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# Study connection between the Laurent series and residues on the $\mathbf{A(z)}$ analytic functions

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**Abstract**

In this paper, we obtain a formula for residues and prove Laurent expansion and expansion to Taylor series for $A(z)$-analytic functions.

*Keywords:* $A$-analytic function, Laurent expansion, Residues Computed.

## 1. Introduction

Let $A(z)$ be an antianalytic function, i.e., $dA/\partial z = 0$ in the domain D $\subset C$; moreover, let $|A(z)| \leq c < 1$ for all $z \in D, c\ is\ constant$. The function $f(z)$ is said to be $A(z)$-analytic in the domain $D$ if for any $z \in D$, the following equality holds:

$$\frac{\partial f}{\partial \overline{z}} = A(z)\frac{\partial f}{\partial z}. \tag{1.1}$$

We denote by $O_A(D)$ the class of all $A(z)$-analytic functions defined in the domain D. Since the antianalytic function is infinitely smooth, $O_A(D)C^{\infty}(D)$ (see [8]).

We will now study the behavior of $f(z)$ at an isolated singularity $z_0$ by expanding (sound familiar) This series will not in general be a Taylor series.

$$a_0 + a_1 (z - z_0) + a_2 (z - z_{0)}{}^2 + \cdots$$

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because Taylor series yield analytic functions , where as $f(z)$ is not analytic at a pole or essential singularity .The series we will obtain will involve negative (as well as positive ) powers of $z - z_0$ . A series consisting of negative powers looks :

$$b_0 + \frac{b_1}{z - z_0} + \frac{b_2}{(z - z_0)^2} + \cdots + \frac{b_k}{(z - z_0)^k} + \cdots , \qquad k \in N_o.$$

**Theorem 1.1 (Analogue to the Cauchy Theorem).** *If* $f \in O_A(D) \cap C(\overline{D})$,

$$\int_{\partial D} f(z)(dz + A(z)d\overline{z}) = 0. \tag{1.2}$$

**Theorem 1.2 (Laurent's growth).** *Let* $f(z)$ *be* $A(z)$ *– analytic in the ring of lemnniscate :* $f(z) \in O_A(L(a, R) \setminus L(a, r))$ , $r < R$ . *Then* $f(z)$ *will be expanded to the Laurent series in the ring* $r < p < R$*:*

$$f(z) = \sum_{k=-\infty}^{\infty} c_{j\psi^k(z,\ a)} \tag{1.3}$$

*where the coefficients of the series are determined by the formula*

$$c_k = \frac{1}{2\pi i} \int_{\partial L(a,p)} \frac{f(\xi)}{[\psi(\xi, a)]^{k+1}} (d\xi + A(\xi) d\xi), \quad k = 0, \pm 1 , \pm 2, \cdots$$

*The series* (1.3) *converges uniformly inside of the ring*

$$L(a, R) \setminus L(a, r) = \{z \in D : r < |\Psi(z, a)| < R \}.$$

**Example 1.3.** *Find the Laurent expansion of the function's two nonzero terms.* $f(z) = \tan z$ *about* $z = \frac{\pi}{2}$.
*Let us call* $z = \frac{\pi}{2} + u$ .
***Solution.*** $f(z) = \frac{\sin(\frac{\pi}{2}+u)}{\cos(\frac{\pi}{2}+u)} = -\frac{\cos u}{\sin u}$
*by using* $\sin(A + B) = \sin A \cos B + \cos A \sin B$ *and* $\cos(A + B) = \cos A \cos B - \sin A \sin B$
*This can be expanded using the Taylor series for* $\sin u$ *and* $\cos u$ *Where*

$$\sin u = \sum_{j=0}^{\infty} \frac{(-1)^j u^{2j+1}}{(2j+1)!}$$

$$\cos u = \sum_{j=0}^{\infty} \frac{(-1)^j u^{2j}}{(2j)!}$$

$$f(z) = -\frac{(1 - \frac{u^2}{2!} + \cdots)}{(u - \frac{u^3}{3!} + \cdots)} = -\frac{1}{u} \frac{(1 - \frac{u^2}{2!} + \cdots)}{(1 - \frac{u^2}{3!} + \cdots)}$$

*The numerator can be increased by using* $\sum_{j=0}^{\infty} u^j = \frac{1}{1-u}$ *, for* $|z| < 1$. *To obtain, for the first two nonzero terms*

$$f(z) = -\frac{1}{u}(1 - \frac{u^2}{2!} + \cdots)(u + \frac{u^2}{3!} + \cdots)$$

$$f(z) = -\frac{1}{u}(1 - \frac{u^2}{3} + \cdots) = -\frac{1}{(z - \frac{\pi}{2})} + \frac{(z - \frac{\pi}{2})}{3} + \cdots$$

**Example 1.4.** $g(z) = \frac{1}{(z^2+1)}$ *convergent in a perforated disc around the pole in the Laurent series* $z_0 = i$.

***Solution.*** *We note first that* $g(z) = \frac{1}{(z-i)(z+i).}$ *We wish to expand this in positive and negative powers of* $z-i$ *. It makes sense to expand the factor* $\frac{1}{(z+i)}$ *in powers of* $z-i$ *and then multiply this expansion by* $\frac{1}{(z-i)}$ *to get the expansion for g(z) .*

*Usually , we alter the geometric series for* $\frac{1}{(1-\tau)}$ *with a shrewdly chosen* $\tau$. *Involving* $-i$ *. We observe that*

$$\frac{1}{z+i} = \frac{1}{2i\ + (z-i)} = \frac{1}{2i} \bullet \frac{1}{1+(\frac{1}{2i)(z-i)}}$$

$$= \frac{-i}{2i} \bullet \frac{1}{1-(\frac{i}{2)(z-i)}} = \frac{-i}{2i}(1+\frac{i}{2}(z-i) - \frac{1}{2^2}(z-i)^2 - \frac{i}{2^3}(z-i)^3 + \cdots)$$

$$= \frac{-i}{2}\sum_{n=0}^{\infty}(\frac{i}{2})^n (z-i)^n.$$

*It follows that*

$$g(z) = \frac{1}{z-i} \bullet \frac{1}{z+i} = -\sum_{n=0}^{\infty}(\frac{i}{2})^{n+1}(z-i)^{n+1}.$$

## 2. Residues of A(z) – analytic function

Let f(z) be an A(z) – analytic function in $D\backslash\{\ a_1,\ a_2,\ \cdots,\ a_n\ \}$ and continuous on $\partial D$, where $a_1,\ a_2,\ \cdots,\ a_n$ are isolated singular points . Then there exists a number r > 0 Such that

$$L\ (\ a_k\ ,\ r\ )\ \cap L\ \ (a_1,\ r\ ) = \ \varnothing\ for\ K\ \neq 1.$$

Assume the following relationships are true:

$$G_r = \{\ z \in D:\ |z-\ \xi| > r\ for\ all\ \xi \in \partial D\ \};\quad \bigcup_{k=1}^{n} L\,(a_k\ ,\ r\ ) \subset\ G_r$$

Where $\partial G_r$ is an arbitrary piecewise – smooth closed contour lying in the domain $D$, and containing the points $a_1,\ a_2,\ \cdots,\ a_n$ inside . Since the function $f(z)$ is A(z) – analytic at each point of the closed domain bounded by the contour $\partial G_r \cup\ \sum_{k=1}^{n}\partial L\,(a_k\ ,\ r\ )$, then by the Cauchy theorem we have

$$\oint_{\partial G_r} f\,(\xi)\,\omega\,(z) = \sum_{k=1}^{n}\oint_{\partial L(a_k\ ,\ r\ )} f\,(\xi)\,\omega\,(z) \tag{2.1}$$

where $(z)\ = dz + A\,(z)\,d\overline{z}$ .

**Definition 2.1.** *The residue of an A(z) – analytic function* $f(z)$ *at a point a is the value of the integral of the function* $f(z)$ *taken over a sufficiently small A(z) – lemniscate* $L(a\ ,\ r)$ *, divided by* $2\pi i\ :\ \underset{z=a}{res_A} f\,(z) = \frac{1}{2\pi i}\ \oint_{\partial L(a_k\ ,\ r\ )} f\,(\xi)\,\omega\,(z).$

**Theorem 2.2 (Analogue to the Cauchy residue theorem).** *Let A(z) be analytic everywhere in a domain for a function f(z).G* $\subset$ *D except for an isolated set of singular points and let its boundary* $\partial G$ *do not contain singular points. Then* $\oint_{\partial G} f\,(\xi)\,\omega\,(z) = 2\pi i\sum_{k=1}^{n} res_A\ z\ \ = a_k f\,(z)\,.$

**Proof .**The proof of this theorem follows from the formula (1.2) and Definition 2.1. □

**Example 2.3.** *We fix* $\xi \in D$ *and consider the kernel* $K_n(\xi, z) = \frac{n!}{2\pi i} \cdot \frac{1}{\psi(\xi,z)^{n+1}}$. *Then*

$$\underset{z=a}{res_A} K_n(\xi, z) = \begin{cases} 0, & n \neq 1, \\ 1, & n = 0. \end{cases} \tag{2.2}$$

*Assume that at the point* $z = a$*, the function* $f(z)$ *can be expanded in a Laurent series:*

$$f(z) = \sum_{k=-\infty}^{\infty} C_k \left( z - a + \overline{\int_{\gamma(a,z)} \overline{A(\tau)} d\tau} \right)^k \tag{2.3}$$

**Theorem 2.4.** *In an isolated singular Point, the residue of an A(z) - analytic function f(z).* $a \in \mathbb{C}$ *is equal to the coefficient* $c_{-1}$ *of the minus first degree of* $\psi(z\ ,\ a\ )$ *in its Laurent expansion in a neighborhood of the* $A(z)$ – *lemniscat* $L(a\ ,\ r\ )$ *at the point* $a$ *:*

$$\underset{z=a}{res_A} f(z) = c_{-1}. \tag{2.4}$$

**Proof .** Equality (2.3) is obtained from Eq. (2.4) by integration over a lemniscat $\partial L(a\ ,\ r)$ using (2.2) :

$$\begin{aligned} \underset{z=a}{\operatorname{res}_A} f(z) &= \frac{1}{2\pi i} \oint_{\partial L(a,r)} \sum_{k=-\infty}^{\infty} C_k \left( z - a + \overline{\int_{\gamma(a,z)} \overline{A(\tau)} d\tau} \right)^k \omega(\xi) \\ &= \frac{1}{2\pi i} \sum_{k=-\infty}^{\infty} C_k \oint_{\partial L(a,r)} \left( z - a + \overline{\int_{\gamma(a,z)} \overline{A(\tau)} d\tau} \right)^k \omega(\xi) \\ &= \frac{1}{2\pi i} 2\pi i c_{-1} = c_{-1}. \end{aligned}$$

□

**Definition 2.5.** *A point* $z = a$ *A(z) – analytic function f(z) of order n is referred to as a zero. if* $f(z) = \left( z - a + \overline{\int_{\gamma(a,z)} \overline{A(\tau)} d\tau} \right)^n .g(z)$, *where* $g(a) \neq 0$ *and* $g(z) \in O_A(D)$.

**Theorem 2.6.** *If the A(z) – analytic function (z) has a point a that is not Identically equal to zero in any neighborhood of* $L(a\ ,\ r\ )$ *, then there exists a natural number n such that* $f(z) = \left( z - a + \overline{\int_{\gamma(a,z)} \overline{A(\tau)} d\tau} \right)^n \phi(z)$, *where the function* $\varphi(z)$ *is* $A(z)$ – *analytic at the point a and is nonzero in some neighborhood of this point .*

**Remark 2.7.** *An isolated singular point* $a \in \mathbb{C}$ *of the function* $f(z)$ *is removable if and only if the Laurent expansion of* $f(z)$ *in a neighborhood of a does not contain the principal part ,* ***i.e.*** $f(z) = \sum_{k=0}^{\infty} C_k \left( z - a + \overline{\int_{\gamma(a,z)} \overline{A(\tau)} d\tau} \right)^k$.

**Definition 2.8.** *A point* $z = a$ *is called a pole of an* $A(z)$ – *analytic function* $f(z)$ *of order n if the point a is a zero of the function* $\frac{1}{f(z)}$ *of order n .*

**Theorem 2.9.** *A pole is an isolated singular point $a \in \mathbb{C}$ of the A(z) – analytic function f(z) if and only if the primary component of the Laurent expansion of the A(z) – analytic function f(z) in the vicinity of the point a contains only a finite(and positive) number of nonzero terms.**i.e.***

$$f\ (z) = \sum_{k=-n}^{\infty} C_k \left( \overline{z - a + \int_{\gamma(a,z)} \overline{A\,(\tau)} d\tau} \right)^k , \quad n \geq 1.$$

**Proof .**
$\Longrightarrow$ Let $a$ be pole ; since $\lim_{z\to a} f\,(z) = \infty$ , there exists a punctured neighborhood of the point $a$ where $f(z)$ is $A(z)$ – analytic and nonzero. In this neighborhood the function $g(z) = \frac{1}{f(z)}$ is $A(z)$ – function analytic and there exists the $\lim_{z\to z} g\,(z) = 0$. Therefore , a is a removable point (zero) of the function $g(z)$ and in the neighborhood $L(a\ ,\ r)$ the following expansion holds :

$$g(z) = \sum_{k=n}^{\infty} b_k \left( \overline{z - a + \int_{\gamma(a,z)} \overline{A\,(\tau)} d\tau} \right)^k .$$

Then in the same neighborhood we obtain the identity

$$f\ (z) = \frac{1}{g\ (z)} = \frac{1}{\left( \overline{z - a + \int_{\gamma(a,z)} \overline{A\,(\tau)} d\tau} \right)^n} \bullet \frac{1}{b_n + b_{n+1} \left( \overline{z - a + \int_{\gamma(a,z)} \overline{A\,(\tau)} d\tau} \right) + \cdots}$$

The second factor is $A(z)$– analytic function at the point , and hence it admits a Taylor expansion , we obtain

$$f\ (z) = \sum_{k=-n}^{\infty} C_k \left( \overline{z - a + \int_{\gamma(a,z)} \overline{A\,(\tau)} d\tau} \right)^k .$$

This is the Laurent expansion of f (z) in the neighborhood $L\,(a\ \ ,\ r\ )\backslash\{a\}$ of the point , and we see that its principal part contains a finite number of terms.

$\Longleftarrow$ Let in a the neighborhood $(a\ \ ,\ r\ )\backslash a, f(z)$ be represented by the Laurent expansion whose principal part contains a finite number of terms and let $c_n \neq 0$.
Then the function f (z) and $g(z) = \psi(z,\ a)^n \bullet$ f (z) are $A(z)$ – analytic in this neighborhood. The function $g(z)$ in the neighborhood considered can represented as follows:

$$g\,(z) = c_{-n} + c_{-n+1} \left( \overline{z - a + \int_{\gamma(a,z)} \overline{A\,(\tau)} d\tau} \right) + c_{-n+2} \left( \overline{z - a + \int_{\gamma(a,z)} \overline{A\,(\tau)} d\tau} \right)^2 \cdots$$

This equality shows that $a$ is a removable point and there exists

$$\lim_{z\to a} g\,(z) = c_{-n} \neq 0.$$

Then the function $f(z) = \frac{g(z)}{\psi(z,\ a)^n}$ tends to infinity as $z \to a$, **i.e.** , $a$ is a pole . The theorem is proved. □

**Definition 2.10.** *If there is a punctured neighborhood of the lemniscate of a point $a \in \mathbb{C}$, it is called an isolated singular point of the function f(z). , **i.e.** ( the set $0 < |\psi\,(z,\ a)\ | < r$ ) if the point a is finite , or a set $R < \left| z + \int_0^z \overline{A\,(\tau)} dt \right| < \infty$ , $A \equiv const$ , $|A| < 1$ if $a = \infty$ in which the function $f\,(z)$ is $A(z)$ – analytic .*

**Definition 2.11.** *An isolated singular point a of a function $f\,(z)$ is called :*

(a) *a pole if* $\lim_{z\to a} f\,(z) = \infty$ ;
(b) *An essential singularity if the limit of $f\,(z)$ as $z \to a$ does not exist .*

## 3. What is the formula for calculating residues?

We illustrate some methods by examples.

**First method** Use the Laurent Expansion.

**Example 3.1.** *Evaluate* $I = \int_{c_0} e^{\frac{1}{z}} dz$ *Where* $C_0$ *is the unit circle* $|z| = 1$
***Solution.*** *The function f(z)* $= e^{\frac{1}{z}}$ *is analytic for at* $z \neq 0$ *inside* $C_0$ *and has the Following Laurent expansion about* $z = 0$

$$e^{\frac{1}{z}} = (1 + \frac{1}{z} + \frac{1}{2!z^2} + \frac{1}{3!z^3} + \cdots)$$

*So that res (f ;0 )* $=$ *the coefficient of* $\frac{1}{z\ =1}$ . *Of course* $z_0 = 0$ *is the only isolated singularity of* $e^{\frac{1}{z}}$ *in* $\mathcal{C}$ .
*By residue theorem* $I = 2\pi i$.

**Second method** Simple poles
A pole $z_0$ of $f(z)$ is said to be simple if its order is **,** that is , if $f(z)$ may be expressed as $f(z) = \frac{c_{-1}}{z-z_0} + c_0 + c_1\ (\ z - \ z_0\ ) + \cdots$

$$\operatorname*{res}_{z=a}{}_A f\ (z) = \lim_{z\to a}[f\ (z) \left( z - a + \overline{\int_{\gamma(a,z)} \overline{A\ (\tau)} d\tau} \right)\ ] \tag{3.1}$$

**Example 3.2.** *Evaluate:* $\int_0^{2\pi} \frac{\cos 2\theta}{5+4\cos\theta}\ d\theta$.
***Solution.*** *Let*

$$I = \int_0^{2\pi} \frac{\cos 2\theta}{5 + 4\cos\theta}\ d\theta = \frac{1}{2}\int_0^{2\pi} \frac{e^{2\theta} + e^{-2\theta}}{5 + 2(e^{i\theta} + e^{-i\theta})}\ d\theta$$

*write*

$$\begin{aligned} z &= e^{i\theta}\ ,\quad d\theta = \frac{dz}{iz} \\ &= \frac{1}{2}\int_C \frac{\left(z^2 + \frac{1}{z^2}\right)}{5 + 2\left(z + \frac{1}{z}\right)}\ \frac{dz}{iz} \\ &= \frac{1}{2i}\int_C \frac{(\ z^4 + 1)}{z^2\,(2z^2 + 5z + 2)} dz \\ &= \frac{1}{2i}\int_C \frac{z^4 + 1}{z^2\,(2z + 1)\,(z + 2)}\ dz \end{aligned}$$

*Where C denotes the unit circle* $|z| = 1$, *the pole of f (z) is* $z^2\ \ (2z + 1)\,(z + 2) = 0 \Longrightarrow z = 0\ ,\ z = -\frac{1}{2}\ ,\ z = -2$
*The poles within the contour C are a simple pole at* $z = -\frac{1}{2}$ , *and a pole of order two at* $z = 0$

*Now , Residue at* $z=-\frac{1}{2}$ *is by* (3.1)

$$\lim_{z\to\frac{1}{2}}\left(z+\frac{1}{2}\right)\frac{1}{2i}\frac{z^4+1}{z^2\left(2z+1\right)\left(z+2\right)}$$
$$=\frac{1}{2}\bullet\frac{1}{2i}\frac{\left(-\frac{1}{2}\right)^4+1}{\left(-\frac{1}{2}\right)^2\left(-\frac{1}{2}+2\right)}$$
$$=\frac{1}{4i}\frac{\frac{1}{16}\ +1}{\frac{1}{4}\bullet\frac{3}{2}}=\mathbf{\frac{17}{24i}}$$

*And residue at* $z=0$ *is coefficient of* $\frac{1}{z}$ *in* $\frac{1}{2i}\frac{z^4+1}{z^2(2z+1)(z+2)}$ *, where z is small*
*Now ,*

$$\frac{1}{2i}\frac{z^4+1}{z^2\left(2z+1\right)\left(z+2\right)}=\frac{1}{4i}\left(1+\frac{1}{z^4}\right)\left(1+\frac{1}{2z}\right)^{-1}\left(1+\frac{2}{z}\right)^{-1}$$
$$=\frac{1}{4i}\left(1+\frac{1}{z^4}\right)\left(1-\frac{1}{2z}+\cdots\right)\left(1-\frac{2}{z}+\cdots\right)$$

*The coefficient of* $\frac{1}{z}$ *is easily seen to be* $\frac{1}{4i}\left(\frac{-5}{2}\right)$ *, ie ,* $\mathbf{\frac{-5}{8i}}$ *Hence by Cauchy's residue theorem*

$$I=2\pi i\bullet\sum_{k=1}^{n}res_A=2\pi i\bullet\left\{\frac{17}{24i}+\left(\frac{-5}{8i}\right)\right\}=\frac{\pi}{6}$$

**Theorem 3.3.** *A nth-order pole of an A(z)-analytic function f(z) is a point z=a. The following formula applies in this case:*

$$\underset{z=a}{res_A}f\left(z\right)=\frac{1}{(n-1)!}\lim_{z\to a}\frac{\partial^{n-1}}{\partial z^{n-1}}[f\left(z\right)\left(z-a+\overline{\int_{\gamma(a,z)}\overline{A\left(\tau\right)}d\tau}\right)^n]. \tag{3.2}$$

**Proof .** Due to Theorem 2.9 , an $A(z)$-analytic function $f(z)$ has the form

$$f\left(z\right)=\sum_{k=-n}^{\infty}C_k\left(z-a+\overline{\int_{\gamma(a,z)}\overline{A\left(\tau\right)}d\tau}\right)^k.$$

Multiplying both sides of this equation by $\left(z-a+\overline{\int_{\gamma(a,z)}\overline{A\left(\tau\right)}d\tau}\right)^n$, we obtain

$$f\left(z\right)\left(z-a+\overline{\int_{\gamma(a,z)}\overline{A\left(\tau\right)}d\tau}\right)^n=c_{-n}+c_{-n+1}\psi(z,\ a)+\cdots+c_{-1}\psi\left(z,\ a\right)^{n-1}+\psi\left(z,\ a\right)^n h(z). \tag{3.3}$$

Here $h(z)=\sum_{k=0}^{\infty}c_{K\psi(z,\ a)^k}$. We take the partial derivative of the function $\psi\left(z,\ a\right)$

$$\frac{\partial\psi^k}{\partial z}=k\psi^{k-1}\frac{\partial\psi^k}{\partial z}=k\psi^{k-1}. \tag{3.4}$$

Using this equation, from (3.3) we obtain

$$\begin{gathered}\frac{\partial^{n-1}}{\partial z^{n-1}}[f\left(z\right)\left(z-a+\overline{\int_{\gamma(a,z)}\overline{A\left(\tau\right)}d\tau}\right)^n=(n-1)!\ c_{-1}+\psi\left(z,\ a\right)^n h_1\left(z\right),\\ h_1(z)=\sum_{k=0}^{\infty}\frac{(n+k-1)!}{(n-1)!}c_k\psi\left(z,\ a\right)^k.\end{gathered} \tag{3.5}$$

Passing to the limit as z→ $a$ in Eq. (3.5), we obtain

$$\underset{z=a}{\mathrm{res}_A} f(z) = \frac{1}{(n-1)!} \lim_{z\to a} \frac{\partial^{n-1}}{\partial z^{n-1}} [f(z) \left( z - a + \overline{\int_{\gamma(a,z)} \overline{A(\tau)} d\tau} \right)^n ].$$

□

**Example 3.4.** *Evaluate:* $\int_0^{2\pi} e^{-\cos\theta} \cos(n\theta + \sin\theta)\, d\theta$*. When* $n$ *is a positive integer.*
***Solution.*** *Consider the integral*

$$\begin{aligned} I &= \int_o^{2\pi} e^{-\cos\theta} [\cos(n\theta + \sin\theta) - i\ \sin(n\theta + \sin\theta)\ ] d\theta \\ &= \int_0^{2\theta} e^{-\cos\theta} e^{-i(n\theta+\sin\theta)} d\theta \\ &= \int_0^{2\pi} e^{-(\cos\theta+\sin\theta)} e^{-in\theta} d\theta \\ &= \int_0^{2\pi} e^{-e^{i\theta}} e^{-in\theta} d\theta \\ &= \int_C \left( e^{-z} \bullet \frac{1}{z^n} \right) \frac{dz}{iz} \end{aligned}$$

*Writing* $e^{i\theta} = z$ , $d\theta = \frac{dz}{iz}$. *Where* $C$ *denotes the unit circle* $|z| = 1$.

$$= \frac{1}{i} \int_C \frac{e^{-z}}{z^{n+1}} dz = \int_C f(z)\, dz,$$

*where* $f(z) = \frac{e^{-z}}{iz^{n+1}}$

$$= 2\pi i \sum_{k=1}^{n} res_A{}^{+} \ \ (By\ Cauchy's\ residue\ theorem\ ).$$

*Obviously the only pole of* $f(z)$ *within the contour* $C$ *is* $z = 0$ *of order* $n{+}1$. *At* $z = 0$, *the residue* $= \frac{1}{n!} \left[ \frac{d^n}{dz^n} \left( \frac{e^{-z}}{i} \right) \right]_{z=0} = \frac{(-1)^n}{i(n)!} = \sum_{k=1}^n res_A{}^{+}$
$\therefore I = 2\pi i \bullet \frac{(-1)^n}{i(n)!} = \frac{2\pi}{n!} (-1)^n$ *i.e.*

$$\int_o^{2\pi} e^{-\cos\theta} [\cos(n\theta + \sin\theta) - i\ \sin(n\theta + \sin\theta)\ ] d\theta = \frac{2\pi}{n!} (-1)^n .$$

*Equating real parts, we have*

$$\int_0^{2\pi} e^{-\cos\theta} \cos(n\theta + \sin\theta)\, d\theta = \frac{2\pi}{n!} (-1)^n .$$

## 4. Conclusions

1. An isolated singular point $a \in \mathbb{C}$ of an $A(z)$-analytic function $f(z)$ is a pole If and only if the principal part of Laurent expansion of the $A(z)$-analytic function $f(z)$ in the neighborhood of the point a contains only a finite (and Positive ) number of nonzero terms, i.e.

$$f(z) = \sum_{k=0}^{\infty} c_k \left( z - a + \overline{\int_{\gamma(a,z)} \overline{A(\tau)} d\tau} \right)^k .$$

2. An isolated singular point $a\in \mathbb{C}$ of an $A(z)$-analytic function $f(z)$ is removable if and only if the Laurent expansion of $f(z)$ in a neighborhood of a dose not contain the principal part, i.e.

$$f(z) = \sum_{k=0}^{\infty} c_k \left( z - a + \overline{\int_{\gamma(a,z)} \overline{A(\tau)} d\tau} \right)^k .$$

3. Prove an analogue to the Cauchy residue theorem.
4. Study $A(z)$-analytical functions in one particular case more often, when the function $A(z)$ is an antianalytic function in the considered domain

## References


[1] Boutet de Monvel, J. Sjöstrand, *Sur la singularité des noyaux de Bergman et de Szegö*, Journ. Équ. Dériv. Partielles (1976) 123–164.

[2] A. L. Bukhgeym and S.G. Kazantsev, *Ellipticheskie sistemy tipa Bel'trami i zadachi tomografii*, Elliptic Syst. Beitrami Type Prob. Doklady, Akademii Nauk SSSR, 315 (1990) 15–19.

[3] D.G. Crighton and J.-L. Lions, *Complex Variables Introduction and Applications*, Cambridge University, 2003.

[4] F. J. Flanigan, *Complex variable harmonic and analytic functions*, 1983.

[5] V. Gutlyanski, V. Ryazanov, U. Srebro and E. Yakubov, *The Beltrami Equation: A Geometric Approach*, Springer-Verlag, 2011.

[6] N.M. Jabborov and T.U. Otaboev, *An analog of the integral Cauchy theorem for A-analytic functions*, Uzbek Mat. Zh. 4 (2016) 50–59.

[7] L. Mejlbro, *Complex Functions Examples c-9 The Argument Principle and Many-Valued Function*, Ventus Publishing ApS, 2008.

[8] S. Narayan and P.K. Mittal, *Theory of Functions of a Complex Variable*, S Chand Pub., 1956.

[9] A. Sadullaev and N. M. Jabborov, *On a class of A-analytic functions*, Zh. Sib. Fed. Univ. Ser. Math. Phys. 9(3) (2016) 374–383.

[10] J. Stewart,*Calculus-Early Transcendental (7th ed.)*, Brooks/Cole Cengage Learning, 2012.

[11] Zh.K. Tishabvev, T. Otabev and Sh. Khursvnov Ya., *Deduction and pricing of arguments for $A(z)$-analytic function*, Results Sci. Technol. Modern Math. Appl. 144 (2018) 56–60.

[12] N. Vekua, *Generalized Analytic Functions*, Pergamon Press, Oxford, 1962.