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Existence and stability analysis for nonlinear Ψ-Hilfer fractional differential equations with nonlocal integral boundary conditions

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Abstract

In this paper, we study the existence and uniqueness of mild solutions for nonlinear fractional differential equations subject to nonlocal integral boundary conditions in the frame of a ψ-Hilfer fractional derivative. Further, we discuss different kinds of stability of Ulam-Hyers for mild solutions to the given problem. Using the fixed point theorems together with generalized Gronwall inequality the desired outcomes are proven. The obtained results generalize many previous works that contain special cases of function ψ. At the end, some pertinent examples demonstrating the effectiveness of the theoretical results are presented.

Keywords: Ψ-Hilfer fractional derivative, mild solution, Ulam-Hyers stability, fixed point theorems
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1. Introduction

In latest years, fractional differential equations (FDE's) theory has received very broad regard in the fields of pure and applied mathematics, see [14, 18, 19, 25] and emerge naturally in diverse scopes of science, with many applications, e.g. [12, 13, 17, 20, 34].

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In 1999 Hilfer introduced the generalization of Riemann-Liouville and Caputo fractional derivatives (FD's) see [17]. The fundamental works on the theory of FDE's with Hilfer derivative can be found in [9, 15, 16, 22, 35]. In 2018, Sousa and Oliveira [31] presented the so-called ψ-Hilfer FD with respect to another function, to unify in one fractional operator a large number of fractional derivatives and thus, window open to new applications. Some works involving the fractional derivative ψ-Hilfer, can be found in [2, 4, 5, 33, 37].

One of the crucial and interesting areas of research in the theory of functional equations is devoted to the stability analysis. Stability analysis is the fundamental property of the mathematical analysis which has got paramount importance in many fields of engineering and science. In the existing literature, there are stabilities such as Mittag Leffler, h-stability, exponential, Lyapunov stability and so on. In the nineteenth-century, Ulam and Hyers presented an interesting type of stability called Ulam-Hyers stability, which, nowadays has been picked up a great deal of consideration due to a wide range of applications in many fields of science such as optimization and mathematical modeling.

The stability of the Ulam can be viewed as a special kind of data dependence which was initiated by the Ulam in [27]. Rassias in [26] extended the concept of UH stability. Many authors subsequently discussed different UH stability problem for various types of fractional integral (FI) and FDE's utilizing various techniques, see [1, 3, 6, 7, 8, 23, 24, 29, 32, 38] and the references therein.

Recently, in [15] the authors prove the existence and uniqueness of global solution for nonlinear FDE's of the type

$$\begin{cases} {}^{H}D^{\mathrm{a}_1,\mathrm{a}_2}\varkappa(\varrho) = f(\varrho,\varkappa(\varrho)), \ \varrho\in(a,b), \ a>0, \\ I_{a+}^{1-\gamma}\varkappa(a) = \varkappa_a, \ \gamma = \mathrm{a}_1+\mathrm{a}_2-\alpha\beta, \end{cases}$$

where ${}^{H}D^{\mathrm{a}_1,\mathrm{a}_2}$ is the Hilfer FD of order $\mathrm{a}_1\in(0,1)$ and type $\mathrm{a}_2\in[0,1]$.

In [9], Asawasamrit et al. they studied the following Hilfer FDE with nonlocal integral boundary conditions

$$\begin{cases} {}^{H}D^{\mathrm{a}_1,\mathrm{a}_2}\varkappa(\varrho) = f(\varrho,\varkappa(\varrho)), \ \varrho\in[a,b], \\ \varkappa(a)=0, \ \varkappa(b) = \sum\limits_{i=1}^{m}\theta_i I_{a+}^{\eta_i}\varkappa(\delta_i), \ \delta_i\in[a,b], \end{cases} \tag{1.1}$$

where $I_{a+}^{\eta_i}$ is the Riemann-Liouville FI of order $\eta_i>0$, $1<\mathrm{a}_1<2$, $0\le \mathrm{a}_2\le 1$ and $\theta_i\in\mathbb{R}$. On the other hand, the authors in [21] have investigated the existence and stability results of implicit problem for FDE (1.1) involving ψ-Hilfer fractional derivative.

Motivated by the aforementioned works and inspired by [31], we study existence, uniqueness and Ulam stability of the following nonlinear FDE involving ψ-Hilfer fractional derivative with nonlocal integral boundary conditions

$$\begin{cases} {}^{H}D_{a+}^{\mathrm{a}_1,\mathrm{a}_2;\psi}\varkappa(\varrho) = f(\varrho,\varkappa(\varrho)), \ \varrho\in(a,b), \ a>0, \\ \varkappa(a)=0, \ I_{a+}^{2-\gamma;\psi}\varkappa(b) = \sum\limits_{i=1}^{m}\theta_i I_{a+}^{\eta_i;\psi}\varkappa(\delta_i), \end{cases} \tag{1.2}$$

where ${}^{H}D_{a+}^{\mathrm{a}_1,\mathrm{a}_2;\psi}$ is the left sided ψ-Hilfer FD of order $\mathrm{a}_1\in(1,2)$ and type $\mathrm{a}_2\in[0,1]$, $I_{a+}^{2-\gamma;\psi}$, $I_{a+}^{\eta_i;\psi}$ are the left sided ψ-Riemann-Liouville FI of orders $2-\gamma$, $\eta_i>0$ respectively, $\gamma=\mathrm{a}_1+\mathrm{a}_2(2-\mathrm{a}_1)\in(1,2]$, $-\infty<a<b<\infty$, $\theta_i\in\mathbb{R}$, $i=1,2,...,m$, $0\le a\le\delta_1<\delta_2<\delta_3<...<\delta_m\le b$, and $f:[a,b]\times\mathbb{R}\to\mathbb{R}$ is a given continuous function.

The considered problem involves a general operator. More precisely, for various values of a_2 and ψ, the problem (1.2) is reduced to FDE's involving the FDs like Hilfer, Caputo, Riemann-Liouville, Katugampola, Erdelyi-Kober, Hadamard, and many other FDs, which yields to generalize many pervious works.

The organization of the rest of the paper divided of four sections. In Section 2, some notations, definitions of fractional calculus, ulam stability and fixed point theorems are presented. In Section 3, Some useful results about the existence, uniqueness and Ulam stability of the nonlinear FDE are obtained. In Section 4, we present some examples to illustrates the effectiveness of the theoretical results.

2. Preliminary notions

In this part, we give some essential ideas of fractional calculus, definitions of various types of Ulam stability and results of nonlinear analysis (fixed point theorems and generalized Gronwall's inequality) that prerequisite in our analysis.

Let $J = [a, b]$, $\mathrm{a}_1 \in (1, 2)$, $\mathrm{a}_2 \in [0, 1]$. By $\mathcal{C} = C(J, \mathbb{R})$ we denote the Banach space of all continuous functions $\varkappa : J \to \mathbb{R}$ with norm

$$\|\varkappa\| = \sup\{|\varkappa(\varrho)| : \varrho \in J\},$$

and $L^1(J, \mathbb{R})$ be the Banach space of Lebesgue integrable functions $\varkappa : J \to \mathbb{R}$ with norm

$$\|\varkappa\|_{L^1} = \int_J |\varkappa(\varrho)|\, d\varrho.$$

Let $\upsilon : J \to \mathbb{R}$ be an integrable function and $\psi \in C^n(J, \mathbb{R})$ an increasing function such that $\psi'(\varrho) \neq 0$, for any $\varrho \in J$.

Definition 2.1 ([18]). *The a_1^{th}-ψ-Riemann-Liouville FI of a function υ is described by*

$$I_{a+}^{\mathrm{a}_1;\psi}\upsilon(\varrho) = \frac{1}{\Gamma(\mathrm{a}_1)}\int_a^\varrho \psi'(\vartheta)(\psi(\varrho) - \psi(\vartheta))^{\mathrm{a}_1 - 1}\upsilon(\vartheta)\, d\vartheta.$$

Definition 2.2 ([18]). *The a_1^{th}-ψ-Riemann-Liouville FD of a function υ is described by*

$$D_{a+}^{\mathrm{a}_1;\psi}\upsilon(\varrho) = \left(\frac{1}{\psi'(\varrho)}\frac{d}{dt}\right)^n I_{a+}^{(n-\mathrm{a}_1);\psi}\upsilon(\varrho),$$

where $n = [\mathrm{a}_1] + 1$, $n \in \mathbb{N}$.

Definition 2.3 ([31]). *The ψ-Hilfer FD of a function υ of order a_1 and type a_2 is described by*

$${}^H D_{a+}^{\mathrm{a}_1,\mathrm{a}_2;\psi}\upsilon(\varrho) = I_{a+}^{\mathrm{a}_2(n-\mathrm{a}_1);\psi}\ D_\psi^{[n]}\ I_{a+}^{(1-\mathrm{a}_2)(n-\mathrm{a}_1);\psi}\upsilon(\varrho),$$

where $D_\psi^{[n]} = \left(\frac{1}{\psi'(\varrho)}\frac{d}{dt}\right)^n$.

Lemma 2.4 ([18, 31]). *Let $\mathrm{a}_1, \mathrm{a}_2, \mu > 0$. Then*
1) $I_{a+}^{\mathrm{a}_1;\psi} I_{a+}^{\mathrm{a}_2;\psi}\varkappa(\varrho) = I_{a+}^{\mathrm{a}_1+\mathrm{a}_2;\psi}\varkappa(\varrho)$.
2) $I_{a+}^{\mathrm{a}_1;\psi}(\psi(\varrho) - \psi(a))^{\mu-1} = \frac{\Gamma(\mu)}{\Gamma(\mathrm{a}_1+\mu)}(\psi(\varrho) - \psi(a))^{\mathrm{a}_1+\mu-1}$.

Lemma 2.5 ([31]). *If $x \in C^n(J, \mathbb{R})$, $\mathrm{a}_1 \in (n-1, n)$ and $\mathrm{a}_2 \in (0, 1)$, then*
1) $I_{a+}^{\mathrm{a}_1;\psi}\ {}^H D_{a+}^{\mathrm{a}_1,\mathrm{a}_2;\psi}\ \varkappa(\varrho) = \varkappa(\varrho) - \sum_{k=1}^n \frac{(\psi(\varrho)-\psi(a))^{\mathfrak{v}-k}}{\Gamma(\mathfrak{v}-k+1)}\left(\frac{1}{\psi'(\varrho)}\frac{d}{dt}\right)^{n-k} I_{a+}^{(1-\mathrm{a}_2)(n-\mathrm{a}_1);\psi}\varkappa(a)$.
2) ${}^H D_{a+}^{\mathrm{a}_1,\mathrm{a}_2;\psi}\ I_{a+}^{\mathrm{a}_1;\psi}\varkappa(\varrho) = \varkappa(\varrho)$.

To define Ulam's stability, we consider the following FDE

$$ {}^{H}D_{a+}^{\mathrm{a}_1,\mathrm{a}_2;\psi}\varkappa(\varrho)=f(\varrho,\varkappa(\varrho)),\ \ \varrho\in J. \tag{2.1}$$

Definition 2.6 ([28]). *The equation (Eq) (2.1) is said to be UH stable if there is a number $k\in\mathbb{R}^*$ such that for each $\epsilon>0$ and for each $\tilde{\varkappa}\in\mathcal{C}$ solution of the inequality*

$$\left|{}^{H}D_{a+}^{\mathrm{a}_1,\mathrm{a}_2;\psi}\tilde{\varkappa}(\varrho)-f(\varrho,\tilde{\varkappa}(\varrho))\right|\le\epsilon,\ \ \varrho\in J, \tag{2.2}$$

there is a solution $\varkappa\in\mathcal{C}$ of the Eq (2.1) with

$$|\tilde{\varkappa}(\varrho)-\varkappa(\varrho)|\le k_f\epsilon,\ \ \varrho\in J.$$

Definition 2.7 ([28]). *Assume that $\tilde{\varkappa}\in\mathcal{C}$ satisfies the inequality in (2.2) and $\varkappa\in\mathcal{C}$ is a solution of the Eq (2.1). If there is a function $\phi_f\in C(\mathbb{R}^+,\mathbb{R}^+)$ with $\phi_f(0)=0$ satisfying*

$$|\tilde{\varkappa}(\varrho)-\varkappa(\varrho)|\le\phi_f(\epsilon),\ \ \varrho\in J.$$

Then the Eq (2.1) is said to be generalized Ulam-Hyres (GUH) stable.

Definition 2.8 ([28]). *The Eq (2.1)is said to be UHR stable with respect to $\phi_f\in C(J,\mathbb{R}^+)$ if there is a number $k\in\mathbb{R}^*$ such that for each $\epsilon>0$ and for each $\tilde{\varkappa}\in\mathcal{C}$ solution of the inequality*

$$\left|{}^{H}D_{a+}^{\mathrm{a}_1,\mathrm{a}_2;\psi}\tilde{\varkappa}(\varrho)-f(\varrho,\tilde{\varkappa}(\varrho))\right|\le\epsilon\phi_f(\varrho),\ \ \varrho\in[0,1], \tag{2.3}$$

there is a solution $\varkappa\in\mathcal{C}$ of the Eq (2.1) with

$$|\tilde{\varkappa}(\varrho)-\varkappa(\varrho)|\le k_{\phi,f}\phi_f(\varrho)\epsilon,\ \ \varrho\in J.$$

Definition 2.9 ([28]). *Assume that $\tilde{\varkappa}\in\mathcal{C}$ satisfies the inequality in (2.3) and $\varkappa\in\mathcal{C}$ is a solution of the Eq (2.1). If there is a constant $k_{\phi,f}>0$ such that*

$$|\tilde{\varkappa}(\varrho)-\varkappa(\varrho)|\le k_{\phi,f}\phi_f(\varrho),\ \ \varrho\in J.$$

Then the Eq (2.1) is said to be generalized Ulam-Hyres-Rassias (GUHR) stable.

Remark 2.10. *If there is a function $v\in\mathcal{C}$ (dependent on $\tilde{\varkappa}$), such that*
1) $|v(\varrho)|\le\epsilon$, for all $\varrho\in J$,
2) ${}^{H}D_{a+}^{\mathrm{a}_1,\mathrm{a}_2;\psi}\tilde{\varkappa}(\varrho)=f(\varrho,\tilde{\varkappa}(\varrho))+v(\varrho)$, $\varrho\in J$.
Then the function $\tilde{\varkappa}\in\mathcal{C}$ is a solution of the inequality (2.2).

We state the following generalization of Gronwall's Lemma.

Lemma 2.11 ([36]). *Let u and v be two integrable functions, z be continuous with domain $[a,b]$ and ψ is defined at the beginning. Suppose that*
1) u and v are nonnegative,
2) z is nonnegative and nondecreasing.
If

$$u(\varrho)\le v(\varrho)+z(\varrho)\int_a^{\varrho}\psi'(\vartheta)(\psi(\varrho)-\psi(\vartheta))^{\mathrm{a}_1-1}u(\vartheta)\,d\vartheta,$$

then

$$u(\varrho)\le v(\varrho)+\int_a^{\varrho}\sum_{k=1}^{\infty}\frac{[z(\varrho)\Gamma(\mathrm{a}_1)]^k}{\Gamma(k\alpha)}\psi'(\vartheta)(\psi(\varrho)-\psi(\vartheta))^{k\alpha-1}v(\vartheta)\,d\vartheta.$$

Corollary 2.12 ([36]). *Under the hypotheses of Lemma 2.11, assume further that $v(\varrho)$ is nondecreasing function for $\varrho \in [a,b]$. Then*

$$u(\varrho) \leq v(\varrho) E_{a_1}\left(z(\varrho) \Gamma(a_1) (\psi(\varrho) - \psi(\vartheta))^{a_1}\right),$$

where $E_{a_1}(.)$ is the Mittag-Leffler function of one parameter, defined as

$$E_{a_1}(\varrho) = \sum_{k=0}^{\infty} \frac{\varrho^k}{\Gamma(k\alpha + 1)}.$$

Theorem 2.13 (Banach fixed point theorem [30]). *Let $\Omega \neq \emptyset$ be a closed subset of a Banach space $(\mathcal{X}, \|.\|)$. If $\tilde{\mathcal{S}} : \Omega \to \Omega$ is a contraction mapping. Then, $\tilde{\mathcal{S}}$ admits a unique fixed point.*

Theorem 2.14 (Schauder fixed point theorem [30]). *Let $\Omega \neq \emptyset$ be a bounded closed convex subset of a Banach space $\mathcal{X}$. If $\tilde{\mathcal{S}} : \Omega \to \Omega$ be a continuous compact operator. Then, $\tilde{\mathcal{S}}$ has a fixed point in Ω.*

To obtain our results, we need the following lemma.

Lemma 2.15. *Let*

$$\Lambda = \frac{(\psi(b) - \psi(a))}{\Gamma(2)} - \sum_{i=1}^{m} \frac{\theta_i}{\Gamma(\gamma + \eta_i)} (\psi(\delta_i) - \psi(a))^{\gamma + \eta_i - 1} \neq 0, \tag{2.4}$$

and for any $q \in C(J)$, then the nonlocal boundary value problem

$$\begin{cases} {}^{H}D_{a+}^{a_1, a_2; \psi} \varkappa(\varrho) = q(\varrho), \ \varrho \in (a,b), \\ \varkappa(a) = 0, \ I_{a+}^{2-\gamma;\psi} \varkappa(b) = \sum_{i=1}^{m} \theta_i I_{a+}^{\eta_i;\psi} \varkappa(\delta_i), \end{cases} \tag{2.5}$$

has a unique mild solution given by

$$\varkappa(\varrho) = \frac{(\psi(\varrho) - \psi(a))^{\gamma-1}}{\Lambda \Gamma(\gamma)} \left(\sum_{i=1}^{m} \theta_i I_{a+}^{a_1+\eta_i;\psi} q(\delta_i) - I_{a+}^{2+a_1-\gamma;\psi} q(b) \right) + I_{a+}^{a_1;\psi} q(\varrho). \tag{2.6}$$

Proof . Taking ψ-FI $I_{a+}^{a_1;\psi}$ to the first equation of (2.5),and from Lemma 2.5, we get

$$\varkappa(\varrho) - \sum_{k=1}^{2} \frac{(\psi(\varrho) - \psi(a))^{\gamma-k}}{\Gamma(\gamma - k + 1)} h_{\psi}^{[2-k]} I_{a+}^{(1-a_2)(2-a_1);\psi} \varkappa(a) = I_{a+}^{a_1;\psi} q(\varrho), \ \varrho \in J. \tag{2.7}$$

We have $(1 - a_2)(2 - a_1) = 2 - \gamma$. Therefore

$$\begin{aligned} \varkappa(\varrho) = & \frac{(\psi(\varrho) - \psi(a))^{\gamma-1}}{\Gamma(\gamma)} \left(\frac{1}{\psi'(\varrho)} \frac{d}{dt} \right) I_{a+}^{2-\gamma;\psi} \varkappa(\varrho) \Big|_{\varrho=a} \\ & + \frac{(\psi(\varrho) - \psi(a))^{\gamma-2}}{\Gamma(\gamma - 1)} I_{a+}^{2-\gamma;\psi} \varkappa(\varrho) \Big|_{\varrho=a} + I_{a+}^{a_1;\psi} q(\varrho) \\ = & \frac{(\psi(\varrho) - \psi(a))^{\gamma-1}}{\Gamma(\gamma)} D^{\gamma-1;\psi} \varkappa(\varrho) \Big|_{\varrho=a} + \frac{(\psi(\varrho) - \psi(a))^{\gamma-2}}{\Gamma(\gamma - 1)} I_{a+}^{2-\gamma;\psi} \varkappa(\varrho) \Big|_{\varrho=a} + I_{a+}^{a_1;\psi} q(\varrho). \end{aligned}$$

Put

$$c_1 = D^{\gamma-1;\psi} \varkappa(\varrho)\Big|_{\varrho=a} \text{ and } c_2 = I_{a+}^{2-\gamma;\psi} \varkappa(\varrho)\Big|_{\varrho=a}, \ \varrho \in J.$$

Then

$$\varkappa(\varrho) = \frac{(\psi(\varrho) - \psi(a))^{\gamma-1}}{\Gamma(\gamma)} c_1 + \frac{(\psi(\varrho) - \psi(a))^{\gamma-2}}{\Gamma(\gamma-1)} c_2 + I_{a+}^{\mathrm{a}_1;\psi} q(\varrho).$$

Because $\lim\limits_{\varrho \to a} (\psi(\varrho) - \psi(a))^{\gamma-2} = \infty$, in the view of boundary conditions $\varkappa(a) = 0$, we must have

$$c_2 = 0.$$

Replacing c_2 by their value in (2.7)), we get

$$\varkappa(\varrho) = \frac{(\psi(\varrho) - \psi(a))^{\gamma-1}}{\Gamma(\gamma)} c_1 + I_{a+}^{\mathrm{a}_1;\psi} q(\varrho). \tag{2.8}$$

Next, we use the second boundary condition to determine the constant c_1 . Applying $I_{a+}^{\eta_i;\psi}$ on both side of equation (2.8), we get

$$I_{a+}^{\eta_i;\psi} \varkappa(\varrho) = \frac{c_1}{\Gamma(\gamma+\eta_i)} (\psi(\varrho) - \psi(a))^{\gamma+\eta_i-1} + I_{a+}^{\mathrm{a}_1+\eta_i;\psi} q(\varrho). \tag{2.9}$$

From the condition $\varkappa(b) = \sum\limits_{i=1}^{m} \theta_i I_{a+}^{\eta_i;\psi} \varkappa(\delta_i)$ and (2.9), we have

$$\begin{aligned} \varkappa(b) &= \sum_{i=1}^{m} \theta_i I_{a+}^{\eta_i;\psi} \varkappa(\delta_i) \\ &= c_1 \sum_{i=1}^{m} \frac{\theta_i}{\Gamma(\gamma+\eta_i)} (\psi(\delta_i) - \psi(a))^{\gamma+\eta_i-1} + \sum_{i=1}^{m} \theta_i I_{a+}^{\mathrm{a}_1+\eta_i;\psi} q(\delta_i). \end{aligned} \tag{2.10}$$

From equations (2.8) and (2.10), we have

$$\begin{aligned} I_{a+}^{2-\gamma;\psi} \varkappa(b) &= \frac{(\psi(b) - \psi(a))}{\Gamma(2)} c_1 + I_{a+}^{2+\mathrm{a}_1-\gamma;\psi} q(b) \\ &= c_1 \sum_{i=1}^{m} \frac{\theta_i}{\Gamma(\gamma+\eta_i)} (\psi(\delta_i) - \psi(a))^{\gamma+\eta_i-1} + \sum_{i=1}^{m} \theta_i I_{a+}^{\mathrm{a}_1+\eta_i;\psi} q(\delta_i). \end{aligned}$$

Thus, we find

$$c_1 = \frac{1}{\Lambda} \left(\sum_{i=1}^{m} \theta_i I_{a+}^{\mathrm{a}_1+\eta_i;\psi} q(\delta_i) - I_{a+}^{2+\mathrm{a}_1-\gamma;\psi} q(b) \right).$$

Substituting the value of c_1 into (2.8), we obtain the fractional integral equation (2.6). □

3. Existence results

In what follows, we apply some fixed point theorems to demonstrate the existence and uniqueness results for problem (1.2).

To obtain our findings, We need the following assumptions.

(As1) There is a constants $l_1 > 0$ such that

$$\left|f(\varrho,\varkappa) - f(\varrho,\tilde{\varkappa})\right| \le l_1 \left|\varkappa - \tilde{\varkappa}\right|.$$

(As2) There is a function $w \in C(J,\mathbb{R}^+)$ such that

$$\left|f(\varrho,\varkappa)\right| \le w(\varrho),\ \forall (\varrho,\varkappa) \in J \times \mathbb{R}.$$

For the sake of convenience, we put

$$k_1 = \sum_{i=1}^{m} |\theta_i| \frac{(\psi(b)-\psi(a))^{\mathrm{a}_1+\eta_i+\gamma-1}}{|\Lambda|\Gamma(\gamma)\Gamma(\mathrm{a}_1+\eta_i+1)},\ k_2 = \frac{(\psi(b)-\psi(a))^{1+\mathrm{a}_1}}{|\Lambda|\Gamma(\gamma)\Gamma(3+\mathrm{a}_1-\gamma)},\ k_3 = \frac{(\psi(b)-\psi(a))^{\mathrm{a}_1}}{\Gamma(\mathrm{a}_1+1)},$$

$$A_{\varkappa} = \left(\sum_{i=1}^{m} \frac{|\theta_i|}{\Gamma(\mathrm{a}_1+\eta_i)} \int_a^{\delta_i} \psi'(\vartheta)(\psi(\delta_i)-\psi(\vartheta))^{\mathrm{a}_1+\eta_i-1} f(\vartheta,\varkappa(\vartheta))\,d\vartheta \right.$$
$$\left. -\frac{1}{\Gamma(2+\mathrm{a}_1-\gamma)} \int_a^{b} \psi'(\vartheta)(\psi(b)-\psi(\vartheta))^{1+\mathrm{a}_1-\gamma} f(\vartheta,\varkappa(\vartheta))\,d\vartheta \right). \tag{3.1}$$

3.1. Existence and uniqueness results via Banach's fixed point theorem

Theorem 3.1. *Let (As1) valid. If*

$$(k_1+k_2+k_3)\, l_1 < 1, \tag{3.2}$$

then, (1.2) has a unique mild solution on J, where k_1, k_2, k_3 are given by (3.1).

Proof . We switch the problem (1.2) into a fixed point problem, we consider the operator $\tilde{\mathcal{S}} : \mathcal{C} \to \mathcal{C}$ as

$$\left(\tilde{\mathcal{S}}\varkappa\right)(\varrho) = \frac{(\psi(\varrho)-\psi(a))^{\gamma-1}}{|\Lambda|\Gamma(\gamma)} \left(\sum_{i=1}^{m} \frac{|\theta_i|}{\Gamma(\mathrm{a}_1+\eta_i)} \int_a^{\delta_i} \psi'(\vartheta)(\psi(\delta_i)-\psi(\vartheta))^{\mathrm{a}_1+\eta_i-1} f(\vartheta,\varkappa(\vartheta))\,d\vartheta \right.$$
$$\left. -\frac{1}{\Gamma(2+\mathrm{a}_1-\gamma)} \int_a^{b} \psi'(\vartheta)(\psi(b)-\psi(\vartheta))^{1+\mathrm{a}_1-\gamma} f(\vartheta,\varkappa(\vartheta))\,d\vartheta \right)$$
$$+\frac{1}{\Gamma(\mathrm{a}_1)} \int_a^{\varrho} \psi'(\vartheta)(\psi(\varrho)-\psi(\vartheta))^{\mathrm{a}_1-1} f(\vartheta,\varkappa(\vartheta))\,d\vartheta.$$

Clearly, the solution of (1.2) is as a fixed point of the operator $\tilde{\mathcal{S}}$.

By (As1), for any $\varkappa, \tilde{\varkappa} \in \mathcal{C}$ and $\varrho \in J$, we get

$$\left|\left(\tilde{\mathcal{S}}\varkappa\right)(\varrho) - \left(\tilde{\mathcal{S}}\tilde{\varkappa}\right)(\varrho)\right|$$
$$\le \frac{(\psi(\varrho)-\psi(a))^{\gamma-1}}{|\Lambda|\Gamma(\gamma)} \left(\sum_{i=1}^{m} \frac{|\theta_i|}{\Gamma(\mathrm{a}_1+\eta_i)} \int_a^{\delta_i} \psi'(\vartheta)(\psi(\delta_i)-\psi(\vartheta))^{\mathrm{a}_1+\eta_i-1} \left|f(\vartheta,\varkappa(\vartheta)) - f(\vartheta,\tilde{\varkappa}(\vartheta))\right| d\vartheta \right.$$
$$\left. +\frac{1}{\Gamma(2+\mathrm{a}_1-\gamma)} \int_a^{b} \psi'(\vartheta)(\psi(b)-\psi(\vartheta))^{1+\mathrm{a}_1-\gamma} \left|f(\vartheta,\varkappa(\vartheta)) - f(\vartheta,\tilde{\varkappa}(\vartheta))\right| d\vartheta \right)$$
$$+\frac{1}{\Gamma(\mathrm{a}_1)} \int_a^{\varrho} \psi'(\vartheta)(\psi(\varrho)-\psi(\vartheta))^{\mathrm{a}_1-1} \left|f(\vartheta,\varkappa(\vartheta)) - f(\vartheta,\tilde{\varkappa}(\vartheta))\right| d\vartheta$$
$$\le l_1 \left(\sum_{i=1}^{m} |\theta_i| \frac{(\psi(b)-\psi(a))^{\mathrm{a}_1+\eta_i+\gamma-1}}{|\Lambda|\Gamma(\gamma)\Gamma(\mathrm{a}_1+\eta_i+1)} + \frac{(\psi(b)-\psi(a))^{1+\mathrm{a}_1}}{|\Lambda|\Gamma(\gamma)\Gamma(3+\mathrm{a}_1-\gamma)} + \frac{(\psi(b)-\psi(a))^{\mathrm{a}_1}}{\Gamma(\mathrm{a}_1+1)}\right) \|\varkappa - \tilde{\varkappa}\|.$$

Thus

$$\left\|\left(\tilde{\mathcal{S}}\varkappa\right)-\left(\tilde{\mathcal{S}}\tilde{\varkappa}\right)\right\|\leq\left(k_1+k_2+k_3\right)l_1\left\|\varkappa-\tilde{\varkappa}\right\|.$$

From (3.2), $\tilde{\mathcal{S}}$ is a contraction. As an outcome of Banach's fixed point theorem, $\tilde{\mathcal{S}}$ has a unique fixed point which is a unique mild solution of (1.2) on J. □

3.2. Existence results via Schauder's fixed point theorem

Theorem 3.2. *Suppose that the hypothesis (As2) is satisfied. Then, (1.2) has at least one mild solution on J.*

Proof . Let $\Omega=\{\varkappa\in\mathcal{C}:\|\varkappa\|\leq M_0\}$ be a non-empty closed bounded convex subset of $\mathcal{C}$, and M_0 is chosen such

$$M_0\geq w^*\left(k_1+k_2+k_3\right),$$

where k_1, k_2, k_3 are given by (3.1), $w^*=\sup\{w(\varrho):\varrho\in J\}$. It is a known that continuity of the functions f implies that the operator $\tilde{\mathcal{S}}$ is continuous. It remain to demonstrate that the operator $\tilde{\mathcal{S}}$ is compact and will be given in the following steps.

Step 1. We show that $\tilde{\mathcal{S}}(\Omega)\subset\Omega$.

In view of (As2) and for each $\varrho\in J$, we have

$$\begin{aligned}
&\left|\left(\tilde{\mathcal{S}}\varkappa\right)(\varrho)\right|\\
&\leq\frac{\left(\psi(\varrho)-\psi(a)\right)^{\gamma-1}}{|\Lambda|\,\Gamma(\gamma)}\left(\sum_{i=1}^{m}\frac{|\theta_i|}{\Gamma(\mathrm{a}_1+\eta_i)}\int_a^{\delta_i}\psi'(\vartheta)\left(\psi(\delta_i)-\psi(\vartheta)\right)^{\mathrm{a}_1+\eta_i-1}|f(\vartheta,\varkappa(\vartheta))|\,d\vartheta\right.\\
&\left.-\frac{1}{\Gamma(2+\mathrm{a}_1-\gamma)}\int_a^{b}\psi'(\vartheta)\left(\psi(b)-\psi(\vartheta)\right)^{1+\mathrm{a}_1-\gamma}|v(\vartheta)|\,d\vartheta\right)\\
&+\frac{1}{\Gamma(\mathrm{a}_1)}\int_a^{\varrho}\psi'(\vartheta)\left(\psi(\varrho)-\psi(\vartheta)\right)^{\mathrm{a}_1-1}|f(\vartheta,\varkappa(\vartheta))|\,d\vartheta\\
&\leq\frac{\left(\psi(\varrho)-\psi(a)\right)^{\gamma-1}w^*}{|\Lambda|\,\Gamma(\gamma)}\left(\sum_{i=1}^{m}\frac{|\theta_i|}{\Gamma(\mathrm{a}_1+\eta_i)}\int_a^{\delta_i}\psi'(\vartheta)\left(\psi(\delta_i)-\psi(\vartheta)\right)^{\mathrm{a}_1+\eta_i-1}d\vartheta\right.\\
&\left.+\frac{1}{\Gamma(2+\mathrm{a}_1-\gamma)}\int_a^{b}\psi'(\vartheta)\left(\psi(b)-\psi(\vartheta)\right)^{1+\mathrm{a}_1-\gamma}d\vartheta\right)\\
&+\frac{w^*}{\Gamma(\mathrm{a}_1)}\int_a^{\varrho}\psi'(\vartheta)\left(\psi(\varrho)-\psi(\vartheta)\right)^{\mathrm{a}_1-1}d\vartheta\\
&\leq w^*\left(\sum_{i=1}^{m}|\theta_i|\frac{\left(\psi(b)-\psi(a)\right)^{\mathrm{a}_1+\eta_i+\gamma-1}}{|\Lambda|\,\Gamma(\gamma)\,\Gamma(\mathrm{a}_1+\eta_i+1)}+\frac{\left(\psi(b)-\psi(a)\right)^{1+\mathrm{a}_1}}{|\Lambda|\,\Gamma(\gamma)\,\Gamma(3+\mathrm{a}_1-\gamma)}+\frac{\left(\psi(b)-\psi(a)\right)^{\mathrm{a}_1}}{\Gamma(\mathrm{a}_1+1)}\right)\\
&\leq w^*\left(k_1+k_2+k_3\right),
\end{aligned}$$

and consequently

$$\left\|\tilde{\mathcal{S}}\varkappa\right\|\leq M_0.$$

Hence, $\tilde{\mathcal{S}}(\Omega)\subset\Omega$ and the set $\tilde{\mathcal{S}}(\Omega)$ is uniformly bounded.

Step 2. $\tilde{\mathcal{S}}$ sends bounded sets of $\mathcal{C}$ into equicontinuous sets.

For $\varrho_1, \varrho_2 \in J$, $\varrho_1 < \varrho_2$ and for $\varkappa \in \Omega$, we have

$$\begin{aligned}
&\left|\left(\tilde{\mathcal{S}}\varkappa\right)(\varrho_2) - \left(\tilde{\mathcal{S}}\varkappa\right)(\varrho_1)\right| \\
&\leq \frac{(\psi(\varrho_2) - \psi(a))^{\gamma-1} - (\psi(\varrho_1) - \psi(a))^{\gamma-1}}{|\Lambda|\,\Gamma(\gamma)} \\
&\times \left(\sum_{i=1}^{m} \frac{|\theta_i|}{\Gamma(\mathrm{a}_1 + \eta_i)} \int_a^{\delta_i} \psi'(\vartheta)(\psi(\delta_i) - \psi(\vartheta))^{\mathrm{a}_1+\eta_i-1} |f(\vartheta, \varkappa(\vartheta))|\, d\vartheta \right. \\
&\left. + \frac{1}{\Gamma(2 + \mathrm{a}_1 - \gamma)} \int_a^{b} \psi'(\vartheta)(\psi(b) - \psi(\vartheta))^{2+\mathrm{a}_1-\gamma} |f(\vartheta, \varkappa(\vartheta))|\, d\vartheta \right) \\
&+ \frac{1}{\Gamma(\mathrm{a}_1)} \int_a^{\varrho_1} \psi'(\vartheta)\left((\psi(\varrho_2) - \psi(\vartheta))^{\mathrm{a}_1-1} - (\psi(\varrho_1) - \psi(\vartheta))^{\mathrm{a}_1-1}\right) |f(\vartheta, \varkappa(\vartheta))|\, d\vartheta \\
&+ \frac{1}{\Gamma(\mathrm{a}_1)} \int_{\varrho_1}^{\varrho_2} \psi'(\vartheta)(\psi(\varrho_2) - \psi(\vartheta))^{\mathrm{a}_1-1} |f(\vartheta, \varkappa(\vartheta))|\, d\vartheta \\
&\leq \frac{\left((\psi(\varrho_2) - \psi(a))^{\gamma-1} - (\psi(\varrho_1) - \psi(a))^{\gamma-1}\right) w^*}{|\Lambda|\,\Gamma(\gamma)} \\
&\times \left(\sum_{i=1}^{m} \frac{|\theta_i|}{\Gamma(\mathrm{a}_1 + \eta_i)} \int_a^{\delta_i} \psi'(\vartheta)(\psi(\delta_i) - \psi(\vartheta))^{\mathrm{a}_1+\eta_i-1}\, d\vartheta \right. \\
&\left. + \frac{1}{\Gamma(2 + \mathrm{a}_1 - \gamma)} \int_a^{b} \psi'(\vartheta)(\psi(b) - \psi(\vartheta))^{1+\mathrm{a}_1-\gamma}\, d\vartheta \right) \\
&+ \frac{w^*}{\Gamma(\mathrm{a}_1 + 1)} \left((\psi(\varrho_2) - \psi(a))^{\mathrm{a}_1} - (\psi(\varrho_1) - \psi(a))^{\mathrm{a}_1}\right).
\end{aligned}$$

As $\varrho_1 \to \varrho_2$, we obtain

$$\left|\left(\tilde{\mathcal{S}}\varkappa\right)(\varrho_2) - \left(\tilde{\mathcal{S}}\varkappa\right)(\varrho_1)\right| \to 0.$$

Hence $\tilde{\mathcal{S}}(\Omega)$ is equicontinuous. The Arzela-Ascoli theorem implies that $\tilde{\mathcal{S}}$ is compact. Thus by Schauder fixed point theorem, we prove that $\tilde{\mathcal{S}}$ has at least one fixed point $\varkappa \in \Omega$ that is in fact a mild solution of (1.2) on J. □

4. Ulam stability results

In this portion, we discuss the various types of Ulam stability for the ψ-Hilfer problem (1.2).

Theorem 4.1. *Suppose that the hypothesis (As1) and condition (3.2) are satisfied. Then, the first Eq of (1.2) is UH stable.*

Proof . Let $\epsilon > 0$. Let $\tilde{\varkappa} \in \mathcal{C}$ be any solution of the inequality

$$\left|{}^{H}D_{a+}^{\mathrm{a}_1,\mathrm{a}_2;\psi}\tilde{\varkappa}(\varrho) - f(\varrho, \tilde{\varkappa}(\varrho))\right| \leq \epsilon, \ \ \varrho \in J.$$

Then, there exists $\upsilon \in \mathcal{C}$ such that

$$ {}^{H}D_{a+}^{\mathrm{a}_1,\mathrm{a}_2;\psi}\tilde{\varkappa}(\varrho) = f(\varrho, \tilde{\varkappa}(\varrho)) + \upsilon(\varrho), \ \ \varrho \in J, \tag{4.1}$$

and $|v(\varrho)| \leq \epsilon$, $\varrho \in J$. In view of Lemma 2.15, we get

$$\tilde{\varkappa}(\varrho) = \frac{(\psi(\varrho)-\psi(a))^{\gamma-1}}{|\Lambda|\Gamma(\gamma)} A_{\tilde{\varkappa}} + I_{a+}^{\mathrm{a}_1;\psi} f(\varrho, \tilde{\varkappa}(\varrho)) + I_{a+}^{\mathrm{a}_1;\psi} v(\varrho), \tag{4.2}$$

is solution of Eq (4.1), where

$$\begin{aligned} A_{\tilde{\varkappa}} = & \left(\sum_{i=1}^{m} \frac{|\theta_i|}{\Gamma(\mathrm{a}_1+\eta_i)} \int_a^{\delta_i} \psi'(\vartheta)(\psi(\delta_i)-\psi(\vartheta))^{\mathrm{a}_1+\eta_i-1} f(\vartheta, \tilde{\varkappa}(\vartheta))\, d\vartheta \right. \\ & \left. - \frac{1}{\Gamma(2+\mathrm{a}_1-\gamma)} \int_a^b \psi'(\vartheta)(\psi(b)-\psi(\vartheta))^{1+\mathrm{a}_1-\gamma} f(\vartheta, \tilde{\varkappa}(\vartheta))\, d\vartheta \right). \end{aligned} \tag{4.3}$$

From Eq (4.2), we have

$$\begin{aligned} & \left| \tilde{\varkappa}(\varrho) - \frac{(\psi(\varrho)-\psi(a))^{\gamma-1}}{|\Lambda|\Gamma(\gamma)} A_{\tilde{\varkappa}} - I_{a+}^{\mathrm{a}_1;\psi} f(\varrho, \tilde{\varkappa}(\varrho)) \right| \\ & \leq I_{a+}^{\mathrm{a}_1;\psi} |v(\varrho)| \leq \epsilon \frac{(\psi(\varrho)-\psi(a))^{\mathrm{a}_1}}{\Gamma(\mathrm{a}_1+1)}. \end{aligned} \tag{4.4}$$

Let $\tilde{\varkappa} \in \mathcal{C}$ be solution of the problem

$$\begin{cases} {}^H D^{\mathrm{a}_1,\mathrm{a}_2;\psi} \tilde{\varkappa}(\varrho) = f(\varrho, \tilde{\varkappa}(\varrho)), \\ \varkappa(a) = \tilde{\varkappa}(a), \ I_{a+}^{2-\gamma;\psi} \varkappa(b) = I_{a+}^{2-\gamma;\psi} \tilde{\varkappa}(b), \end{cases} \tag{4.5}$$

where $I_{a+}^{2-\gamma;\psi}\varkappa(b) = \sum_{i=1}^{m} \theta_i I_{a+}^{\eta_i;\psi} \varkappa(\delta_i)$ and $I_{a+}^{2-\gamma;\psi}\tilde{\varkappa}(b) = \sum_{i=1}^{m} \theta_i I_{a+}^{\eta_i;\psi} \tilde{\varkappa}(\delta_i)$. By Lemma 2.15, the equivalent fractional integral equation of (4.5) is

$$\tilde{\varkappa}(\varrho) = \frac{(\psi(\varrho)-\psi(a))^{\gamma-1}}{|\Lambda|\Gamma(\gamma)} A_{\tilde{\varkappa}} + I_{a+}^{\mathrm{a}_1;\psi} f(\varrho, \tilde{\varkappa}(\varrho)),$$

where $A_{\tilde{\varkappa}}$ is given by (4.3).

Now, by using the assumption (As1), we obtain

$$\begin{aligned} & |A_{\varkappa} - A_{\tilde{\varkappa}}| \\ & \leq \frac{(\psi(\varrho)-\psi(a))^{\gamma-1}}{|\Lambda|\Gamma(\gamma)} \left(\sum_{i=1}^{m} \frac{|\theta_i|}{\Gamma(\mathrm{a}_1+\eta_i)} \int_a^{\delta_i} \psi'(\vartheta)(\psi(\delta_i)-\psi(\vartheta))^{\mathrm{a}_1+\eta_i-1} |f(\vartheta,\varkappa(\vartheta)) - f(\vartheta,\tilde{\varkappa}(\vartheta))|\, d\vartheta \right. \\ & \left. + \frac{1}{\Gamma(2+\mathrm{a}_1-\gamma)} \int_a^b \psi'(\vartheta)(\psi(b)-\psi(\vartheta))^{1+\mathrm{a}_1-\gamma} |f(\vartheta,\varkappa(\vartheta)) - f(\vartheta,\tilde{\varkappa}(\vartheta))|\, d\vartheta \right) \\ & \leq \frac{(\psi(\varrho)-\psi(a))^{\gamma-1}}{|\Lambda|\Gamma(\gamma)} \left(\sum_{i=1}^{m} \frac{|\theta_i|}{\Gamma(\mathrm{a}_1+\eta_i)} \int_a^{\delta_i} \psi'(\vartheta)(\psi(\delta_i)-\psi(\vartheta))^{\mathrm{a}_1+\eta_i-1} |\varkappa(\vartheta) - \tilde{\varkappa}(\vartheta)|\, d\vartheta \right. \\ & \left. + \frac{1}{\Gamma(2+\mathrm{a}_1-\gamma)} \int_a^b \psi'(\vartheta)(\psi(b)-\psi(\vartheta))^{1+\mathrm{a}_1-\gamma} |\varkappa(\vartheta) - \tilde{\varkappa}(\vartheta)|\, d\vartheta \right) \\ & \leq \frac{(\psi(\varrho)-\psi(a))^{\gamma-1} l_1}{|\Lambda|\Gamma(\gamma)} \left(\sum_{i=1}^{m} |\theta_i| \, I_{a+}^{\mathrm{a}_1+\eta_i;\psi} |\varkappa(\delta_i) - \tilde{\varkappa}(\delta_i)| + I_{a+}^{2+\mathrm{a}_1-\gamma;\psi} |\varkappa(b) - \tilde{\varkappa}(b)| \right). \end{aligned} \tag{4.6}$$

Because $\varkappa(b)=\tilde{\varkappa}(b)$, we must have $\varkappa(\delta_i)=\tilde{\varkappa}(\delta_i)$, $i=1,2,...,m$. Therefore, from inequality (4.6), we obtain $A_\varkappa = A_{\tilde{\varkappa}}$. From (4.4) and (As1), we get

$$\begin{aligned}
&|\tilde{\varkappa}(\varrho)-\varkappa(\varrho)| \\
&= \left|\tilde{\varkappa}(\varrho)-\frac{(\psi(\varrho)-\psi(a))^{\gamma-1}}{|\Lambda|\,\Gamma(\gamma)}A_\varkappa - I_{a+}^{\mathrm{a}_1;\psi} f(\varrho,\varkappa(\varrho))\right| \\
&\le \left|\tilde{\varkappa}(\varrho)-\frac{(\psi(\varrho)-\psi(a))^{\gamma-1}}{|\Lambda|\,\Gamma(\gamma)}A_{\tilde{x}} - I_{a+}^{\mathrm{a}_1;\psi} f(\varrho,\tilde{\varkappa}(\varrho))\right| \\
&+ \left|I_{a+}^{\mathrm{a}_1;\psi} f(\varrho,\tilde{\varkappa}(\varrho)) - I_{a+}^{\mathrm{a}_1;\psi} f(\varrho,\varkappa(\varrho))\right| \\
&\le \epsilon\frac{(\psi(b)-\psi(a))^{\mathrm{a}_1}}{\Gamma(\mathrm{a}_1+1)} + \frac{l_1}{\Gamma(\mathrm{a}_1)}\int_a^\varrho \psi'(\vartheta)(\psi(\varrho)-\psi(\vartheta))^{\mathrm{a}_1-1}|\tilde{\varkappa}(\vartheta)-\varkappa(\vartheta)|\,d\vartheta.
\end{aligned}$$

Applying Lemma 2.11 with $u(\varrho)=|\tilde{\varkappa}(\varrho)-\varkappa(\varrho)|$, $v(\varrho)=\epsilon\frac{(\psi(b)-\psi(a))^{\mathrm{a}_1}}{\Gamma(\mathrm{a}_1+1)}$ and $z(\varrho)=\frac{l_1}{\Gamma(\mathrm{a}_1)}$, we obtain

$$\begin{aligned}
&|\tilde{\varkappa}(\varrho)-\varkappa(\varrho)| \\
&\le \epsilon\frac{(\psi(b)-\psi(a))^{\mathrm{a}_1}}{\Gamma(\mathrm{a}_1+1)}\left[1+\int_a^\varrho\sum_{k=1}^{\infty}\frac{[l_1]^k}{\Gamma(k\alpha)}\psi'(\vartheta)(\psi(\varrho)-\psi(\vartheta))^{k\alpha-1}\,d\vartheta\right] \\
&\le \epsilon\frac{(\psi(b)-\psi(a))^{\mathrm{a}_1}}{\Gamma(\mathrm{a}_1+1)}\left[1+\sum_{k=1}^{\infty}\frac{[l_1(\psi(b)-\psi(a))^{\mathrm{a}_1}]^k}{\Gamma(k\alpha+1)}\right] \\
&= \epsilon\frac{(\psi(b)-\psi(a))^{\mathrm{a}_1}}{\Gamma(\mathrm{a}_1+1)}E_{\mathrm{a}_1}\left(l_1(\psi(b)-\psi(a))^{\mathrm{a}_1}\right).
\end{aligned}$$

By setting

$$k_f=\frac{(\psi(b)-\psi(a))^{\mathrm{a}_1}}{\Gamma(\mathrm{a}_1+1)}E_{\mathrm{a}_1}\left(l_1(\psi(b)-\psi(a))^{\mathrm{a}_1}\right).$$

we obtain

$$|\tilde{\varkappa}(\varrho)-\varkappa(\varrho)|\le k_f\epsilon. \tag{4.7}$$

Therefore, the first Eq of (1.2) is UH stable. □

Remark 4.2. *Define $\phi_f:\mathbb{R}^+\to\mathbb{R}^+$ by $\phi_f(\epsilon)=k_f\epsilon$. Then, $\phi_f\in C(\mathbb{R}^+,\mathbb{R}^+)$ and $\phi_f(0)=0$. Then inequality (4.7) can be written as*

$$|\tilde{\varkappa}(\varrho)-\varkappa(\varrho)|\le\phi_f(\epsilon).$$

Thus, the first Eq of (1.2) is GUH stable.

In the next, we introduce the following function.

(As3) The function $\phi\in C([a,b],\mathbb{R}^+)$ is increasing and there is a constant $\lambda_\phi>0$ such that

$$I_{a+}^{\mathrm{a}_1;\psi}\phi(\varrho)\le\lambda_\phi\phi(\varrho),\ \forall\varrho\in J.$$

Theorem 4.3. *Assume that the hypotheses (As1), (As3) and condition (3.2) are satisfied. Then, the first Eq of (1.2) is UHR stable.*

Proof . Let any $\epsilon > 0$. Let $\tilde{\varkappa} \in \mathcal{C}$ be any solution of the inequality

$$\left| {}^{H}D_{a+}^{\mathrm{a}_1,\mathrm{a}_2;\psi} \tilde{\varkappa}(\varrho) - f(\varrho, \tilde{\varkappa}(\varrho)) \right| \leq \epsilon \phi(\varrho), \ \varrho \in J.$$

Then, proceeding as in the proof of Theorem 4.1. From Remark 2.10 , for some continuous function v such that $|v(\varrho)| < \epsilon \phi(\varrho)$, we get

$$\begin{aligned} &\left| \tilde{\varkappa}(\varrho) - \frac{(\psi(\varrho) - \psi(a))^{\gamma-1}}{|\Lambda| \Gamma(\gamma)} A_{\tilde{\varkappa}} - I_{a+}^{\mathrm{a}_1;\psi} f(\varrho, \tilde{\varkappa}(\varrho)) \right| \\ &\leq I_{a+}^{\mathrm{a}_1;\psi} |v(\varrho)| \leq \epsilon I_{a+}^{\mathrm{a}_1;\psi} |\phi(\varrho)| \leq \epsilon \lambda_\phi \phi(\varrho), \ \varrho \in J. \end{aligned}$$

Taking $\tilde{\varkappa} \in \mathcal{C}$ as any solution of (4.5), and following same steps as in the proof of Theorem 4.1,we get

$$\begin{aligned} &|\tilde{\varkappa}(\varrho) - \varkappa(\varrho)| \\ &\leq \epsilon \lambda_\phi \phi(\varrho) + \frac{l_1}{\Gamma(\mathrm{a}_1)} \int_a^\varrho \psi'(\vartheta) (\psi(\varrho) - \psi(\vartheta))^{\mathrm{a}_1 - 1} |\tilde{\varkappa}(\vartheta) - \varkappa(\vartheta)| \, d\vartheta, \ \varrho \in J. \end{aligned}$$

By applying Corollary 2.12, we obtain

$$\begin{aligned} |\tilde{\varkappa}(\varrho) - \varkappa(\varrho)| &\leq \epsilon \lambda_\phi \phi(\varrho) E_{\mathrm{a}_1} \left(l_1 (\psi(\varrho) - \psi(a))^{\mathrm{a}_1} \right) \\ &\leq \epsilon \lambda_\phi \phi(\varrho) E_{\mathrm{a}_1} \left(l_1 (\psi(b) - \psi(a))^{\mathrm{a}_1} \right). \end{aligned}$$

By taking a constant

$$k_{\phi,f} = E_{\mathrm{a}_1} \left(l_1 (\psi(b) - \psi(a))^{\mathrm{a}_1} \right).$$

we obtain

$$|\tilde{\varkappa}(\varrho) - \varkappa(\varrho)| \leq k_{\phi,f} \epsilon \phi(\varrho). \tag{4.8}$$

Therefore, the first Eq (1.2) is UHR stable. □

Remark 4.4. *By putting $\epsilon = 1$ in the inequality (4.8), we deduce that first Eq of (1.2) is GUHR stable.*

5. Examples

In this section, we consider some particular cases of the nonlinear FDE's to apply our results in the study of existence and Ulam stabilities, specifically, UH and UHR.

Consider the following FDE of the form

$$\begin{cases} {}^{H}D_{a+}^{\mathrm{a}_1,\mathrm{a}_2;\psi} \varkappa(\varrho) = f(\varrho, \varkappa), \ \varrho \in (a, b), \\ \varkappa(a) = 0, \ I_{a+}^{2-\gamma;\psi} \varkappa(b) = \sum_{i=1}^{m} \theta_i I_{a+}^{\eta_i;\psi} \varkappa(\delta_i). \end{cases} \tag{5.1}$$

The following examples are particular cases of (5.1).

Example 5.1. *Consider the FDE given by (5.1). Taking $\psi(\varrho) = \log \varrho$, $\mathrm{a}_2 \to 0$, $a = 1$, $b = e$, $\mathrm{a}_1 = \frac{3}{2}$, $\theta_1 = \frac{1}{2}$, $\theta_2 = \frac{1}{10}$, $\eta_1 = \frac{1}{4}$, $\eta_2 = \frac{5}{2}$, $\delta_1 = \frac{3}{2}$, $\delta_2 = 2$. Then, the problem (5.1) reduce to the following problem*

$$\begin{cases} {}^{Ha}D_{1+}^{\frac{3}{2},0;\log \varrho} \varkappa(\varrho) = f(\varrho, \varkappa), \ \varrho \in (1, e), \\ \varkappa(1) = 0, \ I_{a+}^{\frac{1}{2};\log \varrho} \varkappa(e) = \frac{1}{2} I_{a+}^{\frac{1}{4};\log \varrho} \varkappa\left(\frac{3}{2}\right) + \frac{1}{10} I_{a+}^{\frac{5}{2};\log \varrho} \varkappa(2), \end{cases} \tag{5.2}$$

which is the FDE involving Hadamard FD. In this case $\gamma = \frac{3}{2}$ *and with this data we find* $\Lambda = 0.71802 \neq 0$. *Consider the function* $f : [1, e] \times \mathbb{R} \to \mathbb{R}$ *given by*

$$f(\varrho, \varkappa) = \frac{\cos(\varrho)}{\exp(\varrho^2 - 1) + 5} \frac{|\varkappa|}{|\varkappa| + 1}.$$

For $\varkappa, \tilde{\varkappa} \in \mathbb{R}$ *and* $\varrho \in [1, e]$, *we have*

$$\begin{aligned} |f(\varrho, \varkappa) - f(\varrho, \tilde{\varkappa})| &\leq \left| \frac{\cos(\varrho)}{\exp(\varrho^2 - 1) + 5} \left(\frac{|\varkappa|}{|\varkappa| + 1} - \frac{|\tilde{\varkappa}|}{|\tilde{\varkappa}| + 1} \right) \right| \\ &\leq \frac{1}{\exp(\varrho^2 - 1) + 5} \left(\frac{|\varkappa - \tilde{\varkappa}|}{(1 + |\varkappa|)(1 + |\tilde{\varkappa}|)} \right) \\ &\leq \frac{1}{6} |\varkappa - \tilde{\varkappa}|, \end{aligned}$$

thus, the assumption (As1) is satisfied with $l_1 = \frac{1}{6}$. *We will check that condition (3.2) is satisfied. Indeed*

$$(k_1 + k_2 + k_3) l_1 \simeq \frac{0.5 + 0.79 + 0.75}{6} \simeq 0.34 < 1.$$

Then by Theorem 3.1, (5.2) has a unique mild solution on $[1, e]$. *Further, by Theorem 4.1 we conclude that the first Eq of (5.2) is UH stable with*

$$k_f = \frac{1}{\Gamma\left(\frac{5}{2}\right)} E_{\frac{3}{2}} \left(\frac{1}{6} \right).$$

Define

$$\phi(\varrho) = \log(\varrho)^{\frac{3}{2}}, \ \ \varrho \in [1, e].$$

Then, ϕ *is continuous increasing function such that*

$$\begin{aligned} I_{1+}^{\frac{3}{2};\log\varrho} \phi(\varrho) &= \frac{1}{\Gamma\left(\frac{3}{2}\right)} \int_1^{\varrho} \left(\log \frac{\varrho}{\vartheta} \right)^{\frac{1}{2}} \log(\varrho)^{\frac{3}{2}} \frac{d\vartheta}{\vartheta} \\ &\leq \frac{1}{\Gamma\left(\frac{3}{2}\right)} \int_1^{\varrho} \left(\log \frac{\varrho}{\vartheta} \right)^{\frac{1}{2}} \frac{d\vartheta}{\vartheta} \\ &\leq \frac{1}{\Gamma\left(\frac{5}{2}\right)} \log(\varrho)^{\frac{3}{2}}. \end{aligned}$$

Therefore, for $\lambda_\phi = \frac{1}{\Gamma\left(\frac{5}{2}\right)}$ *and* $\phi(\varrho) = \log(\varrho)^{\frac{3}{2}}$, *hypothesis (As3) is satisfied. Hence, by Theorem 4.3 the first Eq of (5.2) is UHR stable.*

Example 5.2. *Consider the FDE given by (5.1). Taking* $\psi(\varrho) = \varrho$, $\mathrm{a}_2 \to 0$, $a = 0$, $b = 1$, $\mathrm{a}_1 = \frac{5}{4}$, $\theta_1 = 3$, $\theta_2 = 5$, $\eta_1 = \frac{1}{4}$, $\eta_2 = \frac{1}{2}$, $\delta_1 = \frac{1}{4}$, $\delta_2 = \frac{1}{2}$. *Then, the problem (5.1) reduce to the following problem*

$$\begin{cases} {}^{RL}D_{0+}^{\frac{5}{4},0;\varrho} \varkappa(\varrho) = f(\varrho, \varkappa), \ \varrho \in (0, 1), \\ \varkappa(0) = 0, \ I_{0+}^{\frac{3}{4};\varrho} \varkappa(1) = 3 I_{0+}^{\frac{1}{4};\varrho} \varkappa\left(\frac{1}{4}\right) + 5 I_{0+}^{\frac{1}{2};\varrho} \varkappa\left(\frac{1}{2}\right), \end{cases} \tag{5.3}$$

which is the FDE involving Riemann-Liouville FD. In this case $\gamma = \frac{5}{4}$ *and with this data we find* $\Lambda = -3.9274 \neq 0$. *Consider the function* $f : [0, 1] \times \mathbb{R} \to \mathbb{R}$ *given by*

$$f(\varrho, x) = \frac{\sin(\varrho)}{\exp(\varrho^2) + 2} \frac{|\varkappa|}{|\varkappa| + 1}.$$

For $\varkappa, \tilde{\varkappa} \in \mathbb{R}$ *and* $\varrho \in [0,1]$, *we have*

$$\begin{aligned}\left|f(\varrho,\varkappa)-f(\varrho,\tilde{\varkappa})\right| &\leq \left|\frac{\sin(\varrho)}{\exp(\varrho^2)+2}\left(\frac{|\varkappa|}{|\varkappa|+1}-\frac{|\tilde{\varkappa}|}{|\tilde{\varkappa}|+1}\right)\right| \\ &\leq \frac{1}{\exp(\varrho^2)+2}\left(\frac{|\varkappa-\tilde{\varkappa}|}{(1+|\varkappa|)(1+|\tilde{\varkappa}|)}\right) \\ &\leq \frac{1}{3}|\varkappa-\tilde{\varkappa}|,\end{aligned}$$

thus, the assumption (As1) is satisfied with $l_1 = \frac{1}{3}$. *We will check that condition (3.2) is satisfied. Indeed*

$$(k_1+k_2+k_3)\, l_1 \simeq \frac{1.51+0.14+0.88}{3} \simeq 0.84 < 1.$$

Then by Theorem 3.1, (5.3) has a unique mild solution on $[0,1]$. *Further, by Theorem 4.1 we conclude that the first Eq of (5.3) is UH stable with*

$$k_f = \frac{1}{\Gamma\left(\frac{9}{4}\right)} E_{\frac{5}{4}}\left(\frac{1}{3}\right).$$

Define

$$\phi(\varrho) = \varrho^{\frac{5}{4}}, \ \ \varrho \in [0,1].$$

Then, ϕ *is continuous increasing function such that*

$$\begin{aligned}I_{0+}^{\frac{5}{4};\varrho}\phi(\varrho) &= \frac{1}{\Gamma\left(\frac{5}{4}\right)}\int_0^{\varrho}(\varrho-\vartheta)^{\frac{1}{4}}\varrho^{\frac{5}{4}}d\vartheta \\ &\leq \frac{1}{\Gamma\left(\frac{5}{4}\right)}\int_0^{\varrho}(\varrho-\vartheta)^{\frac{1}{4}}d\vartheta \\ &\leq \frac{1}{\Gamma\left(\frac{9}{4}\right)}\varrho^{\frac{5}{4}}.\end{aligned}$$

Therefore, for $\lambda_\phi = \frac{1}{\Gamma\left(\frac{9}{4}\right)}$ *and* $\phi(\varrho) = \varrho^{\frac{5}{4}}$, *hypothesis (As3) is satisfied. Hence, by Theorem 4.3 the first Eq of (5.3) is UHR stable.*

Example 5.3. *Consider the FDE given by (5.1). Taking* $\psi(\varrho) = \varrho$, $\mathrm{a}_2 \to \frac{1}{2}$, $a = 0$, $b = 1$, $\mathrm{a}_1 = \frac{7}{4}$, $\theta_1 = 3$, $\theta_2 = 5$, $\eta_1 = \frac{1}{4}$, $\eta_2 = \frac{1}{2}$, $\delta_1 = \frac{1}{4}$, $\delta_2 = \frac{1}{2}$. *Then, the problem (5.1) reduce to the following problem*

$$\begin{cases} {}^{H}D_{0+}^{\frac{7}{4},\frac{1}{2};\varrho}\varkappa(\varrho) = f(\varrho,\varkappa), \ \ \varrho \in (0,1), \\ \varkappa(0) = 0, \ \ I_{0+}^{\frac{1}{8};\varrho}\varkappa(1) = 3I_{0+}^{\frac{1}{4};\varrho}\varkappa\left(\frac{1}{4}\right) + 5I_{0+}^{\frac{1}{2};\varrho}\varkappa\left(\frac{1}{2}\right), \end{cases} \tag{5.4}$$

which is the FDE involving Hilfer FD. In this case $\gamma = \frac{15}{8}$ *and with this data we find* $\Lambda = -1.1725 \neq 0$. *Consider the function* $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ *given by*

$$f(\varrho,\varkappa) = \frac{\sin\varkappa}{\exp(\varrho^2)+6}.$$

For $\varkappa, \tilde{\varkappa} \in \mathbb{R}$ *and* $\varrho \in [0,1]$, *we have*

$$\begin{aligned}\left|f(\varrho,\varkappa)-f(\varrho,\tilde{\varkappa})\right| &\leq \frac{|\varkappa-\tilde{\varkappa}|}{\exp(\varrho^2)+6} \\ &\leq \frac{1}{7}|\varkappa-\tilde{\varkappa}|,\end{aligned}$$

thus, the assumption (As1) is satisfied with $l_1 = \frac{1}{7}$*. We will check that condition (3.2) is satisfied. Indeed*

$$(k_1 + k_2 + k_3)\, l_1 \simeq \frac{3.1 + 0.5 + 0.62}{7} \simeq 0.60 < 1.$$

Then by Theorem 3.1, (5.4) has a unique mild solution on $[0,1]$*. Further, by Theorem 4.1 we conclude that the first Eq of (5.4) is UH stable with*

$$k_f = \frac{1}{\Gamma\left(\frac{11}{4}\right)} E_{\frac{7}{4}}\left(\frac{1}{7}\right).$$

Define

$$\phi(\varrho) = \varrho^{\frac{7}{4}},\ \ \varrho \in [0,1].$$

Then, ϕ *is continuous increasing function such that*

$$\begin{aligned} I_{0^+}^{\frac{7}{4};\varrho}\phi(\varrho) &= \frac{1}{\Gamma\left(\frac{7}{4}\right)} \int_0^{\varrho} (\varrho - \vartheta)^{\frac{3}{4}} \varrho^{\frac{7}{4}} d\vartheta \\ &\leq \frac{1}{\Gamma\left(\frac{7}{4}\right)} \int_0^{\varrho} (\varrho - \vartheta)^{\frac{3}{4}} d\vartheta \\ &\leq \frac{1}{\Gamma\left(\frac{11}{4}\right)} \varrho^{\frac{7}{4}}. \end{aligned}$$

Therefore, for $\lambda_\phi = \frac{1}{\Gamma\left(\frac{11}{4}\right)}$ *and* $\phi(\varrho) = \varrho^{\frac{7}{4}}$*, hypothesis (As3) is satisfied. Hence, by Theorem 4.3 the first Eq of (5.4) is UHR stable.*

6. Conclusions

In this article, we have studied the existence, uniqueness and Ulam-Hyers stability of mild solutions for nonlinear ψ-Hilfer FDE's with nonlocal integral boundary conditions. Our investigations based on the fixed point theorems and generalized Gronwall inequality. The acquired results in this paper are more general and cover many of the parallel problems that contain special cases of function ψ, because our proposed problem contains a general fractional derivative that combines many classic fractional derivatives.

In future works, we will try to extend the existing problem in the present paper to a general structure with the Mittag-Leffler power law [10] and for fractal fractional operators [11].

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