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Existence and uniqueness of weak solution in weighted Sobolev spaces for a class of nonlinear degenerate elliptic problems with measure data

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Abstract

In this paper, we study the existence and uniqueness of weak solution to a Dirichlet boundary value problems for the following nonlinear degenerate elliptic problems

$$-\mathrm{div}\Big[\omega_1\mathcal{A}(x,\nabla u)+\nu_2\mathcal{B}(x,u,\nabla u)\Big]+\nu_1\mathcal{C}(x,u)+\omega_2|u|^{p-2}u=f-\mathrm{div}F,$$

where $1<p<\infty$, ω_1, ν_2, ν_1 and ω_2 are A_p-weight functions, and $\mathcal{A}:\Omega\times\mathbb{R}^n\longrightarrow\mathbb{R}^n$, $\mathcal{B}:\Omega\times\mathbb{R}\times\mathbb{R}^n\longrightarrow\mathbb{R}^n$, $\mathcal{C}:\Omega\times\mathbb{R}\longrightarrow\mathbb{R}$ are Caratéodory functions that satisfy some conditions and the right-hand side term $f-\mathrm{div}F$ belongs to $L^{p'}(\Omega,\omega_2^{1-p'})+\prod_{j=1}^{n}L^{p'}(\Omega,\omega_1^{1-p'})$. We will use the Browder-Minty Theorem and the weighted Sobolev spaces theory to prove the existence and uniqueness of weak solution in the weighted Sobolev space $W_0^{1,p}(\Omega,\omega_1,\omega_2)$.

Keywords: Dirichlet problem, nonlinear degenerate elliptic problems, Browder-Minty Theorem, weighted Sobolev spaces, weak solution.
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1. Introduction

In the past decade, much attention has been devoted to nonlinear elliptic equations because of their wide application to physical models such as non-Newtonian fluids, boundary layer phenomena

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for viscous fluids, chemical heterogenous model, celestial mechanics and reaction-diffusion problems (we refer to [6, 9, 30] where it is possible to find some examples of applications of degenerate elliptic equations).

The Sobolev spaces $W^{k,p}(\Omega)$ without weights, in general, occur as spaces of solutions for elliptic and parabolic partial difierential equations. For degenerate partial differential equations, where we have equations with various types of singularities in the coefficients, it is natural to look for solutions in weighted Sobolev spaces [1, 4, 13, 14, 15, 17, 19, 21, 25, 27]. The type of a weight depends on the equation type.

Our aim in this paper is to prove the existence and uniqueness of weak solution in the weighted Sobolev space $W_0^{1,p}(\Omega,\omega_1,\omega_2)$ (see Definition 2.7) for the Dirichlet problem associated to the degenerate degenerate elliptic equation of the form

$$\begin{cases} -\text{div}\Big[\omega_1\mathcal{A}(x,\nabla u)+\nu_2\mathcal{B}(x,u,\nabla u)\Big]+\nu_1\mathcal{C}(x,u)+\omega_2|u|^{p-2}u=f-\text{div}F & \text{in } \Omega,\\ u(x)=0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where, Ω is a bounded open set in $\mathbb{R}^n$, ω_1, ν_2, ν_1 and ω_2 are A_p-weight functions that will be defined in the Preliminaries, and the functions $\mathcal{B}:\Omega\times\mathbb{R}\times\mathbb{R}^n\longrightarrow\mathbb{R}^n$, $\mathcal{A}:\Omega\times\mathbb{R}^n\longrightarrow\mathbb{R}^n$ and $\mathcal{C}:\Omega\times\mathbb{R}\longrightarrow\mathbb{R}$ are Caratéodory functions that satisfy the assumptions of growth, ellipticity and monotonicity.

Problem like (1.1) have been studied by many authors in the non weighted case (see [3, 7]). For $\omega_1\equiv\nu_2\equiv\nu_1\equiv 1$(the non weighted case), $\omega_2\equiv 0$ and the term $\mathcal{A}(x,\nabla u)$ is equal to zero, existence results for Problem (1.1) have been shown in [5].

When $-\text{div}F = 0$, El Ouaarabi and al. [24] proved in the variational setting, under some assumptions that, for every $f\in L^1(\Omega)$ the Problem (1.1) has a unique solution $u\in W_0^{1,p}(\Omega,\omega_1,\omega_2)$. The degenerate case with difierent conditions haven been studied by many authors (we refer to [2, 11, 26, 32] for more details).

Let us rapidly summarize the work's contents. In Section 2, we give some preliminaries and some technical lemmas. In Section 3, we make precise all the assumptions on $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ and we introduce the notion of weak solution for the Problem (1.1). The main results will be stated and proved in Section 4. Section 5 is devoted to an example which illustrates our main result.

2. Preliminaries

In this section, we present some definitions and preliminary facts which are used throughout this paper. Complete expositions can be found in the monographs by J. Garcia-Cuerva and J. L. Rubio de Prancia [16] and A. Torchinsky [28].

By a weight, we shall mean a locally integrable function ω on $\mathbb{R}^n$ such that $\omega(x)>0$ for a.e. $x\in\mathbb{R}^n$. Every weight ω gives rise to a measure on the measurable subsets on $\mathbb{R}^n$ through integration. This measure will also be denoted by ω. Thus,

$$\omega(E)=\int_E \omega(x)dx \qquad \text{for measurable subset} \quad E\subset\mathbb{R}^n.$$

For $1\le p<\infty$, we denote by $L^p(\Omega,\omega)$ the space of measurable functions f on Ω such that

$$||f||_{L^p(\Omega,\omega)}=\left(\int_\Omega |f(x)|^p\omega(x)dx\right)^{\frac{1}{p}}<\infty,$$

where ω is a weight, and Ω be open in $\mathbb{R}^n$. It is a well-known fact that the space $L^p(\Omega,\omega)$, endowed with this norm is a Banach space. We also have that the dual space of $L^p(\Omega,\omega)$ is the space $L^{p'}(\Omega,\omega^{1-p'})$.

We now determine conditions on the weight ω that guarantee that functions in $L^p(\Omega,\omega)$ are locally integrable on Ω.

Proposition 2.1. *[20, 22] Let $1 \leq p < \infty$. If the weight ω is such that*

$$\omega^{\frac{-1}{p-1}} \in L^1_{loc}(\Omega) \qquad if \quad p > 1,$$

$$ess \sup_{x\in B} \frac{1}{\omega(x)} < +\infty \qquad if \quad p = 1,$$

for every ball $B \subset \Omega$. Then,

$$L^p(\Omega,\omega) \subset L^1_{loc}(\Omega).$$

As a consequence, under conditions of Proposition 2.1, the convergence in $L^p(\Omega,\omega)$ implies convergence in $L^1_{loc}(\Omega)$. Moreover, every function in $L^p(\Omega,\omega)$ has distributional derivatives. It thus makes sense to talk about distributional derivatives of functions in $L^p(\Omega,\omega)$.

A class of weights, which is particularly well understood, is the class of A_p-weights (or Muckenhoupt class) that was introduced by B. Muckenhoupt [23]. These classes have found many useful applications in harmonic analysis [28]. There are many interesting examples of weights (see [19] for p-admissible weights and another examples).

Definition 2.2. *Let $1 \leq p < \infty$. A weight ω is said to be an A_p-weight, or ω belongs to the Muckenhoupt class, if there exists a positive constant $\zeta = \zeta(p,\omega)$ such that, for every ball $B \subset \mathbb{R}^n$*

$$\left(\frac{1}{|B|}\int_B \omega(x)dx\right)\left(\frac{1}{|B|}\int_B (\omega(x))^{\frac{-1}{p-1}}\,dx\right)^{p-1} \leqslant \zeta \qquad if \quad p > 1,$$

$$\left(\frac{1}{|B|}\int_B \omega(x)dx\right) ess \sup_{x\in B} \frac{1}{\omega(x)} \leqslant \zeta \qquad if \quad p = 1,$$

where $|.|$ denotes the n-dimensional Lebesgue measure in $\mathbb{R}^n$.

The infimum over all such constants ζ is called the A_p constant of ω. We denote by A_p, $1 \leq p < \infty$, the set of all A_p weights.

If $1 \leq q \leq p < \infty$, then $A_1 \subset A_q \subset A_p$ and the A_q constant of ω equals the A_p constant of ω (we refer to [18, 19, 29] for more informations about A_p-weights).

Example 2.3. *(Example of A_p-weights)*

(i) *If ω is a weight such that $C \leqslant \omega(y) \leqslant D$ for a.e. $y \in \mathbb{R}^n$, where C and D are positive constants. Then $\omega \in A_p$ for $1 \leqslant p < \infty$.*

(ii) *If $\omega(y) = |y|^\eta$, $y \in \mathbb{R}^n$. Then $\omega \in A_p$ if and only if $-n < \eta < n(p-1)$ for $1 \leqslant p < \infty$ (see Corollary 4.4 in [28]).*

(iii) *Let Ω be an open subset of $\mathbb{R}^n$. Then $\omega(y) = e^{\lambda v(y)} \in A_2$, with $v \in W^{1,n}(\Omega)$ and λ is sufficiently small (see Corollary 2.18 in [23]).*

Definition 2.4. *A weight ω is said to be doubling, if there exists a positive constant C such that*

$$\omega(2B) \leqslant C\omega(B),$$

for every ball $B = B(x,r) \subset \mathbb{R}^n$, where $\omega(B) = \int_B \omega(x)dx$ and $2B$ denotes the ball with the same center as B which is twice as large. The infimum over all constants C is called the doubling constant of ω.

It follows directly from the A_p condition and Hölder inequality that an A_p-weight has the following strong doubling property. In particular, every A_p-weight is doubling (see Corollary 15.7 in [18]).

Proposition 2.5. *[30] Let $\omega \in A_p$ with $1 \leqslant p < \infty$ and let E be a measurable subset of a ball $B \subset \mathbb{R}^n$. Then*

$$\left(\frac{|E|}{|B|}\right)^p \leqslant C\frac{\omega(E)}{\omega(B)}$$

where C is the A_p constant of ω.

Remark 2.6. *If $\omega(E) = 0$ then $|E| = 0$. The measure ω and the Lebesgue measure $|.|$ are mutually absolutely continuous, that is they have the same zero sets $\big(\omega(E) = 0$ if and only if $|E| = 0\big)$; so there is no need to specify the measure when using the ubiquitous expression almost everywhere and almost every, both abbreviated a.e..*

The weighted Sobolev space $W^{1,p}(\Omega, \omega, v)$ is defined as follows.

Definition 2.7. *Let $\Omega \subset \mathbb{R}^n$ be open, and let ω and v be A_p-weights, $1 \leqslant p < \infty$. We define the weighted Sobolev space $W^{1,p}(\Omega, \omega, v)$ as the set of functions $u \in L^p(\Omega, v)$ with $D_k u \in L^p(\Omega, \omega)$, for $k = 1, ..., n$. The norm of u in $W^{1,p}(\Omega, \omega, v)$ is given by*

$$||u||_{W^{1,p}(\Omega,\omega,v)} = \left(\int_\Omega |u(x)|^p v(x)dx + \int_\Omega |\nabla u(x)|^p \omega(x)dx\right)^{\frac{1}{p}}. \tag{2.1}$$

We also define $W_0^{1,p}(\Omega, \omega, v)$ as the closure of $\mathcal{C}_0^\infty(\Omega)$ in $W^{1,p}(\Omega, \omega, v)$ with respect to the norm (2.1).

Equipped by this norm, $W^{1,p}(\Omega, \omega, v)$ and $W_0^{1,p}(\Omega, \omega, v)$ are separable and reflexive Banach spaces $\big($see Proposition 2.1.2. in [20] and see [19, 22] for more informations about the spaces $W^{1,p}(\Omega, \omega, v)\big)$. The dual of space $W_0^{1,p}(\Omega, \omega, v)$ is the space defined as

$$\Big[W_0^{1,p}(\Omega, \omega, v)\Big]^* = \left\{f - \sum_{i=1}^n D_i f_i : \frac{f}{v} \in L^{p'}(\Omega, v), \frac{f_i}{\omega} \in L^{p'}(\Omega, \omega), i = 1, ..., n\right\}.$$

To prove the main result of this paper, we use the following results.

Proposition 2.8. *[31](Convergence Principles). A sequence (x_n) in a Banach space X has the following convergence properties.*

(1) *Strong convergence. Let x be a fixed element of X. If every subsequence of (x_n) has, in turn, a subsequence which converges strongly to x, then the original sequence converges strongly to x.*

(2) *Weak convergence. Let x be a fixed element of X. If every subsequence of (x_n) has, in turn, a subsequence which converges weakly to x, then the original sequence converges weakly to x.*

Theorem 2.9. *[15] Let $\omega \in A_p$, $1 \leqslant p < \infty$, and let Ω be a bounded open set in $\mathbb{R}^n$. If $u_m \longrightarrow u$ in $L^p(\Omega,\omega)$, then there exist a subsequence (u_{i_j}) and $\psi \in L^p(\Omega,\omega)$ such that*

(i) *$u_{i_j}(x) \longrightarrow u(x)$, $i_j \longrightarrow \infty$, almost everywhere on Ω.*

(ii) *$|u_{i_j}(x)| \leqslant \psi(x)$, almost everywhere on Ω.*

Theorem 2.10. *[10] (The weighted Sobolev inequality) Let $\omega \in A_p$, $1 < p < \infty$, and let Ω be a bounded open set in $\mathbb{R}^n$. There exist constants C_Ω and δ positive such that for all $\varphi \in W_0^{1,p}(\Omega,\omega)$ and all ν satisfying $1 \leqslant \nu \leqslant \frac{n}{n-1} + \delta$,*

$$||\varphi||_{L^{\nu p}(\Omega,\omega)} \leqslant C_\Omega ||\nabla\varphi||_{L^p(\Omega,\omega)},$$

where C_Ω depends only on n, p, the A_p constant of ω and the diameter of Ω.

Remark 2.11. *Let $\omega, v \in A_p$. then,*

(i) *If $\omega = v$, then $C_0^\infty(\Omega)$ is dense in $W_0^{1,p}(\Omega,\omega) = W_0^{1,p}(\Omega,\omega,\omega)$.*

(ii) *If $\varphi \in W_0^{1,p}(\Omega,\omega,v)$, then by Theorem 2.10 (with $\nu = 1$), it holds that*

$$||\varphi||_{L^p(\Omega,\omega)} \leqslant C_\Omega ||\nabla\varphi||_{L^p(\Omega,\omega)} \leqslant C_\Omega ||\varphi||_{W_0^{1,p}(\Omega,\omega,v)}.$$

Hence, $W_0^{1,p}(\Omega,\omega,v) \subset W_0^{1,p}(\Omega,\omega)$.

Proposition 2.12. *[8] Let $1 < p < \infty$.*

(i) *There exists a positive constant C_p such that for all $\eta, \xi \in \mathbb{R}^n$, we have*

$$\Big||\xi|^{p-2}\xi - |\eta|^{p-2}\eta\Big| \leq C_p|\xi - \eta|\Big(|\xi| + |\eta|\Big)^{p-2}.$$

(ii) *There exist two positive constants β_p and γ_p such that for every $x, y \in \mathbb{R}^n$, it holds that*

$$\beta_p\Big(|x| + |y|\Big)^{p-2}|x-y|^2 \leq \Big\langle |x|^{p-2}x - |y|^{p-2}y, x - y\Big\rangle \leq \gamma_p\Big(|x| + |y|\Big)^{p-2}|x-y|^2.$$

The Browder-Minty Theorem is stated as follows.

Theorem 2.13. *[32] Let $A : Y \longrightarrow Y^*$ be a monotone, coercive and hemicontinuous operator on the real, separable, reflexive Banach space Y. Then the following assertions hold:*

(a) *For each $T \in Y^*$, the equation $Au = T$ has a solution $u \in Y$.*

(b) *If the operator A is strictly monotone, then equation $Au = T$ has a unique solution $u \in Y$.*

3. Basic assumptions and notion of solutions

3.1. Basic assumptions

Let us now give the precise hypotheses on the Problem (1.1), we assume that the following assumptions: Ω be a bounded open subset of $\mathbb{R}^n(n \geq 2)$, $1 < q, s < p < \infty$, let ω_1, ν_2, ν_1 and ω_2 are A_p-weight functions, and let $\mathcal{A} : \Omega \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$, $\mathcal{B} : \Omega \times \mathbb{R} \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$, with $\mathcal{A}(x, \xi) = \Big(\mathcal{A}_1(x, \xi), ..., \mathcal{A}_n(x, \xi)\Big)$ and $\mathcal{B}(x, \eta, \xi) = \Big(\mathcal{B}_1(x, \eta, \xi), ..., \mathcal{B}_n(x, \eta, \xi)\Big)$ and $\mathcal{C} : \Omega \times \mathbb{R} \longrightarrow \mathbb{R}$ satisfying the following assumptions:

(A1) For $k = 1, ..., n$, $\mathcal{A}_k$, $\mathcal{B}_k$ and $\mathcal{C}$ are Caratéodory functions.

(A2) There are positive functions h_1, h_2, h_3, $h_4 \in L^\infty(\Omega)$ and $\gamma_1 \in L^{p'}(\Omega, \omega_1)$ $\Big($with $\frac{1}{p} + \frac{1}{p'} = 1\Big)$, $\gamma_2 \in L^{q'}(\Omega, \nu_2)\Big($with $\frac{1}{q} + \frac{1}{q'} = 1\Big)$ and $\gamma_3 \in L^{s'}(\Omega, \nu_1)\Big($with $\frac{1}{s} + \frac{1}{s'} = 1\Big)$ such that

$$|\mathcal{A}(x, \xi)| \leq \gamma_1(x) + h_1(x)|\xi|^{p-1},$$

$$|\mathcal{B}(x, \eta, \xi)| \leq \gamma_2(x) + h_2(x)|\eta|^{q-1} + h_3(x)|\xi|^{q-1},$$

and

$$|\mathcal{C}(x, \eta)| \leq \gamma_3(x) + h_4(x)|\eta|^{s-1},$$

where $(\eta, \xi) \in \mathbb{R} \times \mathbb{R}^n$.

(A3) There exists a constant $\alpha > 0$ such that

$$\langle \mathcal{A}(x, \xi) - \mathcal{A}(x, \xi^{'}), \xi - \xi^{'} \rangle \geqslant \alpha|\xi - \xi^{'}|^p,$$

$$\langle \mathcal{B}(x, \eta, \xi) - \mathcal{B}(x, \eta^{'}, \xi^{'}), \xi - \xi^{'} \rangle \geqslant 0,$$

and

$$\Big(\mathcal{C}(x, \eta) - \mathcal{C}(x, \eta^{'})\Big)\Big(\eta - \eta^{'}\Big) \geqslant 0,$$

whenever $\eta, \eta' \in \mathbb{R}$ and $\xi, \xi' \in \mathbb{R}^n$ with $\eta \neq \eta^{'}$ and $\xi \neq \xi^{'}$ $\Big($where $\langle ., . \rangle$ denotes here the usual inner product in $\mathbb{R}^n\Big)$.

(A4) There are constants β_1, β_2, $\beta_3 > 0$ such that

$$\langle \mathcal{A}(x, \xi), \xi \rangle \geqslant \beta_1|\xi|^p,$$

$$\langle \mathcal{B}(x, \eta, \xi), \xi \rangle \geqslant \beta_2|\xi|^q + \beta_3|\eta|^q,$$

and

$$\mathcal{C}(x, \eta)\eta \geqslant 0,$$

for all $(\eta, \xi) \in \mathbb{R} \times \mathbb{R}^n$.

3.2. Notions of solutions

The definition of a weak solution for Problem (1.1) can be stated as follows.

Definition 3.1. *One says* $u \in W_0^{1,p}(\Omega, \omega_1, \omega_2)$ *is a weak solution to Problem* (1.1), *provided that*

$$\int_\Omega \langle \mathcal{A}(x, \nabla u), \nabla v\rangle \,\omega_1\, dx + \int_\Omega \langle \mathcal{B}(x, u, \nabla u), \nabla v\rangle \,\nu_2\, dx + \int_\Omega \mathcal{C}(x, u)\, v\, \nu_1\, dx$$
$$+ \int_\Omega |u|^{p-2} u\, v\, \omega_2\, dx = \int_\Omega f v dx + \sum_{j=1}^{n} \int_\Omega f_j D_j v dx,$$

for all $v \in W_0^{1,p}(\Omega, \omega_1, \omega_2)$.

Remark 3.2. *We seek to establish a relationship between* ω_1, ν_2 *and* ν_1, *in order to ensure the existence and uniqueness of solution for our Problem* (1.1). *At first we notice, for all* ω_1, ν_2, $\nu_1 \in A_p$ *:*

(i) *If* $\frac{\nu_2}{\omega_1} \in L^{r_1}(\Omega, \omega_1)$ *where* $r_1 = \frac{p}{p-q}$ *and* $1 < q < p < \infty$, *then, by Hölder inequality we obtain*

$$||u||_{L^q(\Omega,\nu_2)} \leqslant C_{p,q} ||u||_{L^p(\Omega,\omega_1)},$$

where $C_{p,q} = ||\frac{\nu_2}{\omega_1}||^{1/q}_{L^{r_1}(\Omega,\omega_1)}$.

(ii) *Analogously, if* $\frac{\nu_1}{\omega_1} \in L^{r_2}(\Omega, \omega_1)$ *where* $r_2 = \frac{p}{p-s}$ *and* $1 < s < p < \infty$, *then*

$$||u||_{L^s(\Omega,\nu_1)} \leqslant C_{p,s} ||u||_{L^p(\Omega,\omega_1)},$$

where $C_{p,s} = ||\frac{\nu_1}{\omega_1}||^{1/s}_{L^{r_2}(\Omega,\omega_1)}$.

4. Main result

4.1. Result on the existence and uniqueness

The main result of this article is given in the next theorem.

Theorem 4.1. *Let* $\omega_i, \nu_i \in A_p (i = 1, 2)$, $1 < q, s < p < \infty$ *and assume that the assumptions* ***(A1)** – **(A4)*** *hold. If*

1. $f \in L^{p'}(\Omega, \omega_2^{1-p'})$ *and* $f_j \in L^{p'}(\Omega, \omega_1^{1-p'})$ *for* $j = 1, ..., n$,
2. $\frac{\nu_2}{\omega_1} \in L^{p/(p-q)}(\Omega, \omega_1)$ *and* $\frac{\nu_1}{\omega_1} \in L^{p/(p-s)}(\Omega, \omega_1)$,

then the problem (1.1) *has a unique solution* $u \in W_0^{1,p}(\Omega, \omega_1, \omega_2)$.

4.2. Proof of Theorem 4.1

The essential one of our proof is to reduce the (1.1) to an operator problem $\mathbf{A}u = \mathbf{T}$ and apply the Theorem 2.13.

We define

$$\mathbf{O} : W_0^{1,p}(\Omega, \omega_1, \omega_2) \times W_0^{1,p}(\Omega, \omega_1, \omega_2) \longrightarrow \mathbb{R}$$

and

$$\mathbf{T} : W_0^{1,p}(\Omega, \omega_1, \omega_2) \longrightarrow \mathbb{R},$$

where $\mathbf{O}$ and $\mathbf{T}$ are defined below.
Then $u \in W_0^{1,p}(\Omega, \omega_1, \omega_2)$ is a weak solution of (1.1) if and only if

$$\mathbf{O}(u, v) = \mathbf{T}(v), \qquad \text{for all } v \in W_0^{1,p}(\Omega, \omega_1, \omega_2).$$

The proof of Theorem 4.1 is divided into several steps.

4.2.1. Equivalent operator equation

In this subsection, we prove that the Problem (1.1) is equivalent to an operator equation $\mathbf{A}u = \mathbf{T}$. We define the operator $\mathbf{T}$ by $\mathbf{T} = \int_\Omega fv dx + \sum_{j=1}^{n} \int_\Omega f_j D_j v dx$.

Using Hölder inequality, Theorem 2.10 and Remark 2.11 (ii), we obtain

$$\begin{aligned} |\mathbf{T}(v)| &\leq \int_\Omega \frac{|f|}{\omega_2} |v|\omega_2 \, dx + \sum_{j=1}^{n} \int_\Omega \frac{|f_j|}{\omega_1} |D_j v| \, \omega_1 dx \\ &\leq ||f/\omega_2||_{L^{p'}(\Omega,\omega_2)} ||v||_{L^p(\Omega,\omega_2)} + \sum_{j=1}^{n} ||f_j/\omega_1||_{L^{p'}(\Omega,\omega_1)} ||D_j v||_{L^p(\Omega,\omega_1)} \\ &\leq \left(C_\Omega ||f/\omega_2||_{L^{p'}(\Omega,\omega_2)} + \sum_{j=1}^{n} ||f_j/\omega_1||_{L^{p'}(\Omega,\omega_1)} \right) ||v||_{W_0^{1,p}(\Omega,\omega_1,\omega_2)}. \end{aligned}$$

According to $f \in L^{p'}(\Omega, \omega_2^{1-p'})$ and $f_j \in L^{p'}(\Omega, \omega_1^{1-p'})$ for $j = 1, ..., n$, we deduce that $\mathbf{T} \in \left[W_0^{1,p}(\Omega, \omega_1, \omega_2) \right]^*$.

The operator $\mathbf{F}$ is broken down into the from

$$\mathbf{O}(u, v) = \mathbf{O}_1(u, v) + \mathbf{O}_2(u, v) + \mathbf{O}_3(u, v) + \mathbf{O}_4(u, v),$$

where $\mathbf{O}_i : W_0^{1,p}(\Omega, \omega_1, \omega_2) \times W_0^{1,p}(\Omega, \omega_1, \omega_2) \longrightarrow \mathbb{R}$, for $i = 1, 2, 3, 4$, are defined as

$$\mathbf{O}_1(u, v) = \int_\Omega \langle \mathcal{A}(x, \nabla u), \nabla v \rangle \omega_1 dx, \quad \mathbf{O}_2(u, v) = \int_\Omega \langle \mathcal{B}(x, u, \nabla u), \nabla v \rangle \nu_2 dx,$$

$$\mathbf{O}_3(u, v) = \int_\Omega \mathcal{C}(x, u) v \, \nu_1 dx \quad \text{and} \quad \mathbf{O}_4(u, v) = \int_\Omega |u|^{p-2} u \, v \, \omega_2 \, dx.$$

Then, we have

$$|\mathbf{O}(u, v)| \leq |\mathbf{O}_1(u, v)| + |\mathbf{O}_2(u, v)| + |\mathbf{O}_3(u, v)| + |\mathbf{O}_4(u, v)|. \tag{4.1}$$

On the other hand, we get by using **(A2)**, Hölder inequality, Remark 3.2 (i) and Theorem 2.10,

$$\begin{aligned} &|\mathbf{O}_1(u, v)| \\ &\leq \int_\Omega |\mathcal{A}(x, \nabla u)| |\nabla v| \omega_1 dx \\ &\leq \int_\Omega \left(\gamma_1 + h_1 |\nabla u|^{p-1} \right) |\nabla v| \omega_1 dx \\ &\leq ||\gamma_1||_{L^{p'}(\Omega,\omega_1)} ||\nabla v||_{L^p(\Omega,\omega_1)} + ||h_1||_{L^\infty(\Omega)} ||\nabla u||^{p-1}_{L^p(\Omega,\omega_1)} ||\nabla v||_{L^p(\Omega,\omega_1)} \\ &\leq \left(||\gamma_1||_{L^{p'}(\Omega,\omega_1)} + ||h_1||_{L^\infty(\Omega)} ||u||^{p-1}_{W_0^{1,p}(\Omega,\omega_1,\omega_2)} \right) ||v||_{W_0^{1,p}(\Omega,\omega_1,\omega_2)}, \end{aligned}$$

and

$$
\begin{aligned}
&|\mathbf{O}_2(u,v)| \\
&\leq \int_\Omega |\mathcal{B}(x,u,\nabla u)||\nabla v|\nu_2 dx \\
&\leq \int_\Omega \Big(\gamma_2 + h_2|u|^{q-1} + h_3|\nabla u|^{q-1}\Big)|\nabla v|\nu_2 dx \\
&\leq ||\gamma_2||_{L^{q'}(\Omega,\nu_2)}||\nabla v||_{L^q(\Omega,\nu_2)} + ||h_2||_{L^\infty(\Omega)}||u||^{q-1}_{L^q(\Omega,\nu_2)}||\nabla v||_{L^q(\Omega,\nu_2)} \\
&\quad + ||h_3||_{L^\infty(\Omega)}||\nabla u||^{q-1}_{L^q(\Omega,\nu_2)}||\nabla v||_{L^q(\Omega,\nu_2)} \\
&\leq ||\gamma_2||_{L^{q'}(\Omega,\nu_2)}C_{p,q}||\nabla v||_{L^p(\Omega,\omega_1)} + ||h_2||_{L^\infty(\Omega)}C^{q-1}_{p,q}||u||^{q-1}_{L^p(\Omega,\omega_1)}C_{p,q}||\nabla v||_{L^p(\Omega,\omega_1)} \\
&\quad + ||h_3||_{L^\infty(\Omega)}C^{q-1}_{p,q}||\nabla u||^{q-1}_{L^p(\Omega,\omega_1)}C_{p,q}||\nabla v||_{L^p(\Omega,\omega_1)} \\
&\leq \Big[C^q_{p,q}\left(C^{q-1}_\Omega||h_2||_{L^\infty(\Omega)} + ||h_3||_{L^\infty(\Omega)}\right)||u||^{q-1}_{W^{1,p}_0(\Omega,\omega_1,\omega_2)} \\
&\quad + C_{p,q}||\gamma_2||_{L^{q'}(\Omega,\nu_2)}\Big]||v||_{W^{1,p}_0(\Omega,\omega_1,\omega_2)}.
\end{aligned}
$$

Analogously, using **(A2)**, Hölder inequality, Remark 3.2 (ii) and Theorem 2.10, we obtain

$$
\begin{aligned}
&|\mathbf{O}_3(u,v)| \\
&\leq \int_\Omega |\mathcal{C}(x,u)||v|\nu_1 dx \\
&\leq \Big[C_\Omega C_{p,s}||\gamma_3||_{L^{s'}(\Omega,\nu_1)} + C^s_{p,s}C^s_\Omega||h_4||_{L^\infty(\Omega)}||u||^{s-1}_{W^{1,p}_0(\Omega,\omega_1,\omega_2)}\Big]||v||_{W^{1,p}_0(\Omega,\omega_1,\omega_2)}.
\end{aligned}
$$

Next, by applying Hölder inequality and Remark 2.11 (ii), we get

$$
\begin{aligned}
|\mathbf{O}_4(u,v)| &\leq \int_\Omega |u|^{p-1}|v|\omega_2 dx \\
&\leq \Big(\int_\Omega |u|^p\omega_2 dx\Big)^{1/p'}\Big(\int_\Omega |v|^p\omega_2 dx\Big)^{1/p} \\
&= ||u||^{p-1}_{L^p(\Omega,\omega_2)}||v||_{L^p(\Omega,\omega_2)} \\
&\leq C_\Omega||u||^{p-1}_{W^{1,p}_0(\Omega,\omega_1,\omega_2)}||v||_{W^{1,p}_0(\Omega,\omega_1,\omega_2)}
\end{aligned}
$$

Hence, in (4.1) we obtain, for all $u,v\in W^{1,p}_0(\Omega,\omega_1,\omega_2)$

$$
\begin{aligned}
&|\mathbf{O}(u,v)| \\
&\leq \Big[||\gamma_1||_{L^{p'}(\Omega,\omega_1)} + ||h_1||_{L^\infty(\Omega)}||u||^{p-1}_{W^{1,p}_0(\Omega,\omega_1,\omega_2)} + C_\Omega C_{p,s}||\gamma_3||_{L^{s'}(\Omega,\nu_1)} \\
&\quad + C_{p,q}||\gamma_2||_{L^{q'}(\Omega,\nu_2)} + C^q_{p,q}\left(C^{q-1}_\Omega||h_2||_{L^\infty(\Omega)} + ||h_3||_{L^\infty(\Omega)}\right)||u||^{q-1}_{W^{1,p}_0(\Omega,\omega_1,\omega_2)} \\
&\quad + C^s_{p,s}C^s_\Omega||h_4||_{L^\infty(\Omega)}||u||^{s-1}_{W^{1,p}_0(\Omega,\omega_1,\omega_2)} + C_\Omega||u||^{p-1}_{W^{1,p}_0(\Omega,\omega_1,\omega_2)}\Big]||v||_{W^{1,p}_0(\Omega,\omega_1,\omega_2)}.
\end{aligned}
$$

Then $\mathbf{O}(u,.)$ is linear and continuous, for each $u\in W^{1,p}_0(\Omega,\omega_1,\omega_2)$. Thus, there exists a linear and continuous operator on $W^{1,p}_0(\Omega,\omega_1,\omega_2)$ denoted by $\mathbf{A}$ such that

$$
\langle \mathbf{A}u, v\rangle = \mathbf{O}(u,v), \quad \text{for all } u,v\in W^{1,p}_0(\Omega,\omega_1,\omega_2).
$$

Moreover, we have

$$
\begin{aligned}
&\|\mathbf{A}u\|_* \\
&\leq ||\gamma_1||_{L^{p'}(\Omega,\omega_1)} + ||h_1||_{L^\infty(\Omega)}||u||^{p-1}_{W^{1,p}_0(\Omega,\omega_1,\omega_2)} + C_\Omega C_{p,s}||\gamma_3||_{L^{s'}(\Omega,\nu_1)} \\
&\quad + C_{p,q}||\gamma_2||_{L^{q'}(\Omega,\nu_2)} + C^q_{p,q}\left(C^{q-1}_\Omega||h_2||_{L^\infty(\Omega)} + ||h_3||_{L^\infty(\Omega)}\right)||u||^{q-1}_{W^{1,p}_0(\Omega,\omega_1,\omega_2)} \\
&\quad + C^s_{p,s}C^s_\Omega||h_4||_{L^\infty(\Omega)}||u||^{s-1}_{W^{1,p}_0(\Omega,\omega_1,\omega_2)} + C_\Omega||u||^{p-1}_{W^{1,p}_0(\Omega,\omega_1,\omega_2)},
\end{aligned}
$$

where

$$\|\mathbf{A}u\|_* := \sup\Big\{|\langle \mathbf{A}u, v\rangle| = |\mathbf{O}(u,v)| : v \in W_0^{1,p}(\Omega,\omega_1,\omega_2), \|v\|_{W_0^{1,p}(\Omega,\omega_1,\omega_2)} = 1\Big\},$$

is the norm in $\Big[W_0^{1,p}(\Omega,\omega_1,\omega_2)\Big]^*$. Hence, we obtain the operator

$$\begin{aligned}\mathbf{A} : W_0^{1,p}(\Omega,\omega_1,\omega_2) &\longrightarrow \Big[W_0^{1,p}(\Omega,\omega_1,\omega_2)\Big]^* \\ u &\longmapsto \mathbf{A}u.\end{aligned}$$

However, the Problem (1.1) is equivalent to the operator equation

$$\mathbf{A}u = \mathbf{T}, \quad u \in W_0^{1,p}(\Omega,\omega_1,\omega_2).$$

*4.2.2. Monotonicity of the operator **A***

The operator $\mathbf{A}$ is strictly monotone. In fact.
Let $v_1, v_2 \in W_0^{1,p}(\Omega,\omega_1,\omega_2)$ with $v_1 \neq v_2$. We have

$$\begin{aligned}&\langle \mathbf{A}v_1 - \mathbf{A}v_2, v_1 - v_2\rangle \\ &= \mathbf{O}(v_1, v_1 - v_2) - \mathbf{O}(v_2, v_1 - v_2) \\ &= \int_\Omega \langle \mathcal{A}(x,\nabla v_1), \nabla(v_1 - v_2)\rangle \omega_1 dx - \int_\Omega \langle \mathcal{A}(x,\nabla v_2), \nabla(v_1 - v_2)\rangle \omega_1 dx \\ &\quad + \int_\Omega \langle \mathcal{B}(x,v_1,\nabla v_1), \nabla(v_1 - v_2)\rangle \nu_2 dx - \int_\Omega \langle \mathcal{B}(x,v_2,\nabla v_2), \nabla(v_1 - v_2)\rangle \nu_2 dx \\ &\quad + \int_\Omega \mathcal{C}(x,v_1)(v_1 - v_2)\nu_1 dx - \int_\Omega \mathcal{C}(x,v_2)(v_1 - v_2)\nu_1 dx \\ &\quad + \int_\Omega |v_1|^{p-2}v_1(v_1 - v_2)\omega_2 dx - \int_\Omega |v_2|^{p-2}v_2(v_1 - v_2)\omega_2 dx \\ &= \int_\Omega \langle \mathcal{A}(x,\nabla v_1) - \mathcal{A}(x,\nabla v_2), \nabla(v_1 - v_2)\rangle \omega_1 dx \\ &\quad + \int_\Omega \langle \mathcal{B}(x,v_1,\nabla v_1) - \mathcal{B}(x,v_2,\nabla v_2), \nabla(v_1 - v_2)\rangle \nu_2 dx \\ &\quad + \int_\Omega \Big(\mathcal{C}(x,v_1) - \mathcal{C}(x,v_2)\Big)\Big(v_1 - v_2\Big)\nu_1 dx \\ &\quad + \int_\Omega \Big(|v_1|^{p-2}v_1 - |v_2|^{p-2}v_2\Big)\Big(v_1 - v_2\Big)\omega_2 dx\end{aligned}$$

Thanks to **(A3)** and Proposition 2.12 (ii), we obtain

$$\begin{aligned}&\langle \mathbf{A}v_1 - \mathbf{A}v_2, v_1 - v_2\rangle \\ &\geq \alpha \int_\Omega |\nabla(v_1 - v_2)|^p\, \omega_1\, dx + \beta_p \int_\Omega \Big(|v_1| + |v_2|\Big)^{p-2} |v_1 - v_2|^2\, \omega_2\, dx \\ &\geq \alpha \int_\Omega |\nabla(v_1 - v_2)|^p \omega_1 dx \\ &\geq \alpha \|\nabla(v_1 - v_2)\|^p_{L^p(\Omega,\omega_1)}.\end{aligned}$$

Therefore, the operator $\mathbf{A}$ is strictly monotone.

*4.2.3. Coercivity of the operator **A***

In this step, we prove that the operator $\mathbf{A}$ is coercive. To this purpose let $u \in W_0^{1,p}(\Omega, \omega_1, \omega_2)$, we have

$$\begin{aligned}\langle \mathbf{A}u, u\rangle &= \mathbf{O}(u,u)\\ &= \mathbf{O}_1(u,u) + \mathbf{O}_2(u,u) + \mathbf{O}_3(u,u) + \mathbf{O}_4(u,u)\\ &= \int_\Omega \langle \mathcal{A}(x,\nabla u), \nabla u\rangle \omega_1 dx + \int_\Omega \langle \mathcal{B}(x,u,\nabla u), \nabla u\rangle \nu_2 dx + \int_\Omega \mathcal{C}(x,u) u\, \nu_1 dx + \int_\Omega |u|^p \omega_2 dx.\end{aligned}$$

Moreover, from **(A4)** and Theorem 2.10(with $\nu = 1$), we obtain

$$\begin{aligned}\langle \mathbf{A}u, u\rangle &\geq \beta_1 \int_\Omega |\nabla u|^p \omega_1 dx + \beta_2 \int_\Omega |\nabla u|^q \nu_2 dx + \beta_3 \int_\Omega |u|^q \nu_2 dx + \int_\Omega |u|^p \omega_2 dx\\ &\geq min(\beta_1, 1)\left[\int_\Omega |\nabla u|^p \omega_1 dx + \int_\Omega |u|^p \omega_2 dx\right] + min(\beta_2, \beta_3)\left[\int_\Omega |\nabla u|^q \nu_2 dx + \int_\Omega |u|^q \nu_2 dx\right]\\ &\geq min(\beta_1, 1)\|u\|^p_{W_0^{1,p}(\Omega,\omega_1,\omega_2)}.\end{aligned}$$

Hence, we obtain

$$\frac{\langle \mathbf{A}u, u\rangle}{\|u\|_{W_0^{1,p}(\Omega,\omega_1,\omega_2)}} \geq min(\beta_1, 1)\|u\|^{p-1}_{W_0^{1,p}(\Omega,\omega_1,\omega_2)}.$$

Therefore, since $p > 1$, we have

$$\frac{\langle \mathbf{A}u, u\rangle}{\|u\|_{W_0^{1,p}(\Omega,\omega_1,\omega_2)}} \longrightarrow +\infty \text{ as } \|u\|_{W_0^{1,p}(\Omega,\omega_1,\omega_2)} \longrightarrow +\infty,$$

that is, $\mathbf{A}$ is coercive.

*4.2.4. Continuity of the operator **A***

We need to show that the operator $\mathbf{A}$ is continuous. To this purpose let $u_i \longrightarrow u$ in $W_0^{1,p}(\Omega, \omega_1, \omega_2)$ as $i \longrightarrow \infty$. Then $\nabla u_i \longrightarrow \nabla u$ in $(L^p(\Omega, \omega_1))^n$. Hence, thanks to Theorem 2.9, there exist a subsequence (u_{i_j}) and $\psi \in L^p(\Omega, \omega_1)$ such that

$$\begin{aligned}&\nabla u_{i_j}(x) \longrightarrow \nabla u(x), \quad \text{a.e. in } \Omega\\ &|\nabla u_{i_j}(x)| \leq \psi(x), \quad \text{a.e. in } \Omega.\end{aligned} \tag{4.2}$$

We will show that $\mathbf{A}u_i \longrightarrow \mathbf{A}u$ in $\left[W_0^{1,p}(\Omega, \omega_1, \omega_2)\right]^*$. In order to prove this convergence we proceed in several steps.

Step 1:

For $k = 1, ..., n$, we define the operator

$$\begin{aligned}&B_k : W_0^{1,p}(\Omega, \omega_1, \omega_2) \longrightarrow L^{p'}(\Omega, \omega_1)\\ &(B_k u)(x) = \mathcal{A}_k(x, \nabla u(x)).\end{aligned}$$

We need to show that $B_k u_i \longrightarrow B_k u$ in $L^{p'}(\Omega, \omega_1)$. We will apply the Lebesgue's theorem and the convergence principle in Banach spaces.

(i) Let $u \in W_0^{1,p}(\Omega, \omega_1, \omega_2)$. Using **(A2)** and Theorem 2.10(with $\nu = 1$), we obtain

$$\begin{aligned}
\|B_k u\|^{p'}_{L^{p'}(\Omega,\omega_1)} &= \int_\Omega |B_k u(x)|^{p'} \omega_1 dx = \int_\Omega |\mathcal{A}_k(x, \nabla u)|^{p'} \omega_1 dx \\
&\leq \int_\Omega \left(\gamma_1 + h_1 |\nabla u|^{p-1}\right)^{p'} \omega_1 dx \\
&\leq C_p \int_\Omega \left(\gamma_1^{p'} + h_1^{p'} |\nabla u|^p\right) \omega_1 dx \\
&\leq C_p \left[\|\gamma_1\|^{p'}_{L^{p'}(\Omega,\omega_1)} + \|h_1\|^{p'}_{L^\infty(\Omega)} \|\nabla u\|^p_{L^p(\Omega,\omega_1)}\right] \\
&\leq C_p \left[\|\gamma_1\|^{p'}_{L^{p'}(\Omega,\omega_1)} + \|h_1\|^{p'}_{L^\infty(\Omega)} \|u\|^p_{W_0^{1,p}(\Omega,\omega_1,\omega_2)}\right],
\end{aligned}$$

where the constant C_p depends only on p.

(ii) Let $u_i \longrightarrow u$ in $W_0^{1,p}(\Omega, \omega_1, \omega_2)$ as $i \longrightarrow \infty$. By **(A2)** and (4.2), we obtain

$$\begin{aligned}
&\|B_k u_{i_j} - B_k u\|^{p'}_{L^{p'}(\Omega,\omega_1)} \\
&= \int_\Omega |B_k u_{i_j}(x) - B_k u(x)|^{p'} \omega_1 dx \\
&\leq \int_\Omega \Big(|\mathcal{A}_k(x, \nabla u_{i_j})| + |\mathcal{A}_k(x, \nabla u)|\Big)^{p'} \omega_1 dx \\
&\leq C_p \int_\Omega \Big(|\mathcal{A}_k(x, \nabla u_{i_j})|^{p'} + |\mathcal{A}_k(x, \nabla u)|^{p'}\Big) \omega_1 dx \\
&\leq C_p \int_\Omega \left[\left(\gamma_1 + h_1 |\nabla u_{i_j}|^{p-1}\right)^{p'} + \left(\gamma_1 + h_1 |\nabla u|^{p-1}\right)^{p'}\right] \omega_1 dx \\
&\leq C_p \int_\Omega \left[\left(\gamma_1 + h_1 \psi^{p-1}\right)^{p'} + \left(\gamma_1 + h_1 \psi^{p-1}\right)^{p'}\right] \omega_1 dx \\
&\leq 2 C_p C'_p \int_\Omega \left(\gamma_1^{p'} + h_1^{p'} \psi^p\right) \omega_1 dx \\
&\leq 2 C_p C'_p \left[\|\gamma_1\|^{p'}_{L^{p'}(\Omega,\omega_1)} + \|h_1\|^{p'}_{L^\infty(\Omega)} \|\psi\|^p_{L^p(\Omega,\omega_1)}\right].
\end{aligned}$$

Hence, thanks to **(A1)**, we get, as $i \longrightarrow \infty$

$$B_k u_{i_j}(x) = \mathcal{A}_k(x, \nabla u_{i_j}(x)) \longrightarrow \mathcal{A}_k(x, \nabla u(x)) = B_k u(x), \quad \text{a.e. } x \in \Omega.$$

Therefore, by Lebesgue's theorem, we obtain

$$\|B_k u_{i_j} - B_k u\|_{L^{p'}(\Omega,\omega_1)} \longrightarrow 0,$$

that is,

$$B_k u_{i_j} \longrightarrow B_k u \quad \text{in} \quad L^{p'}(\Omega, \omega_1).$$

Finally, in view to convergence principle in Banach spaces, we have

$$B_k u_i \longrightarrow B_k u \quad \text{in} \quad L^{p'}(\Omega, \omega_1). \tag{4.3}$$

Step 2:
For $k = 1, ..., n$, we define the operator

$$\begin{aligned}
M_k &: W_0^{1,p}(\Omega, \omega_1, \omega_2) \longrightarrow L^{q'}(\Omega, \nu_2) \\
(M_k u)(x) &= \mathcal{B}_k(x, u(x), \nabla u(x)).
\end{aligned}$$

We will prove that $M_k u_i \longrightarrow M_k u$ in $L^{q'}(\Omega, \nu_2)$.

(i) Let $u \in W_0^{1,p}(\Omega,\omega_1,\omega_2)$. Using **(A2)**, Remark 3.2 **(i)** and Theorem 2.10(with $\nu = 1$), we obtain

$$\begin{aligned}
&\|M_k u\|^{q'}_{L^{q'}(\Omega,\nu_2)}\\
&= \int_\Omega |\mathcal{B}_k(x,u,\nabla u)|^{q'}\nu_2 dx\\
&\le \int_\Omega \left(\gamma_2 + h_2|u|^{q-1} + h_3|\nabla u|^{q-1}\right)^{q'}\nu_2 dx\\
&\le C_q \int_\Omega \left[\gamma_2^{q'} + h_2^{q'}|u|^q + h_3^{q'}|\nabla u|^q\right]\nu_2 dx\\
&\le C_q \left[\|\gamma_2\|^{q'}_{L^{q'}(\Omega,\nu_2)} + \|h_2\|^{q'}_{L^\infty(\Omega)}\|u\|^q_{L^q(\Omega,\nu_2)} + \|h_3\|^{q'}_{L^\infty(\Omega)}\|\nabla u\|^q_{L^q(\Omega,\nu_2)}\right]\\
&\le C_q \left[\|\gamma_2\|^{q'}_{L^{q'}(\Omega,\nu_2)} + \|h_2\|^{q'}_{L^\infty(\Omega)}C^q_{p,q}\|u\|^q_{L^p(\Omega,\omega_1)} + \|h_3\|^{q'}_{L^\infty(\Omega)}C^q_{p,q}\|\nabla u\|^q_{L^p(\Omega,\omega_1)}\right]\\
&\le C_q \left[\|\gamma_2\|^{q'}_{L^{q'}(\Omega,\nu_2)} + C^q_{p,q}\left(C^q_\Omega\|h_2\|^{q'}_{L^\infty(\Omega)} + \|h_3\|^{q'}_{L^\infty(\Omega)}\right)\|u\|^q_{W_0^{1,p}(\Omega,\omega_1,\omega_2)}\right],
\end{aligned}$$

where the constant C_q depends only on q.

(ii) Let $u_i \longrightarrow u$ in $W_0^{1,p}(\Omega,\omega_1,\omega_2)$ as $i \longrightarrow \infty$. According to **(A2)**, Remark 3.2 (i) and the same arguments used in Step 1 **(ii)**, we obtain analogously,

$$M_k u_i \longrightarrow M_k u \quad \text{in} \quad L^{q'}(\Omega,\nu_2). \tag{4.4}$$

Step 3:
We define the operator

$$\begin{aligned}
&H : W_0^{1,p}(\Omega,\omega_1,\omega_2) \longrightarrow L^{s'}(\Omega,\nu_1)\\
&(Hu)(x) = \mathcal{C}(x,u(x)).
\end{aligned}$$

In this step, we will show that $Hu_i \longrightarrow Hu$ in $L^{s'}(\Omega,\nu_1)$.

(i) Let $u \in W_0^{1,p}(\Omega,\omega_1,\omega_2)$. Using **(A2)** and Remark 3.2 (ii), we obtain

$$\begin{aligned}
\|Hu\|^{s'}_{L^{s'}(\Omega,\nu_1)} &= \int_\Omega |\mathcal{C}(x,u)|^{s'}\nu_1 dx\\
&\le \int_\Omega \left(\gamma_3 + h_4|u|^{s-1}\right)^{s'}\nu_1 dx\\
&\le C_s \int_\Omega \left(\gamma_3^{s'} + h_4^{s'}|u|^s\right)\nu_1 dx\\
&\le C_s \left[\|\gamma_3\|^{s'}_{L^{s'}(\Omega,\nu_1)} + \|h_4\|^{p'}_{L^\infty(\Omega)}\|u\|^s_{L^s(\Omega,\nu_1)}\right]\\
&\le C_s \left[\|\gamma_3\|^{s'}_{L^{s'}(\Omega,\nu_1)} + C^s_{p,s}\|h_4\|^{p'}_{L^\infty(\Omega)}\|u\|^s_{L^p(\Omega,\omega_1)}\right]\\
&\le C_s \left[\|\gamma_3\|_{L^{s'}(\Omega,\omega_1)} + C^s_{p,s}C^s_\Omega\|h_4\|^{s'}_{L^\infty(\Omega)}\|u\|^s_{W_0^{1,p}(\Omega,\omega_1,\omega_2)}\right],
\end{aligned}$$

where the constant C_s depends only on s.

(ii) Let $u_i \longrightarrow u$ in $W_0^{1,p}(\Omega,\omega_1,\omega_2)$ as $i \longrightarrow \infty$. By **(A2)** and Remark 3.2 (ii), we get

$$\begin{aligned}
&\|Hu_{i_j} - Hu\|^{s'}_{L^{s'}(\Omega,\nu_1)}\\
&= \int_\Omega \Big|Hu_{i_j}(x) - Hu(x)\Big|^{p'} \nu_1 dx\\
&\le \int_\Omega \Big(|\mathcal{C}(x,u_{i_j})| + |\mathcal{C}(x,u)|\Big)^{s'} \nu_1 dx\\
&\le C_s \int_\Omega \Big(|\mathcal{C}(x,u_{i_j})|^{s'} + |\mathcal{C}(x,u)|^{s'}\Big)\nu_1 dx\\
&\le C_s \int_\Omega \Big[\Big(\gamma_3 + h_4|u_{i_j}|^{s-1}\Big)^{s'} + \Big(\gamma_3 + h_4|u|^{s-1}\Big)^{s'}\Big]\nu_1 dx\\
&\le C_s \int_\Omega \Big[\Big(\gamma_3 + h_4|\psi|^{s-1}\Big)^{s'} + \Big(\gamma_3 + h_4\psi^{s-1}\Big)^{s'}\Big]\nu_1 dx\\
&\le 2C_sC'_s\Big[\|\gamma_3\|^{s'}_{L^{s'}(\Omega,\nu_1)} + \|h_4\|^{s'}_{L^\infty(\Omega)}\|\psi\|^{s}_{L^s(\Omega,\nu_1)}\Big]\\
&\le 2C_sC'_s\Big[\|\gamma_3\|^{s'}_{L^{s'}(\Omega,\nu_1)} + C^s_{p,s}\|h_4\|^{s'}_{L^\infty(\Omega)}\|\psi\|^{s}_{L^p(\Omega,\omega_1)}\Big],
\end{aligned}$$

next, using condition **(A1)**, we deduce, as $i \longrightarrow \infty$

$$Hu_{i_j}(x) = \mathcal{C}(x,u_{i_j}(x)) \longrightarrow \mathcal{C}(x,u(x)) = Hu(x), \quad \text{a.e. } x \in \Omega.$$

Therefore, by the Lebesgue's theorem, we obtain

$$\|Hu_{i_j} - Hu\|_{L^{s'}(\Omega,\nu_1)} \longrightarrow 0,$$

that is,

$$Hu_{i_j} \longrightarrow Hu \quad \text{in} \quad L^{s'}(\Omega,\nu_1).$$

We conclude, from the convergence principle in Banach spaces, that

$$Hu_i \longrightarrow Hu \quad \text{in} \quad L^{s'}(\Omega,\nu_1). \tag{4.5}$$

Step 4:
We define the operator

$$\begin{aligned}
&J : W_0^{1,p}(\Omega,\omega_1,\omega_2) \longrightarrow L^{p'}(\Omega,\omega_2)\\
&(Ju)(x) = |u(x)|^{p-2}u(x).
\end{aligned}$$

In this step, we will demonstrate that $Ju_i \longrightarrow Ju$ in $L^{p'}(\Omega,\omega_2)$.

(i) Let $u \in W_0^{1,p}(\Omega,\omega_1,\omega_2)$. We have

$$\begin{aligned}
\|Ju\|^{p'}_{L^{p'}(\Omega,\omega_2)} &= \int_\Omega |Ju|^{p'}\omega_2 dx\\
&= \int_\Omega |u|^{(p-1)p'}\omega_2 dx\\
&= \int_\Omega |u|^{p}\omega_2 dx\\
&\le \|u\|^{p}_{W_0^{1,p}(\Omega,\omega_1,\omega_2)}.
\end{aligned}$$

(ii) Let $u_i \longrightarrow u$ in $W_0^{1,p}(\Omega,\omega_1,\omega_2)$ as $i \longrightarrow \infty$. Then $u_i \longrightarrow u$ in $L^p(\Omega,\omega_2)$. Hence, thanks to Theorem 2.9, there exist a subsequence (u_{i_j}) and $\varphi \in L^p(\Omega,\omega_2)$ such that

$$u_{i_j}(x) \longrightarrow u(x), \quad \text{a.e. in } \Omega$$

$$|u_{i_j}(x)| \le \varphi(x), \quad \text{a.e. in } \Omega.$$

Next, we get

$$\begin{aligned}
\|Ju_{i_j} - Ju\|^{p'}_{L^{p'}(\Omega,\omega_2)} &= \int_\Omega \Big|Ju_{i_j}(x) - Ju(x)\Big|^{p'} \omega_2 dx \\
&\le \int_\Omega \Big(|Ju_{i_j}(x)| + |Ju(x)|\Big)^{p'} \omega_2 dx \\
&\le C_p \int_\Omega \Big(|Ju_{i_j}(x)|^{p'} + |Ju(x)|^{p'}\Big)\omega_2 dx \\
&\le C_p \int_\Omega \Big(||u_{i_j}|^{p-2}u_{i_j}|^{p'} + ||u|^{p-2}u|^{p'}\Big)\omega_2 dx \\
&\le C_p \int_\Omega \Big(|u_{i_j}|^{(p-1)p'} + |u|^{(p-1)p'}\Big)\omega_2 dx \\
&\le C_p \int_\Omega \Big(|u_{i_j}|^{p} + |u|^{p}\Big)\omega_2 dx \\
&\le C_p \int_\Omega \Big(|\varphi|^{p} + |\varphi|^{p}\Big)\omega_2 dx \\
&\le 2C_p \int_\Omega |\varphi|^p \omega_2 dx \\
&\le 2C_p \|\varphi\|^p_{L^p(\Omega,\omega_2)}.
\end{aligned}$$

Therefore, by Lebesgue's theorem, we obtain

$$\|Ju_{i_j} - Ju\|_{L^{p'}(\Omega,\omega_2)} \longrightarrow 0,$$

that is,

$$Ju_{i_j} \longrightarrow Ju \quad \text{in} \quad L^{p'}(\Omega,\omega_2).$$

We conclude, in view to convergence principle in Banach spaces, that

$$Ju_i \longrightarrow Ju \quad \text{in} \quad L^{p'}(\Omega,\omega_2). \tag{4.6}$$

Finally, let $v \in W_0^{1,p}(\Omega,\omega_1,\omega_2)$ and using Hölder inequality, we obtain

$$\begin{aligned}
|\mathbf{O}_1(u_i,v) - \mathbf{O}_1(u,v)| &= \Big|\int_\Omega \langle \mathcal{A}(x,\nabla u_i) - \mathcal{A}(x,\nabla u), \nabla v\rangle \omega_1 dx\Big| \\
&\le \sum_{k=1}^n \int_\Omega |\mathcal{A}_k(x,\nabla u_i) - \mathcal{A}_k(x,\nabla u)||D_k v|\omega_1 dx \\
&= \sum_{k=1}^n \int_\Omega |B_k u_i - B_k u||D_k v|\omega_1 dx \\
&\le \sum_{k=1}^n \|B_k u_i - B_k u\|_{L^{p'}(\Omega,\omega_1)} \|D_k v\|_{L^p(\Omega,\omega_1)} \\
&\le \left(\sum_{k=1}^n \|B_k u_i - B_k u\|_{L^{p'}(\Omega,\omega_1)}\right) \|v\|_{W_0^{1,p}(\Omega,\omega_1,\omega_2)},
\end{aligned}$$

and by Remark 3.2 (i), we get

$$\begin{aligned}
|\mathbf{O}_2(u_i,v)-\mathbf{O}_2(u,v)| &= |\int_\Omega \langle \mathcal{B}(x,u_i,\nabla u_i)-\mathcal{B}(x,u,\nabla u),\nabla v\rangle \nu_2 dx| \\
&\leq \sum_{k=1}^n \int_\Omega |\mathcal{B}_k(x,u_i,\nabla u_i)-\mathcal{B}_k(x,u,\nabla u)||D_k v|\nu_2 dx \\
&= \sum_{k=1}^n \int_\Omega |M_k u_i - M_k u||D_k v|\nu_2 dx \\
&\leq \left(\sum_{k=1}^n \|M_k u_i - M_k u\|_{L^{q'}(\Omega,\nu_2)}\right)\|\nabla v\|_{L^q(\Omega,\nu_2)} \\
&\leq C_{p,q}\left(\sum_{k=1}^n \|M_k u_i - M_k u\|_{L^{q'}(\Omega,\nu_2)}\right)\|\nabla v\|_{L^p(\Omega,\omega_1)} \\
&\leq C_{p,q}\left(\sum_{k=1}^n \|M_k u_i - M_k u\|_{L^{q'}(\Omega,\nu_2)}\right)\|v\|_{W_0^{1,p}(\Omega,\omega_1,\omega_2)},
\end{aligned}$$

and by Remark 3.2 (ii), we get

$$\begin{aligned}
|\mathbf{O}_3(u_i,v)-\mathbf{O}_3(u,v)| &\leq \int_\Omega |g(x,u_i)-g(x,u)||v|\nu_1 dx \\
&= \int_\Omega |Hu_i - Hu||v|\nu_1 dx \\
&\leq \|Hu_i - Hu\|_{L^{s'}(\Omega,\nu_1)}\|v\|_{L^s(\Omega,\nu_1)} \\
&\leq C_{p,s}\|Hu_i - Hu\|_{L^{s'}(\Omega,\nu_1)}\|v\|_{L^p(\Omega,\omega_1)} \\
&\leq C_{p,s}C_\Omega\|Hu_i - Hu\|_{L^{s'}(\Omega,\nu_1)}\|v\|_{W_0^{1,p}(\Omega,\omega_1,\omega_2)}.
\end{aligned}$$

and by Step 4, we obtain

$$\begin{aligned}
|\mathbf{O}_4(u_i,v)-\mathbf{O}_4(u,v)| &\leq \int_\Omega \Big||u_i|^{p-2}u_i - |u|^{p-2}u\Big||v|\omega_2 dx \\
&= \int_\Omega |Ju_i - Ju||v|\omega_2 dx \\
&\leq \|Ju_i - Ju\|_{L^{p'}(\Omega,\omega_2)}\|v\|_{W_0^{1,p}(\Omega,\omega_1,\omega_2)}.
\end{aligned}$$

Hence, for all $v \in W_0^{1,p}(\Omega,\omega_1,\omega_2)$, we have

$$\begin{aligned}
&|\mathbf{O}(u_i,v)-\mathbf{O}(u,v)| \\
&\leq \sum_{j=1}^4 \Big|\mathbf{O}_j(u_i,v)-\mathbf{O}_j(u,v)\Big| \\
&\leq \Big[\sum_{k=1}^n \Big(\|B_k u_i - B_k u\|_{L^{p'}(\Omega,\omega_1)} + C_{p,q}\|M_k u_i - M_k u\|_{L^{q'}(\Omega,\nu_2)}\Big) \\
&\quad + C_{p,s}C_\Omega\|Hu_i - Hu\|_{L^{s'}(\Omega,\nu_1)} + \|Ju_i - Ju\|_{L^{p'}(\Omega,\omega_2)}\Big]\|v\|_{W_0^{1,p}(\Omega,\omega_1,\omega_2)}.
\end{aligned}$$

Then, we get

$$\begin{aligned}
\|\mathbf{A}u_i - \mathbf{A}u\|_* \leq& \sum_{k=1}^n \Big(\|B_k u_i - B_k u\|_{L^{p'}(\Omega,\omega_1)} + C_{p,q}\|M_k u_i - M_k u\|_{L^{q'}(\Omega,\nu_2)}\Big) \\
&+ C_{p,s}C_\Omega\|Hu_i - Hu\|_{L^{s'}(\Omega,\nu_1)} + \|Ju_i - Ju\|_{L^{p'}(\Omega,\omega_2)}.
\end{aligned}$$

Combining (4.3), (4.4), (4.5) and (4.6), we deduce that

$$\|\mathbf{A}u_i - \mathbf{A}u\|_* \longrightarrow 0 \text{ as } i \longrightarrow \infty,$$

that is, $\mathbf{A}u_i \longrightarrow \mathbf{A}u$ in $\Big[W_0^{1,p}(\Omega, \omega_1, \omega_2)\Big]^*$. Hence, $\mathbf{A}$ is continuous and this implies that $\mathbf{A}$ is hemicontinuous.

Therefore, by Theorem 2.13, the operator equation $\mathbf{A}u = \mathbf{T}$ has exactly one solution $u \in W_0^{1,p}(\Omega, \omega_1, \omega_2)$ and it is the unique solution for problem (1.1).

With this last step the proof of Theorem 4.1 is completed.

5. Example

Take $\Omega = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 1\}$, and consider the weight functions $\omega_1(x, y) = (x^2 + y^2)^{-1/2}$, $\nu_2(x, y) = (x^2 + y^2)^{-1/3}$, $\nu_1(x, y) = (x^2 + y^2)^{-1}$ and $\omega_2(x, y) = (x^2 + y^2)^{-3/2}$ $\Big($we have that ω_1, ν_2, ν_1, and ω_2 are A_4−weight, $p = 4$, $q = 3$ and $s = 2\Big)$, and the functions $\mathcal{B} : \Omega \times \mathbb{R} \times \mathbb{R}^2 \longrightarrow \mathbb{R}^2$, $\mathcal{A} : \Omega \times \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ and $\mathcal{C} : \Omega \times \mathbb{R} \longrightarrow \mathbb{R}$ defined by

$$\mathcal{A}\Big((x, y), \xi\Big) = h_1(x, y)|\xi|^2\xi,$$

where $h_1(x, y) = 4e^{(x^2+y^2)}$, and

$$\mathcal{B}\Big((x, y), \eta, \xi\Big) = h_3(x, y)|\xi|\xi,$$

where $h_3(x, y) = 1 + cos^2(xy)$, and

$$\mathcal{C}\Big((x, y), \eta\Big) = h_4(x, y)\eta,$$

where $h_4(x, y) = 2 - cos^2(xy)$.

Let us consider the operator

$$\begin{aligned}\mathbf{L}u(x, y) \quad &= \quad -\text{div}\Big[\omega_1(x, y)\mathcal{A}\Big((x, y), \nabla u\Big) + \nu_2(x, y)\mathcal{B}\Big((x, y), u, \nabla u(x, y)\Big)\Big] \\ &\quad + \nu_1(x, y)\mathcal{C}\Big((x, y), u\Big) + \omega_2(x, y)|u|^{p-2}u\end{aligned}$$

Therefore, by Theorem 4.1, the problem

$$\begin{cases}\mathbf{L}u(x, y) = \frac{cos(x+y)}{(x^2+y^2)} - \frac{\partial}{\partial x}\left(\frac{sin(x+y)}{(x^2+y^2)}\right) - \frac{\partial}{\partial y}\left(\frac{sin(x+y)}{(x^2+y^2)}\right) & \text{in } \Omega, \\ u(x, y) = 0 & \text{on } \partial\Omega,\end{cases}$$

has exactly one solution $u \in W_0^{1,4}(\Omega, \omega_1, \omega_2)$.

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References

[1] A. Abbassi, C. Allalou and A. Kassidi, *Topological degree methods for a Neumann problem governed by nonlinear elliptic equation*, Moroccan J. of Pure and Appl. Anal., 6(2) (2020) 231–242.

[2] A. Abbassi, C. Allalou and A. Kassidi, *Existence results for some nonlinear elliptic equations via topological degree methods*, J Elliptic Parabol Equ., 7(1) (2021) 121–136.

[3] A. Abbassi, C. Allalou and A. Kassidi, *Existence of weak solutions for nonlinear p-elliptic problem by topological degree*, Nonlinear Dyn. Syst. Theory., 20(3) (2020) 229–241.

[4] F. Behboudi and A. Razani, *Two weak solutions for a singular $(p, q)-$Laplacian problem*, Filomat, 33(11) (2019) 3399–3407.

[5] A. Bensoussan, L. Boccardo and F. Murat, *On a non linear partial differential equation having natural growth terms and unbounded solution*, Annales de l'Institut Henri Poincare (C) Non Linear Anal., 5(4) (1988) 347–364.

[6] D. Bresch, J. Lemoine, F. Guillen-Gonzalez, *A note on a degenerate elliptic equations with applications to lake and seas*, Electron. J. Differ. Equations, 2004(42) (2004) 1–13.

[7] F. Chiarenza, *Regularity for Solutions of Quasilinear Elliptic Equations Under Minimal Assumptions*, Potential Theory Degenerate Partial Differ. Operators. Springer, Dordrecht (1995) 325–334

[8] M. Chipot, Elliptic Equations: *An Introductory Course*, Birkh¨auser, Berlin, 2009.

[9] M. Colombo, *Flows of Non-Smooth Vector Fields and Degenerate Elliptic Equations: With Applications to the Vlasov-Poisson and Semigeostrophic Systems*, Publications on the Scuola Normale Superiore Pisa, Pisa. 22 (2017).

[10] P. Drabek, A. Kufner, V. Mustonen, *Pseudo-monotonicity and degenerated or singular elliptic operators*, Bull. Aust. Math. Soc., 58(2) (1998) 213–221.

[11] P. Drabek, A. Kufner and F. Nicolosi. Quasilinear elliptic equations with degenerations and singularities, Walter de Gruyter. 5(2011).

[12] E. Fabes, D. S. Jerison and C. E. Kenig, *The Wiener test for degenerate elliptic equations*, Annales de l'institut Fourier. 32(3) (1982) 151–182.

[13] E. Fabes, C. E. Kenig and R. P. Serapioni, T*he local regularity of solutions of degenerate elliptic equations*, Commun. Stat.- Theory Methods, 7(1) (1982) 77–116.

[14] B. Franchi and R. Serapioni, *Pointwise estimates for a class of strongly degenerate elliptic operators: a geometrical approach*, Annali della Scuola Normale Superiore di Pisa-Classe di Scienze. 14(4) (1987) 527–568.

[15] S. Fucik, O. John and A. Kufner, *Function Spaces*, Noordhoff International Publishing, Leyden. Academia, Publishing House of the Czechoslovak Academy of Sciences, Prague, 1977.

[16] J. Garcia-Cuerva and J.L. Rubio de Francia, *Weighted norm inequalities and related topics*, Elsevier, 2011.

[17] S. Heidari and A. Razani, *Infinitely many solutions for $(p(x), q(x))-$Laplacian-like systems*, Commun. Korean Math. Soc., 36(1) (2021) 51—62.

[18] J. Heinonen, T. Kilpelainen and O. Martio, *Nonlinear Potential Theory of Degenerate Elliptic Equations*, Oxford Math, Monographs, Clarendon Press, Oxford, 1993.

[19] A. Kufner, *Weighted sobolev spaces*, John Wiley and Sons Incorporated 31 (1985).

[20] A. Kufner and B. Opic, *How to define reasonably weighted Sobolev spaces* , Commentationes Mathematicae Universitatis Carolinae., 25(3) (1984) 537–554.

[21] M. Makvand Chaharlang and A. Razani, *Two weak solutions for some Kirchhoff-type problem with Neumann boundary condition*, Georgian Math. J., (2020) 1–10.

[22] B. Muckenhoupt, *Weighted norm inequalities for the hardy maximal function*, Trans. Am. Math. Soc., 165 (1972) 207–226.

[23] B. Muckenhoupt, *The equivalence of two conditions for weight functions*, Studia Math., 49(1) (1974) 101–106.

[24] M. E. Ouaarabi, A. Abbassi and C. Allalou, *Existence result for a Dirichlet problem governed by nonlinear degenerate elliptic equation in weighted Sobolev spaces*, J. Elliptic. Parabol. Equ., 7(1) (2021) 221–242.

[25] M. E. Ouaarabi, C. Allalou and A. Abbassi, *On the Dirichlet Problem for some Nonlinear Degenerated Elliptic Equations with Weight*, 2021 7th International Conference on Optimization and Applications (ICOA), 2021, pp. 1–6. doi: 10.1109/ICOA51614.2021.9442620.

[26] M. E. Ouaarabi, A. Abbassi and C. Allalou, *Existence Result for a General Nonlinear Degenerate Elliptic Problems with Measure Datum in Weighted Sobolev Spaces*, Int. J. Optim. Appl. , 1(2) (2021) 1–9.

[27] F. Safari and A. Razani, *Positive weak solutions of a generalized supercritical Neumann problem, Iranian Journal of Science and Technology*, Trans. A: Sci.. 44(6) (2020) 1891–1898.

[28] A. Torchinsky, *Real-variable methods in harmonic analysis*, Academic Press, 1986.

[29] B.O.Turesson, *Nonlinear potential theory and weighted Sobolev spaces*, Springer Science and Business Media. 1736 (2000).

[30] X. Xu, *A local partial regularity theorem for weak solutions of degenerate elliptic equations and its application to the thermistor problem*, Differ. Integral Equ. 12(1) (1999) 83–100.
[31] E. Zeidler, *Nonlinear Functional Analysis and its Applications*, vol.I, Springer-Verlag, Berlin, 1990.
[32] E. Zeidler, *Nonlinear Functional Analysis and its Applications*, vol.II/B, Springer-Verlag, New York, 1990.