r})$, which shows the symmetry.
3. Let $\mathcal{C}([\mathfrak{p}, \mathfrak{q}], \mathbf{r}) \overset{\mathcal{R}}{\approx} \mathcal{C}([\mathfrak{r}, \mathfrak{s}], \mathbf{r})$ and $\mathcal{C}([\mathfrak{r}, \mathfrak{s}], \mathbf{r}) \overset{\mathcal{R}}{\approx} \mathcal{C}([\mathfrak{t}, \mathfrak{u}], \mathbf{r})$. We assert that $\mathcal{C}([\mathfrak{p}, \mathfrak{q}], \mathbf{r}) \overset{\mathcal{R}}{\approx} \mathcal{C}([\mathfrak{t}, \mathfrak{u}], \mathbf{r})$.
   Since, $\mathcal{C}([\mathfrak{p}, \mathfrak{q}], \mathbf{r}) = \mathcal{C}([\mathfrak{r}, \mathfrak{s}], \mathbf{r})$, $[\mathfrak{p}, \mathfrak{q}] + \mathfrak{n}_1 = [\mathfrak{r}, \mathfrak{s}] + \mathfrak{n}_2$
   and since, $\mathcal{C}([\mathfrak{r}, \mathfrak{s}], \mathbf{r}) = \mathcal{C}([\mathfrak{t}, \mathfrak{u}], \mathbf{r})$, $[\mathfrak{r}, \mathfrak{s}] + \mathfrak{n}_3 = [\mathfrak{t}, \mathfrak{u}] + \mathfrak{n}_4$, for some $\mathfrak{n}_1, \mathfrak{n}_2, \mathfrak{n}_3, \mathfrak{n}_4 \in \mathcal{N}$.
   Hence, $\mathcal{C}([\mathfrak{p}, \mathfrak{q}], \mathbf{r}) = \mathcal{C}([\mathfrak{t}, \mathfrak{u}], \mathbf{r})$ and $[\mathfrak{p}, \mathfrak{q}] + \mathfrak{n}_1 + \mathfrak{n}_3 = [\mathfrak{r}, \mathfrak{s}] + \mathfrak{n}_2 + \mathfrak{n}_3 = [\mathfrak{t}, \mathfrak{u}] + \mathfrak{n}_4 + \mathfrak{n}_2$,
   $\Longrightarrow \mathcal{C}([\mathfrak{p}, \mathfrak{q}], \mathbf{r}) \overset{\mathcal{R}}{\approx} \mathcal{C}([\mathfrak{t}, \mathfrak{u}], \mathbf{r})$, which shows transitivity.

Hence, the $\mathcal{C}-$classes of interval circles defined in (4.2) constitute the equivalence $\mathcal{C}-$classes of interval circles. The family $\langle \breve{\mathbf{C}} \rangle$, of equivalence $\mathcal{C}-$classes of interval circles, is known as the quotient set of $\breve{\mathbf{C}}$. It is interesting to mention that a quotient set $\langle \breve{\mathbf{C}} \rangle$ is still not a conventional vector space. Further, $\mathcal{C}([\mathfrak{r}, \mathfrak{s}], \mathbf{r}) \in \langle \mathcal{C}([\mathfrak{p}, \mathfrak{q}], \mathbf{r}) \rangle \Longrightarrow \mathcal{C}([\mathfrak{r}, \mathfrak{s}], \mathbf{r}) = \mathcal{C}([\mathfrak{p}, \mathfrak{q}], \mathbf{r})$ and $[\mathfrak{p}, \mathfrak{q}] \overset{\mathcal{N}}{=} [\mathfrak{r}, \mathfrak{s}]$. Equivalently, the family of equivalence $\mathcal{C}-$classes constitutes a partition of the entire set of all $\mathcal{C}-$classes of interval circles defined on elements of $\mathcal{U}$. □

Definition 4.5. *Let $\mathcal{U}$ be the set of closed and bounded intervals and $\mathcal{N}$ be the null set. Let $\mathcal{M}$ be a self map defined on $\mathcal{U}$. An interval circle $\mathcal{C}([\mathfrak{p}_\circ, \mathfrak{q}_\circ], \boldsymbol{r})$ is known as a near fixed interval circle of $\mathcal{M}$ if and only if $\mathcal{M}[\mathfrak{p}, \mathfrak{q}] \stackrel{\mathcal{N}}{=} [\mathfrak{p}, \mathfrak{q}]$, $[\mathfrak{p}, \mathfrak{q}] \in \mathcal{C}([\mathfrak{p}_\circ, \mathfrak{q}_\circ], \boldsymbol{r})$.*

Next, we establish a result utilizing a $b-$interval metric variant of the classical Caristi map [4] for determining near fixed interval circle of the function $\mathcal{M}$ and its equivalence class.

Theorem 4.6. *Let $\mathcal{C}_b([\mathfrak{p}_\circ, \mathfrak{q}_\circ], \boldsymbol{r})$ be an interval circle in a $b-$interval metric space $(\mathcal{U}, d_b)$, $\boldsymbol{s} \geq 1$ and $\mathcal{N}$ be the null set. Define $\zeta : \mathcal{U} \to [0, \infty)$ as:*

$$\zeta([\mathfrak{p}, \mathfrak{q}]) = d_b([\mathfrak{p}, \mathfrak{q}], [\mathfrak{p}_\circ, \mathfrak{q}_\circ]), \ [\mathfrak{p}, \mathfrak{q}] \in \mathcal{U}. \tag{4.3}$$

If there exists a self map $\mathcal{M} : \mathcal{U} \to \mathcal{U}$ so that

1. $d_b([\mathfrak{p}, \mathfrak{q}], \mathcal{M}[\mathfrak{p}, \mathfrak{q}]) \leq \zeta([\mathfrak{p}, \mathfrak{q}]) - \zeta(\mathcal{M}[\mathfrak{p}, \mathfrak{q}])$;
2. $d_b(\mathcal{M}[\mathfrak{p}, \mathfrak{q}], [\mathfrak{p}_\circ, \mathfrak{q}_\circ]) \geq \boldsymbol{r}$, $[\mathfrak{p}, \mathfrak{q}] \in \mathcal{U}$,
   then $\mathcal{C}([\mathfrak{p}_\circ, \mathfrak{q}_\circ], \boldsymbol{r})$ is a near fixed interval circle of $\mathcal{M}$.
3. $$\text{If } \ d_b(\mathcal{M}[\mathfrak{p}, \mathfrak{q}], \mathcal{M}[\mathfrak{r}, \mathfrak{s}]) \leq \eta d_b([\mathfrak{p}, \mathfrak{q}], [\mathfrak{r}, \mathfrak{s}]), \quad \eta \in [0, 1) \ \text{ and } \ [\mathfrak{p}, \mathfrak{q}], [\mathfrak{r}, \mathfrak{s}] \in \mathcal{U}, \tag{4.4}$$
   then $\mathcal{M}[\mathfrak{p}, \mathfrak{q}] \stackrel{\mathcal{N}}{=} [\mathfrak{p}, \mathfrak{q}]$ and $\mathcal{M}[\bar{\mathfrak{p}}, \bar{\mathfrak{q}}] \stackrel{\mathcal{N}}{=} [\bar{\mathfrak{p}}, \bar{\mathfrak{q}}] \Longrightarrow [\bar{\mathfrak{p}}, \bar{\mathfrak{q}}] \in \langle [\mathfrak{p}, \mathfrak{q}] \rangle$.
4. *Further, if for $[\mathfrak{p}, \mathfrak{q}] \in \mathcal{C}([\mathfrak{p}_\circ, \mathfrak{q}_\circ], \boldsymbol{r})$ and $[\mathfrak{r}, \mathfrak{s}] \in \mathcal{U} \setminus \mathcal{C}([\mathfrak{p}_\circ, \mathfrak{q}_\circ], \boldsymbol{r})$, contraction condition (4.4) is satisfied, then $\mathcal{M}$ has a unique equivalence interval $\mathcal{C}-$class of near fixed interval circles $\langle \mathcal{C}([\mathfrak{p}_\circ, \mathfrak{q}_\circ], \boldsymbol{r}) \rangle$, that is, if $\mathcal{C}([\bar{\mathfrak{p}}_\circ, \bar{\mathfrak{q}}_\circ], \boldsymbol{r})$ is a near fixed interval circle of $\mathcal{M}$, then $\mathcal{C}([\bar{\mathfrak{p}}_\circ, \bar{\mathfrak{q}}_\circ], \boldsymbol{r}) \in \langle \mathcal{C}([\mathfrak{p}_\circ, \mathfrak{q}_\circ], \boldsymbol{r}) \rangle$ or $\langle \mathcal{C}([\mathfrak{p}_\circ, \mathfrak{q}_\circ], \boldsymbol{r}) \rangle = \langle \mathcal{C}([\bar{\mathfrak{p}}_\circ, \bar{\mathfrak{q}}_\circ], \boldsymbol{r}) \rangle$. Equivalently, if $\mathcal{C}([\mathfrak{p}_\circ, \mathfrak{q}_\circ], \boldsymbol{r})$ and $\mathcal{C}([\bar{\mathfrak{p}}_\circ, \bar{\mathfrak{q}}_\circ], \boldsymbol{r})$ are the near fixed interval circles of $\mathcal{M}$, then $\mathcal{C}([\mathfrak{p}_\circ, \mathfrak{q}_\circ], \boldsymbol{r}) \stackrel{\mathcal{R}}{\approx} \mathcal{C}([\bar{\mathfrak{p}}_\circ, \bar{\mathfrak{q}}_\circ], \boldsymbol{r})$.*

Proof . Let $[\mathfrak{p}, \mathfrak{q}] \in \mathcal{C}([\mathfrak{p}_\circ, \mathfrak{q}_\circ], \mathbf{r})$ be any arbitrary point. Using 1 and equation (4.3)

$$\begin{aligned} d_b([\mathfrak{p}, \mathfrak{q}], \mathcal{M}[\mathfrak{p}, \mathfrak{q}]) &\leq \zeta([\mathfrak{p}, \mathfrak{q}]) - \zeta(\mathcal{M}[\mathfrak{p}, \mathfrak{q}]) \\ &= d_b([\mathfrak{p}, \mathfrak{q}], [\mathfrak{p}_\circ, \mathfrak{q}_\circ]) - d_b(\mathcal{M}[\mathfrak{p}, \mathfrak{q}], [\mathfrak{p}_\circ, \mathfrak{q}_\circ]) \\ &= \mathbf{r} - d_b(\mathcal{M}[\mathfrak{p}, \mathfrak{q}], [\mathfrak{p}_\circ, \mathfrak{q}_\circ]) \\ &\leq 0, \quad \text{using (2)} \end{aligned}$$

and so $\mathcal{M}[\mathfrak{p}, \mathfrak{q}] \stackrel{\mathcal{N}}{=} [\mathfrak{p}, \mathfrak{q}]$, that is, $[\mathfrak{p}, \mathfrak{q}]$ is a near fixed point of $\mathcal{M}$. We assert that for point $[\bar{\mathfrak{p}}, \bar{\mathfrak{q}}] \in \langle [\mathfrak{p}, \mathfrak{q}] \rangle$, $\mathcal{M}[\bar{\mathfrak{p}}, \bar{\mathfrak{q}}] \stackrel{\mathcal{N}}{=} [\bar{\mathfrak{p}}, \bar{\mathfrak{q}}]$. Since, d_b satisfies null equalities, so $[\bar{\mathfrak{p}}, \bar{\mathfrak{q}}] \oplus \mathfrak{n}_1 = [\mathfrak{p}, \mathfrak{q}] \oplus \mathfrak{n}_2$, $\mathfrak{n}_1, \mathfrak{n}_2 \in \mathcal{N}$. Now,

$$\begin{aligned} d_b(\mathcal{M}[\bar{\mathfrak{p}}, \bar{\mathfrak{q}}], [\bar{\mathfrak{p}}, \bar{\mathfrak{q}}]) &= d_b(\mathcal{M}[\bar{\mathfrak{p}}, \bar{\mathfrak{q}}], [\bar{\mathfrak{p}}, \bar{\mathfrak{q}}] \oplus \mathfrak{n}_1) \\ &\leq \mathbf{s}\big[d_b(\mathcal{M}[\bar{\mathfrak{p}}, \bar{\mathfrak{q}}], [\mathfrak{p}_\mathfrak{n}, \mathfrak{q}_\mathfrak{n}]) + d_b([\mathfrak{p}_\mathfrak{n}, \mathfrak{q}_\mathfrak{n}], [\bar{\mathfrak{p}}, \bar{\mathfrak{q}}] \oplus \mathfrak{n}_1)\big] \\ &= \mathbf{s}\big[d_b(\mathcal{M}[\bar{\mathfrak{p}}, \bar{\mathfrak{q}}], \mathcal{M}[\mathfrak{p}_{\mathfrak{n}-1}, \mathfrak{q}_{\mathfrak{n}-1}]) + d_b([\mathfrak{p}_\mathfrak{n}, \mathfrak{q}_\mathfrak{n}], [\bar{\mathfrak{p}}, \bar{\mathfrak{q}}] \oplus \mathfrak{n}_1)\big] \\ &\leq \mathbf{s}\big[\eta d_b([\bar{\mathfrak{p}}, \bar{\mathfrak{q}}], [\mathfrak{p}_\mathfrak{n}, \mathfrak{q}_\mathfrak{n}]) + d_b([\mathfrak{p}_\mathfrak{n}, \mathfrak{q}_\mathfrak{n}], [\mathfrak{p}, \mathfrak{q}] \oplus \mathfrak{n}_2)\big] \\ &= \mathbf{s}\big[\eta d_b([\bar{\mathfrak{p}}, \bar{\mathfrak{q}}] \oplus \mathfrak{n}_1, [\mathfrak{p}_\mathfrak{n}, \mathfrak{q}_\mathfrak{n}]) + d_b([\mathfrak{p}_\mathfrak{n}, \mathfrak{q}_\mathfrak{n}], [\mathfrak{p}, \mathfrak{q}] \oplus \mathfrak{n}_2)\big] \\ &= \mathbf{s}\big[\eta d_b([\mathfrak{p}, \mathfrak{q}] \oplus \mathfrak{n}_2, [\mathfrak{p}_\mathfrak{n}, \mathfrak{q}_\mathfrak{n}]) + d_b([\mathfrak{p}_\mathfrak{n}, \mathfrak{q}_\mathfrak{n}], [\mathfrak{p}, \mathfrak{q}] \oplus \mathfrak{n}_2)\big] \\ &= \mathbf{s}\big[\eta d_b([\mathfrak{p}, \mathfrak{q}], [\mathfrak{p}_\mathfrak{n}, \mathfrak{q}_\mathfrak{n}]) + d_b([\mathfrak{p}_\mathfrak{n}, \mathfrak{q}_\mathfrak{n}], [\mathfrak{p}, \mathfrak{q}])\big] \\ &\to 0, \quad \text{as } \ \mathfrak{n} \to \infty, \end{aligned}$$

that is, $\mathcal{M}[\bar{\mathfrak{p}},\bar{\mathfrak{q}}] \overset{\mathcal{N}}{=} [\bar{\mathfrak{p}},\bar{\mathfrak{q}}]$, for any $[\bar{\mathfrak{p}},\bar{\mathfrak{q}}] \in \langle[\mathfrak{p},\mathfrak{q}]\rangle$ and for all $[\mathfrak{p},\mathfrak{q}] \in \mathcal{C}([\mathfrak{p}_\circ,\mathfrak{q}_\circ],\mathbf{r})$,
that is, $\mathcal{C}([\mathfrak{p}_\circ,\mathfrak{q}_\circ],\mathbf{r})$ is a near fixed interval circle of $\mathcal{M}$.
Let there exists two equivalence classes of near fixed interval circles $\langle\mathcal{C}([\mathfrak{p}_\circ,\mathfrak{q}_\circ],\mathbf{r})\rangle$ and $\langle\mathcal{C}([\bar{\mathfrak{p}}_\circ,\bar{\mathfrak{q}}_\circ],\mathbf{r})\rangle$ of $\mathcal{M}$, that is, $\mathcal{M}$ satisfies conditions (1) and (2) for each of the near fixed interval circles $\mathcal{C}([\mathfrak{p}_\circ,\mathfrak{q}_\circ],\mathbf{r})$ and $\mathcal{C}([\bar{\mathfrak{p}}_\circ,\bar{\mathfrak{q}}_\circ],\mathbf{r})$. Now, for $[\mathfrak{p},\mathfrak{q}] \in \mathcal{C}([\mathfrak{p}_\circ,\mathfrak{q}_\circ],\mathbf{r})$ and $[\mathfrak{r},\mathfrak{s}] \in \mathcal{U}\setminus\mathcal{C}([\mathfrak{p}_\circ,\mathfrak{q}_\circ],\mathbf{r})$, $\mathcal{M}[\mathfrak{p},\mathfrak{q}] \overset{\mathcal{N}}{=} [\mathfrak{p},\mathfrak{q}]$ and $\mathcal{M}[\mathfrak{r},\mathfrak{s}] \overset{\mathcal{N}}{=} [\mathfrak{r},\mathfrak{s}]$.
Then $\mathcal{M}[\mathfrak{p},\mathfrak{q}] \oplus \mathfrak{n}_1 = [\mathfrak{p},\mathfrak{q}] \oplus \mathfrak{n}_2$ and $\mathcal{M}[\mathfrak{r},\mathfrak{s}] \oplus \mathfrak{n}_3 = [\mathfrak{r},\mathfrak{s}] \oplus \mathfrak{n}_4$, for some $\mathfrak{n}_1,\mathfrak{n}_2,\mathfrak{n}_3,\mathfrak{n}_4 \in \mathcal{N}$.
Now,

$$\begin{aligned} d_b([\mathfrak{p},\mathfrak{q}],[\mathfrak{r},\mathfrak{s}]) &= d_b([\mathfrak{p},\mathfrak{q}]\oplus\mathfrak{n}_2,[\mathfrak{r},\mathfrak{s}]\oplus\mathfrak{n}_4) \\ &= d_b(\mathcal{M}[\mathfrak{p},\mathfrak{q}]\oplus\mathfrak{n}_1,\mathcal{M}[\mathfrak{r},\mathfrak{s}]\oplus\mathfrak{n}_3) \\ &= d_b(\mathcal{M}[\mathfrak{p},\mathfrak{q}],\mathcal{M}[\mathfrak{r},\mathfrak{s}]) \\ &\le \eta d_b([\mathfrak{p},\mathfrak{q}],[\mathfrak{r},\mathfrak{s}]), \end{aligned}$$

a contradiction. Hence, $\langle\mathcal{C}([\mathfrak{p}_\circ,\mathfrak{q}_\circ],\mathbf{r})\rangle$ is a unique equivalence $\mathcal{C}-$class of a near fixed interval circles of $\mathcal{M}$. □

Remark 4.7. *It is obvious that geometrically (1) implies that $\mathcal{M}[\mathfrak{p},\mathfrak{q}]$ is in the interior of an interval circle and (2) implies that $\mathcal{M}[\mathfrak{p},\mathfrak{q}]$ is in the exterior of an interval circle, that is, $\mathcal{M}\big(\mathcal{C}([\mathfrak{p}_\circ,\mathfrak{q}_\circ],\boldsymbol{r})\big) \subseteq \mathcal{C}([\mathfrak{p}_\circ,\mathfrak{q}_\circ],\boldsymbol{r})$.*

The following example illustrates Theorem 4.6.

Example 4.8. *Let $\mathcal{U}$ be the set of closed and bounded intervals and $\mathcal{N}$ be the null set. Let a $b-$interval metric $d_b : \mathcal{U} \to \mathcal{U}$ be defined as $d_b([\mathfrak{p},\mathfrak{q}],[\mathfrak{r},\mathfrak{s}]) = (\mathfrak{p}+\mathfrak{q}-\mathfrak{r}-\mathfrak{s})^2$ with $\boldsymbol{s} = 2$. Choose $[\alpha,\beta] \in \mathcal{U}$ such that $d_b([-2,2],[\alpha,\beta]) > 4$. The circle*

$$\begin{aligned} \mathcal{C}([-2,2],4) &= \{[\mathfrak{p},\mathfrak{q}] \in \mathcal{U} : d_b([\mathfrak{p},\mathfrak{q}],[-2,2]) = 4\} \\ &= \{[\mathfrak{p},\mathfrak{q}] \in \mathcal{U} : (\mathfrak{p}+\mathfrak{q})^2 = 4\}. \end{aligned}$$

Define a self map $\mathcal{M} : \mathcal{U} \longrightarrow \mathcal{U}$ as $\mathcal{M}[\mathfrak{p},\mathfrak{q}] = \begin{cases} [\mathfrak{p},\mathfrak{q}], & [\mathfrak{p},\mathfrak{q}] \in \mathcal{C}([-2,2],4) \\ [\alpha,\beta], & [\mathfrak{p},\mathfrak{q}] \notin \mathcal{C}([-2,2],4) \end{cases}$
and $d_b([\mathfrak{p},\mathfrak{q}],[\alpha,\beta]) \le \eta d_b([\mathfrak{p},\mathfrak{q}],[\mathfrak{r},\mathfrak{s}])$, where, $\eta \in [0,1)$, $[\mathfrak{p},\mathfrak{q}] \in \mathcal{C}([-2,2],4)$ and $[\mathfrak{r},\mathfrak{s}] \notin \mathcal{C}([-2,2],4)$. Then the self map $\mathcal{M}$ verifies all the postulates of Theorem 4.6, that is, the set of non-unique near fixed points of $\mathcal{M}$, $\{[\mathfrak{p},\mathfrak{q}] \in \mathcal{U} : (\mathfrak{p}+\mathfrak{q})^2 = 4\}$ contains a near fixed interval circle $\mathcal{C}([-2,2],4)$. However, one may notice that there are infinitely many near fixed interval circles contained in the unique equivalence $\mathcal{C}-$class $\langle\mathcal{C}([-2,2],4)\rangle$ of near fixed interval circles of $\mathcal{M}$.

The following examples depict the significance of conditions (1) and (2) in the survival of a near fixed interval circle and a unique equivalence $\mathcal{C}-$class of near fixed interval circles.

Example 4.9. *Let $b-$interval metric be defined as in Example 4.8 and $\mathcal{C}([\mathfrak{p}_\circ,\mathfrak{q}_\circ],\boldsymbol{r})$ be an interval circle defined on $\mathcal{U}$. Now, define a self map $\mathcal{M} : \mathcal{U} \longrightarrow \mathcal{U}$ as $\mathcal{M}[\mathfrak{p},\mathfrak{q}] = [\mathfrak{p}_\circ,\mathfrak{q}_\circ]$, $[\mathfrak{p},\mathfrak{q}] \in \mathcal{U}$.
Then the map $\mathcal{M}$ verifies condition (1) but does not verify conditions (2), (3), and (4) of Theorem 4.6. One may notice that $\mathcal{M}$ does not nearly fix the interval circle $\mathcal{C}([\mathfrak{p}_\circ,\mathfrak{q}_\circ],\boldsymbol{r})$.*

Example 4.10. *Let $b-$interval metric be defined as in Example 4.8 and $\mathcal{C}([\mathfrak{p}_\circ,\mathfrak{q}_\circ],\boldsymbol{r})$ be an interval circle defined on $\mathcal{U}$. Choose a point $[\alpha,\beta] \in \mathcal{U}$ such that $d_b([\mathfrak{p}_\circ,\mathfrak{q}_\circ],[\alpha,\beta]) = \rho > \boldsymbol{r}$. Now, define a self map $\mathcal{M} : \mathcal{U} \longrightarrow \mathcal{U}$ as $\mathcal{M}[\mathfrak{p},\mathfrak{q}] = [\alpha,\beta]$, $[\mathfrak{p},\mathfrak{q}] \in \mathcal{U}$.*

Then the map $\mathcal{M}$ verifies condition (2) but does not verify conditions (1), (3), and (4) of Theorem 4.6. One may notice that $\mathcal{M}$ does not nearly fix the interval circle $\mathcal{C}([\mathfrak{p}_\circ,\mathfrak{q}_\circ],\boldsymbol{r})$.

Theorem 4.11. *Theorem 4.6 still holds even if we substitute* (1) *by* $(1)'$, (2) *by* $(2)'$.

$(1)'$ $d_b([\mathfrak{p},\mathfrak{q}],\mathcal{M}[\mathfrak{p},\mathfrak{q}]) \le \zeta([\mathfrak{p},\mathfrak{q}]) + \zeta(\mathcal{M}[\mathfrak{p},\mathfrak{q}]) - 2\boldsymbol{r}$;

$(2)'$ $d_b(\mathcal{M}[\mathfrak{p},\mathfrak{q}],[\mathfrak{p}_\circ,\mathfrak{q}_\circ]) \le \boldsymbol{r}$.

Proof . Let $[\mathfrak{p},\mathfrak{q}] \in \mathcal{C}([\mathfrak{p}_\circ,\mathfrak{q}_\circ],\mathbf{r})$ be any arbitrary point. Using $(1)'$ and equation (4.3)

$$\begin{aligned} d_b([\mathfrak{p},\mathfrak{q}],\mathcal{M}[\mathfrak{p},\mathfrak{q}]) &\le d_b([\mathfrak{p},\mathfrak{q}],[\mathfrak{p}_\circ,\mathfrak{q}_\circ]) + d_b(\mathcal{M}[\mathfrak{p},\mathfrak{q}] - 2\mathbf{r},[\mathfrak{p}_\circ,\mathfrak{q}_\circ]) \\ &= \mathbf{r} + d_b(\mathcal{M}[\mathfrak{p},\mathfrak{q}],[\mathfrak{p}_\circ,\mathfrak{q}_\circ]) - 2\mathbf{r} \\ &\le \mathbf{r}+\mathbf{r} - 2\mathbf{r} = 0, \quad \text{using} \quad (2)', \end{aligned}$$

and so $\mathcal{M}[\mathfrak{p},\mathfrak{q}] \overset{\mathcal{N}}{=} [\mathfrak{p},\mathfrak{q}]$. Now, $\mathcal{C}([\mathfrak{p}_\circ,\mathfrak{q}_\circ],\mathbf{r})$ is near fixed circle of $\mathcal{M}$ and uniqueness of equivalence interval $\mathcal{C}-$class of near fixed interval circle of $\mathcal{M}$ may be proved as in Theorem 4.6. □

Remark 4.12. *It is obvious that geometrically* $(1)'$ *implies that* $\mathcal{M}[\mathfrak{p},\mathfrak{q}]$ *is in the exterior of an interval circle and* $(2)'$ *implies that* $\mathcal{M}[\mathfrak{p},\mathfrak{q}]$ *is in the interior of an interval circle, that is* $\mathcal{M}\big(\mathcal{C}([\mathfrak{p}_\circ,\mathfrak{q}_\circ],\boldsymbol{r})\big) \subseteq \mathcal{C}([\mathfrak{p}_\circ,\mathfrak{q}_\circ],\boldsymbol{r})$.

Example 4.13. *Let* $\mathcal{U}$ *be the set of closed and bounded intervals and* $\mathcal{N}$ *be the null set. Let a* $b-$*interval metric* $d_b : \mathcal{U} \to \mathcal{U}$ *be defined as* $d_b([\mathfrak{p},\mathfrak{q}],[\mathfrak{r},\mathfrak{s}]) = |\mathfrak{p}+\mathfrak{q}-\mathfrak{r}-\mathfrak{s}|^3$ *with* $\boldsymbol{s} = 4$. *Choose,* $[\alpha,\beta] \in \mathcal{U}$ *such that* $d_b([2,9],[\alpha,\beta]) < 10$. *The interval circle*

$$\begin{aligned} \mathcal{C}([2,9],10) &= \{[\mathfrak{p},\mathfrak{q}] \in \mathcal{U} : d_b([\mathfrak{p},\mathfrak{q}],[2,9]) = 10\} \\ &= \{[\mathfrak{p},\mathfrak{q}] \in \mathcal{U} : |\mathfrak{p}+\mathfrak{q}-11|^3 = 10\}. \end{aligned}$$

Define a self map $\mathcal{M} : \mathcal{U} \longrightarrow \mathcal{U}$ *as* $\mathcal{M}[\mathfrak{p},\mathfrak{q}] = \begin{cases} [\mathfrak{p},\mathfrak{q}], & [\mathfrak{p},\mathfrak{q}] \in \mathcal{C}([2,9],10) \\ [\alpha,\beta], & [\mathfrak{p},\mathfrak{q}] \notin \mathcal{C}([2,9],10) \end{cases}.$
Then the self map $\mathcal{M}$ *verifies all the postulates* $(1)'$ *and* $(2)'$ *of Theorem 4.11 except* (3), *that is, the set of non-unique near fixed points of* $\mathcal{M}$, $\{[\mathfrak{p},\mathfrak{q}] \in \mathcal{U} : |\mathfrak{p}+\mathfrak{q}-\ 11|^3 = 10\}$ *contains a near fixed interval circle* $\mathcal{C}([-2,2],4)$. *However, one may notice that there does not exist any* $\eta \in [0,1)$ *such that postulate* (3) *is satisfied, even then there are infinitely many near fixed interval circles contained in the unique equivalence* $\mathcal{C}-$*class* $\big\langle \mathcal{C}([2,9],10) \big\rangle$ *of near fixed interval circle of* $\mathcal{M}$.

The following example depicts the significance of conditions $(1)'$ and $(2)'$ in the survival of a near fixed interval circle.

Example 4.14. *Let a* $b-$*interval metric be defined as in Example 4.13 and* $\mathcal{C}([\mathfrak{p}_\circ,\mathfrak{q}_\circ],\boldsymbol{r})$ *be a near fixed interval circle defined on* $\mathcal{U}$. *Choose, a point* $[\alpha,\beta] \in \mathcal{U}$ *such that* $d_b([\mathfrak{p}_\circ,\mathfrak{q}_\circ],[\alpha,\beta]) = \rho < \boldsymbol{r}$. *Now, define a self map* $\mathcal{M} : \mathcal{U} \longrightarrow \mathcal{U}$ *as* $\mathcal{M}[\mathfrak{p},\mathfrak{q}] = [\alpha,\beta],\ [\mathfrak{p},\mathfrak{q}] \in \mathcal{U}$.

Then the map $\mathcal{M}$ *verifies condition* $(2)'$, *but does not verify conditions* $(1)'$, (3), *and* (4) *of Theorem 4.11. One may notice that* $\mathcal{M}$ *does not nearly fix the interval circle* $\mathcal{C}([\mathfrak{p}_\circ,\mathfrak{q}_\circ],\boldsymbol{r})$.

5. Conclusion

Acknowledging the work of Wu [20] and motivated by the fact that the set of closed and bounded intervals on the collection of real numbers may not be a vector space, we have equipped $b-$interval metric to the set of closed and bounded intervals on the collection of real numbers by utilizing the notion of a null set. Our novel distance structure, $b-$interval metric space, is an extension and a generalization of an interval metric space [20], which is different from the standard metric space [7]. We have also included examples to demonstrate that a $b-$interval metric is neither a $b-$metric ([1],[5]) nor an interval metric [20]. Further, we have introduced notions like $b-$convergence, $b-$Cauchy sequence, completeness and demonstrated that the set of open balls, which forms a basis on a $b-$interval metric space, generates a $\mathcal{T}_0-$topology on it. Example 3.17 demonstrates that a conventional Banach contraction principle may not be proved in a $b-$interval metric space. Also, Example 3.22 demonstrates that the Theorem 5 of Sehgal [17] may not be proved via $b-$interval metric. Thereby, we have deduced that the celebrated results in metric fixed point theory may not be proved in a novel $b-$interval metric space. However, Examples 3.17 and 3.22 demonstrated the significant fact that $b-$interval metric space have created an environment for the survival of a near fixed point and a unique equivalence class of near fixed point. We have concluded the paper by introducing notions of interval circle, near fixed interval circle, its equivalence class, and establishing the existence of near fixed interval circle and its equivalence class.

6. Competing interest

All the authors declare that there is no conflict of interest.

7. Authors' contributions

All authors contributed equally and significantly in writing this article.

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