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A new general polynomial transforms for solving ordinary differential equations with variable coefficients

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Abstract

In this paper, we introduce the definition of a new general integral transform which call it "A new general polynomial transform ". Also, we introduce properties, theorems, proofs and transforms of the Logarithmic functions, polynomials functions and other functions. As well as, we discuss how we can apply this integral transform and its inverse to solve the ordinary differential equations with variable coefficients.

Keywords: Integral transforms, Inverse integral transforms, Polynomial transform, Inverse polynomial transform

1. Introduction

Laplace, Sumudu, Natural, Elzaki, Aboodh, SEE, complex SEE, polynomial transforms [3, 2, 7, 4, 8, 1, 5] which is known to find the solution the linear ordinary differential equations with constant coefficients ($L.O.DE^s$) with one condition (data) that integral transforms of the function $f(t)$ is defined. In this paper, we define and present a new integral transform which is work to find the solution the linear ordinary differential equations with variables coefficients [6]:

$$a_0 t^n \frac{dy^{(n)}}{dt} + a_1 t^{n-1} \frac{dy^{(n-1)}}{dt} + \cdots + a_{n-1} t \frac{dy}{dt} + a_n y(t) = f(t)$$

$a_0, a_1, \ldots, a_n$ are constants.

This integral transform is defined for some functions, the kernel of this transform $k(p, x) = x^{-q(p)-1}$, where $q(p)$ is a function of parameter p, and the interval of the new transform is $[a, b] = [1, \infty)$.

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2. The general polynomial transform

A new integral transform "A new general polynomial transform" for the function $f(x)$, where $x \geq 1$ is defined by the integral

$$P_g\{f(x)\} = \int_{x=1}^{\infty} x^{-q(p)-1} f(x)dx = \int_{x=1}^{\infty} x^{-(q(p)+1)} f(x)dx = F(q(p)),$$

such that the above integral is convergent, and $q(p)$ is a function of a parameter p. (for example $q(p) = \frac{1}{p}, q(p) = p^2, q(p) = \frac{p}{1-p}$ and so on).

2.1. Property of a new general polynomial transform

A new general integral transform is characterized by the linear property that is

$$P_g(Af(x) \pm Bg(x)) = AP_g(f(x)) \pm BP_g(g(x))$$

where A and B are constants, the functions $f(x)$ and $g(x)$ are defined when $x > 1$.

Proof .

$$\begin{aligned} P_g(Af(x) \pm Bg(x)) &= \int_1^{\infty} x^{-(q(p)+1)}(Af(x) \pm Bg(x))dx \\ &= A\int_1^{\infty} x^{-(q(p)+1)} f(x)dx \pm B\int_1^{\infty} x^{-(q(p)+1)} g(x)dx \\ &= AP_g(f(x) \pm BP_g(g(x)). \end{aligned}$$

□

2.2. Transformations for some basic functions

For any function $f(x)$, we assume the above integral exists. The Sufficient conditions for the existence of a new general polynomial transform are that $f(x)$ for $x > 1$ be piecewise continuous otherwise general polynomial transform may or may not exist. In this part of section, we find general polynomial transform of simple functions

1. If $f(x) = k$, k is constant, then $P_g\{k\} = \frac{k}{q(p)}, q(p) > 0$.

   Proof .

$$\begin{aligned} P_g\{k\} &= \int_1^{\infty} x^{-(q(p)+1)} kdx \\ &= k\int_1^{\infty} x^{-(q(p)+1)} dx \\ &= \frac{k}{-q(p)-1+1}[x^{-(q(p)+1)+1}]_1^{\infty} \\ &= \frac{-k}{q(p)}[0-1] = \frac{k}{q(p)}, q(p) > 0 \end{aligned}$$

□

2. If $f(x) = x^n, n \in R, x > 1$, then $P_g\{x^n\} = \dfrac{1}{q(p) - n}, q(p) > n$.

Proof .

$$\begin{aligned} P_g\{x^n\} &= \int_1^{\infty} x^{-(q(p)+1)} x^n dx \\ &= \int_1^{\infty} x^{-(q(p)+1-n)} dx \\ &= \frac{1}{-q(p) - 1 + n + 1} [x^{-(q(p)+1-n)+1}]_1^{\infty} \\ &= \frac{-1}{q(p) - n}[0 - 1] = \frac{1}{q(p) - n} \end{aligned}$$

□

3. If $f(x) = \ln(x), x > 1$, then $P_g\{\ln x\} = \dfrac{1}{(q(p))^2}, q(p) > 0$.

Proof . $P_g\{\ln x\} = \int_1^{\infty} x^{-(q(p)+1)} \ln x dx$. Integration by parts, we get $P_g\{lnx\} = \dfrac{1}{(q(p))^2}$. □

4. If $f(x) = x^n \ln(x), n \in R, x > 1$, then

$$P_g\{x^n \ln x\} = \int_1^{\infty} x^{-(q(p)+1)} x^n \ln x dx = \int_1^{\infty} x^{-q(p)-1+n} dx.$$

Integration by parts, after simple computations, we obtain:

$$P_g\{x^n \ln x\} = \frac{1}{(q(p) - n)^2}, q(p) > n.$$

5. If $f(x) = \sin(a \ln(x)), x > 1$ and a is a constant then $P_g\{\sin(a \ln(x))\} = \dfrac{a}{(q(p))^2 + a^2}$.

Proof . $P_g\{\sin(a \ln(x))\} = \int_1^{\infty} x^{-(q(p)+1)} \sin(a \ln(x)) dx$. Integration by parts, we get:

$$u = \sin(a \ln(x)), dv = x^{-q(p)-1} dx, du = \frac{a \cos(a \ln(x))}{x}, v = \frac{-x^{-q(p)}}{q(p)}.$$

$P_g\{sin(aln(x))\} = \dfrac{a}{q(p)} \int_1^{\infty} x^{-q(p)-1} cos(aln(x)dx$, also Integration by parts, we obtain

$u = \cos(a\ln(x)), dv = x^{-q(p)-1}dx, du = \dfrac{-a\sin(a\ln(x))}{x}, v = \dfrac{-x^{-q(p)}}{q(p)}$, then

$$P_g\{\sin(a\ln(x))\} = \frac{a}{q(p)}\frac{-x^{-q(p)}}{q(p)}\cos(a\ln(x))|_1^\infty - \frac{a^2}{(q(p))^2}\int_1^\infty x^{-q(p)-1}sin(aln(x))dx$$

$$P_g\{sin(aln(x))\} = \frac{a}{q(p)}[0+\frac{1}{q(p)}] - \frac{a^2}{(q(p))^2}P_g\{\sin(a\ln(x))\},$$

$$(1+\frac{a^2}{(q(p))^2})P_g\{\sin(a\ln(x))\} = \frac{a}{(q(p))^2},$$

$$(\frac{q(p))^2+a^2}{(q(p))^2})P_g\{\sin(a\ln(x))\} = \frac{a}{(q(p))^2},$$

$$Then\ P_g\{\sin(a\ln(x))\} = \frac{a}{(q(p))^2+a^2}$$

□Similarty:

6. If $f(x) = \cos(a\ln(x)), x > 1$ and a is a constant then $P_g\{\cos(a\ln(x))\} = \dfrac{q(p)}{(q(p))^2+a^2}$.
7. If $f(x) = \sinh(a\ln(x)), x > 1$ and a is a constant then $P_g\{\sinh(a\ln(x))\} = \dfrac{a}{(q(p))^2-a^2}$, $|q(p)| > a$

Proof .

$$P_g\{\sinh(a\ln(x))\} = \int_1^\infty x^{-q(p)-1}\sinh(a\ln(x)dx$$
$$= \int_1^\infty (\frac{e^{a\ln x} - e^{-a\ln x}}{2})x^{-q(p)-1}dx$$

After simple computations, we get:

$$P_g\{sinh(aln(x))\} = \frac{a}{(q(p))^2-a^2}, |q(p)| > a.$$

□

Similarly,

8. Since $\{cosh(aln(x))\} = (\dfrac{e^{alnx}+e^{-alnx}}{2})$, we have

$$P_g\{cosh(aln(x))\} = \frac{q(p)}{(q(p))^2-a^2}, \quad |q(p)| > a.$$

Theorem 2.1. *If $P_g\{f(x)\} = F(q(p))$, then $P_g\{x^{-\alpha}f(x)\} = F(q(p)+\alpha)$ and α is a constant number.*

Proof . Since by definition we have

$$P_g\{x^{-\alpha}f(x)\} = \int_1^\infty x^{-q(p)-1}x^{-\alpha}f(x)dx$$
$$= \int_1^\infty x^{-(q(p)+1+\alpha)}f(x)dx$$
$$= F(q(p)+\alpha)$$

□

Definition 2.2. *Let $g(x)$ be a function, where $x > 1$ and $P_g\{f(x)\} = F(q(p)), g(x)$ is called an inverse for general polynomial integral transform and written as $P_g^{-1}\{F(q(p))\} = g(x)$, where P_g^{-1} returns the integral transform to the ordinal function $(g(x))$.*

2.3. Inverse general polynomail integral transform

1. $P_g^{-1}\{\dfrac{k}{q(p)}\} = k, q(p) > 0.$

2. $P_g^{-1}\{\dfrac{1}{q(p)-n}\} = x^n, q(p) > n, n \in R, x > 1.$

3. $P_g^{-1}\{\dfrac{1}{(q(p))^2}\} = \ln x, q(p) > 0, x > 1.$

4. $P_g^{-1}\{\dfrac{1}{(q(p)-n)^2}\} = x^n lnx, q(p) > n, n \in R, x > 1.$

5. $P_g^{-1}\{\dfrac{a}{((q(p))^2+a^2)}\} = \sin(a\ln x), x > 1.$

6. $P_g^{-1}\{\dfrac{q(p)}{((q(p))^2+a^2)}\} = \cos(a\ln x), x > 1.$

7. $P_g^{-1}\{\dfrac{a}{((q(p))^2-a^2)}\} = \sinh(a\ lnx), |q(p)| > a, x > 1.$

8. $P_g^{-1}\{\dfrac{q(p)}{(q(p))^2-a^2}\} = \cosh(a\ln x), |q(p)| > a, x > 1.$

A property of P_g^{-1} is a linear property as it is for the transform P_g. Now if

$$P_g^{-1}\{F_1(q(p))\} = g_1(x), \ldots, P_g^{-1}\{F_n(q(p))\} = g_n(x)$$

and $A_1, A_2, \ldots, A_n$ are constant, then

$$\begin{aligned}&P_g^{-1}\{A_1F_1(q(p)) + A_2F_2(q(p)) + \cdots + A_nF_n(q(p))\}\\&= A_1P_g^{-1}\{F_1(q(p))\} + A_2P_g^{-1}\{F_2(q(p))\} + \cdots + A_nP_g^{-1}\{F_n(q(p))\}\end{aligned}$$

Theorem 2.3. *If $P_g^{-1}\{F(q(p))\} = g(x)$, then $P_g^{-1}\{F(q(p)+\alpha)\} = x^{-\alpha}P_g^{-1}\{F(q(p))\}$, where α is a constant number.*

Proof . The proof holds, because $P_g^{-1}\{F(q(p)+\alpha)\} = x^{-\alpha}g(x) = x^{-\alpha}P_g^{-1}\{F(q(p))\}$. □

3. Solution of linear ordinary differential equation with variable coefficients

Definition 3.1. *[6] The linear ordinary differential equation*

$$a_0x^ny^{(n)} + a_1x^{n-1}y^{(n-1)} + \cdots + a_{n-1}xy^{'} + a_ny = h(x)$$

$a_0, a_1, \ldots, a_n$ *are constants and* $h(x)$ *is a function of* x*, is called Euler's equation.*

Theorem 3.2. *If the function* $h(x)$ *is defined for* $x > 1$ *and its derivatives* $h^{'}(x), h^{''}(x), \ldots, h^n(x)$ *are exist, then*

$$\begin{aligned}P_g\{x^nh^{(n)}(x)\} =& -h^{(n-1)}(1) - (q(p) - (n-1))h^{(n-2)}(1)\\ &- (q(p) - (n-1))(q(p) - (n-2))h^{(n-3)}(1)\\ &- \cdots - (q(p) - (n-1))(q(p) - (n-2))(q(p) - (n-3))\ldots(q(p) - 1)h(1)\\ &+ (q(p) - (n-1))!F(q(p)\end{aligned}$$

Proof . Since $P_g\{xh^{'}(x)\} = \int_1^\infty x^{-q(p)}h^{'}(x)dx$, by integration by parts, we obtain:

$$\begin{aligned}&u = x^{-q(p)}, dv = h^{'}(x)dx, du = -q(p)x^{-q(p)-1}dx, v = h(x),\\ &P_g\{xh^{'}(x)\} = h(x)x^{-q(p)}|_1^\infty + q(p)\int_1^\infty x^{-q(p)-1}h(x)dx,\\ &P_g\{xh^{'}(x)\} = -h(1) + q(p)P_g\{h(x)\} = -h(1) + q(p)F(q(p)),\end{aligned}$$

and

$$P_g\{x^2h^{''}(x)\} = \int_1^\infty x^{-q(p)+1}h^{''}(x)dx.$$

Also, by integration by parts, we obtain:

$$\begin{aligned}&u = x^{-q(p)+1}, dv = h^{''}(x)dx, du = (-q(p)+1)x^{-q(p)}dx, v = h^{'}(x),\\ &P_g\{x^2h^{''}(x)\} = h^{'}(x)x^{-q(p)+1}|_1^\infty + (q(p)-1)\int_1^\infty x^{-q(p)}h^{'}(x)dx,\\ &= -h^{'}(1) + (q(p)-1)P_g\{h^{'}(x)\} = -h^{'}(1) - (q(p)-1)h(1) + (q(p)-1)q(p)F(q(p)).\end{aligned}$$

Also

$$\begin{aligned}P_g\{x^3h^{'''}(x)\} =& \int_1^\infty x^{-q(p)+2}h^{'''}(x)dx,\\ =& -h^{''}(1) - (q(p)-2)h^{'}(1) - (q(p)-2)(q(p)-1)h(1) + (q(p)-2)(q(p)-1)q(p)F(q(p)).\end{aligned}$$

Thus, by repeating this technique for n^{th}-times, we obtain

$$\begin{aligned}&P_g\{x^nh^{(n)}(x)\} =\\ &-h^{(n-1)}(1) - (q(p) - (n-1))h^{(n-2)}(1) - (q(p) - (n-1))(q(p) - (n-2))h^{(n-3)}(1) - \ldots\\ &- (q(p) - (n-1))(q(p) - (n-2))(q(p) - (n-3))\ldots(q(p) - 1)h(1) + (q(p) - (n-1))!F(q(p)).\end{aligned}$$

□

4. Application

In this section, we represent three application:

Application 4.1. We solve the differential equation $x^2y^{''}+4xy^{'}+2y=x\ln x$ with condition, $y(1)=y^{'}(1)=0$.

Taking P_g to both side of above equation, we get

$$-y^{'}(1)-(q(p)-1)y(1)+(q(p)-1)(q(p))P_g\{y(x)\}+4[-y(1)+q(p)P_g\{y(x)\}]+2P_g\{y(x)\}$$
$$=\frac{1}{(q(p)-1)^2}.$$

Applying the conditions of the above differential equation, we have:

$$(q(p)-1)(q(p))P_g\{y(x)\}+4(q(p))P_g\{y(x)\}+2P_g\{y(x)\}=\frac{1}{(q(p)-1)^2},$$
$$[(q(p)-1)(q(p))+4(q(p))+2]P_g\{y(x)\}=\frac{1}{(q(p)-1)^2},$$
$$[(q(p))^2+3(q(p))+2]P_g\{y(x)\}=\frac{1}{(q(p)-1)^2}.$$

By using partial fraction for the right sides of the above equation, we have:

$$P_g\{y(x)\}=\frac{A}{(q(p)+1)}+\frac{B}{(q(p)+2)}+\frac{C}{(q(p)-1)}+\frac{D}{(q(p)-1)^2}.$$

After simple computations, we obtain

$$A=\frac{1}{4},B=\frac{-1}{9},C=\frac{-5}{36},D=\frac{1}{6}$$

and by taking P_g^{-1} to both side of the above equation, we obtain

$$y(x)=P_g^{-1}\{\frac{\frac{1}{4}}{(q(p)+1)}\}+P_g^{-1}\{\frac{\frac{-1}{9}}{(q(p)+2)}\}+P_g^{-1}\{\frac{\frac{-5}{36}}{(q(p)-1)}\}+P_g^{-1}\{\frac{\frac{1}{6}}{(q(p)-1)^2}\}.$$

Then the exact solution is

$$y(x)=\frac{1}{4}x^{-1}+(\frac{-1}{9})x^{-2}-\frac{5}{36}x+\frac{1}{6}x\ln x,$$
$$y(x)=\frac{1}{4x}-\frac{1}{9x^2}-\frac{5}{36}x+\frac{1}{6}x\ln x.$$

Application 4.2. We solve the ordinary differential equation:

$$x\frac{dy}{dx}+y=16\sin(\ln x),\ \text{and}\ y(1)=-7.$$

Taking P_g to both side of above differential equation, we get

$$\begin{aligned}
&[-y(1) + q(p)F(q(p))] + F(q(p)) = \frac{16}{(q(p))^2 + 1},\\
&[7 + q(p)F(q(p))] + F(q(p)) = \frac{16}{(q(p))^2 + 1},\\
&(q(p) + 1)F(q(p)) = \frac{16}{(q(p))^2 + 1} - 7,\\
&F(q(p)) = \frac{16 - 7(q(p))^2 - 7}{((q(p))^2 + 1)((q(p)) + 1)}\\
&F(q(p)) = \frac{-7(q(p))^2 + 9}{((q(p))^2 + 1)((q(p)) + 1)}.
\end{aligned}$$

By using partial fraction for the right side of the above equation, we obtain:

$$P_g\{y(x)\} = \frac{A}{(q(p) + 1)} + \frac{B(q(p)) + c}{(q(p))^2 + 1}.$$

After simple computations, we get:

$$A = -1, B = -8 \textit{ and } C = 16.$$

We substitute this values in above equation, we get:

$$F(q(p)) = P_g\{y(x)\} = \frac{1}{(q(p) + 1)} + \frac{-8(q(p)) + 16}{(q(p))^2 + 1}.$$

And taking the inverse transformation of the last algebraic equation, we get the exact solution of the required differential equation

$$y(x) = x^{-1} + 16\sin(\ln(x)) - 8\cos(\ln(x)).$$

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