On some topological concepts via grill

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Abstract

In this work, the new grill concepts are studied using grill topological spaces and by using some defined sets where the set \( \alpha \)-open sets are defined. Properties of this set and some relationships were presented, in addition to studying a set of functions, including open, closed and continuous functions, finding the relationship between them and giving examples and properties that belong to this set, which will be a starting point for studying many topological properties using this set.

Keywords: Grill, \( \alpha \)-open sets, \( \alpha \)-closed sets, \( \mathcal{C} \)-\( \alpha \)-open function, \( \mathcal{C} \)-\( \alpha \)-c function.

1. Introduction

Choquet [1] developed the concept of a grill on a topological space, and it has proven to be a useful tool for exploring several topological problems. A grill on \( \mathcal{Q} \) is a family of non-empty subsets of a topological space \( (\mathcal{Q}, \tau) \). If (i) \( \mathcal{A} \in \mathcal{C} \) and \( \mathcal{A} \subseteq \mathcal{B} \) so \( \mathcal{B} \in \mathcal{C} \), and (ii) \( \mathcal{A}, \mathcal{B} \subseteq \mathcal{Q} \) and \( \mathcal{A} \cup \mathcal{B} \in \mathcal{C} \), then \( \mathcal{A} \in \mathcal{C} \) or \( \mathcal{B} \in \mathcal{C} \). A triple \( (\mathcal{Q}, \tau, \mathcal{C}) \) is said to be a grill topological space.

Roy and Mukherjee [6] used a grill to define a unique topology and researched topological ideas. For any topological space point \( x \), \( (\mathcal{Q}, \tau), \tau(q) \) represents a compilation of \( x \)'s open neighborhoods. A mapping \( \phi: \mathcal{P}(\mathcal{Q}) \rightarrow \mathcal{P}(\mathcal{Q}) \) is defined as \( \phi(\mathcal{A}) = \{ q \in \mathcal{Q}: \mathcal{A} \cap \mathcal{S} \in \mathcal{C} \text{ for all } \mathcal{S} \in \tau(q) \} \) for each \( \mathcal{A} \in \mathcal{P}(\mathcal{Q}) \). A mapping \( \Psi: \mathcal{P}(\mathcal{Q}) \rightarrow \mathcal{P}(\mathcal{Q}) \) is defined as \( \Psi(\mathcal{A}) = \mathcal{A} \cup \phi(\mathcal{A}) \) for all \( \mathcal{A} \in \mathcal{P}(\mathcal{Q}) \). The map \( \Psi \) satisfies Kuratowski closure axioms:

i. \( \Psi(\phi) = \phi \),

ii. If \( \mathcal{A} \subseteq \mathcal{B} \) so \( \Psi(\mathcal{A}) \subseteq \Psi \),

iii. If \( \mathcal{A} \subseteq \mathcal{X} \), so \( \Psi(\Psi(\mathcal{A})) = \Psi(\mathcal{A}) \),

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iv. If $A, B \subseteq X$, so $\Psi(A \cup B) = \Psi(A) \cup (B)$.

There are some types of a grill topological space as like a cofinite topology and discrete topology \cite{6}. In the shape of a grill $G$ on a topological space $(Q, \tau)$, there is a one kind of a topology. \( \tau_G \) on $Q$ a gift $\tau_G = \{ s \subseteq Q: \Psi(Q_{-s}) = Q_{-s} \}$, for any reason $A \subseteq Q$, $\Psi(A) = A \cup \phi(A) = \tau_G \cdot \text{cl}(A)$ and $\tau \subseteq \tau_G$. We can find $\tau_G$ by used the base as following $\beta(\tau_G, \emptyset) = \{ V - A; V \in \tau, A \notin \} \ [9]$.

In any topological space $(Q, \tau)$, there is a grill $\tau \subseteq \beta(\emptyset, \tau) \subseteq \tau_G$, where $\beta(\emptyset, \tau) = \{ V - A: V \in \tau, A \notin \tau, A \notin G \}$ is open base for $\tau_G$.

As an example, let $(Q, \tau)$ be a topological space, if $G = P(\emptyset)$, then, $\tau_G = \tau$, because for any $\tau_G$ basic open set $V = Q - A$ with $u \in \tau$ and $A \notin \tau_G$. We have $A = \emptyset$, in order for $V = s \in \tau$, so we have this case $\tau = \beta(\emptyset, \tau) = \tau_G$. A subset $A$ of a topological space $Q$ is alleged to be: $\alpha$-open \cite{2} if $A \subseteq \text{Int}(\text{cl}(A))$. The family of all $\alpha$-open set denoted by $\tau$.

There are many researchers who have used these combinations to obtain new generalizations \cite{3,4}. In this research used the symbol $\text{Int}(A)$ to interior of the set $A$ and the symbol $\text{cl}(A)$ is the closure of $A$.

2. On $\alpha$-open sets in topological spaces

Definition 2.1. The set $A$ is said to be grill $\alpha$-open if there exists $s \in \tau$ such that $s \cdot A \notin G$ and $A - \text{Int}cl_G(s) \notin G$, and as indicated by $G^\alpha$-open the complement $G^\alpha$-open is $G^\alpha$-closed. The set of all $G^\alpha$-open symbolized by $G^\alpha \circ o(q)$ and the ensemble first and foremost $G^\alpha$-closed shortly $G^\alpha c(q)$.

Example 2.2. Let $(Q, \tau, G)$ be a grill topological space, and let $Q = \{ q, q_2, q_3 \}, \tau = \{ Q, \phi,\{ q_1 \},\{ q_1, q_2 \} \}, \mathcal{F} = \{ Q, \phi,\{ q_3 \},\{ q_2, q_3 \} \}, G = \{ s \subseteq Q; q_2 \notin s \}, \phi: P(Q) \rightarrow P(Q), \phi(A) = (q \in Q; \forall s \in \tau_x; s \cap A \notin \emptyset), \Psi(A) = A \cup \phi, \tau_G =\{ Q, \phi,\{ q_1, q_2, q_3 \}\} , \mathcal{F}_G = \{ Q, \phi,\{ q_3 \},\{ q_1, q_2 \}\}$, then $G^\alpha \circ o(q) = \{ Q, \phi,\{ q_3 \},\{ q_1, q_2 \}\}$.

Remark 2.3.

(i) Every set that is open is $G^\alpha \alpha$-open set.

(ii) Every set that is closed is $G^\alpha \alpha$-closed set.

Proof. (i) Let $A \in \tau$, then there exists $s \in A$ such that $u \subseteq \text{Int} cl_G(\emptyset)$, but $s \in A \in \tau$, so $s - A = \emptyset \notin G \wedge A - \text{Int} cl_G(A) = \emptyset \notin G$.

(ii) Let $A$ be a closed set. Then $A^c \in \tau, A^c$ is $G^\alpha \circ o(q)$ so, $A$ is $G^\alpha c(q)$. \hfill \Box

The converse of Remark 2.3 is not true, see Example 2.4.

Example 2.4. Let $Q = \{ q_1, q_2, q_3, q_4 \}, \tau = \{ Q, \phi,\{ q_3 \},\{ q_2, q_3 \}\}, G = P(Q) \setminus \{ \emptyset \}$. It's clear that $\{ q_1, q_2 \} \in G^\alpha \circ o(K)$, but $\{ q_1, q_2 \} \notin \tau$. And $\{ q_3 \} \in G^\alpha c(K)$ but $\{ q_1, q_2 \}$ is not a closed set.

Theorem 2.5. Every $G^\alpha \circ o$-open set is a $G \alpha$-open.

Proof. Let $s \in G$, so, $\text{Int} \circ cl_G(\emptyset) \subseteq \text{Int} \circ cl_G(s) \subseteq A - \text{Int} \circ cl_G(s) \notin G$. Let $A - \text{Int} \circ cl_G(s) \notin G$, thus by the definition of a grill we will have $A - \text{Int} \circ cl_G(s) \notin G$, a contradiction, so $A - \text{Int} \circ cl_G(s) \notin G$. \hfill \Box
Proposition 2.6. For any grill topology \((Q, \tau, G)\), \(A\) is a \(G^\alpha\)-open set if and only if \(A\) is a \(G^\ast\alpha\)-open set whenever \(G = p(Q) \setminus \{\emptyset\}\).

Proof. Let \(G\) be an \(\alpha\)-open set, then there exists \(s \in \tau\) such that \(s \subseteq A \subseteq \text{Int}_l(s)\), so \(s \cdot A = \emptyset \notin G\) and \(A - \text{Int}_l(s) = \emptyset \notin G\). Now since \(\tau = \tau_{\emptyset}\), we have \(A - \text{Int}_l_G(s) \notin G\).

Conversely, \(s \cdot A = \emptyset \land A - \text{Int}_l(s) \notin G(\tau) = \tau_{\emptyset}\). Since \(G = \rho(Q) \setminus \{\emptyset\}\), \(u \subseteq A\) and \(A \subseteq \text{Int}_l(s)\). So \(s \subseteq A \subseteq \text{Int}_l(s)\). □

Proposition 2.7. Let \(A\) be a \(G^\ast\alpha\)-open set and \(B \subseteq Q\) such that \(A \subseteq B \subseteq \text{Int}_l(e)(A)\), then \(B\) is a \(G^\ast\alpha\)-open set.

Proof. Suppose that \(s\) is an open set, as a result \(s \cdot A \notin G \land A - \text{Int}_l(e)(s) \notin G, (A\) is \(G^\ast\alpha\)-open set).

Since \(A \subseteq B \subseteq \text{Int}_l_G(A), s \cdot \text{Int}_l_G(A) \notin G \subseteq s \cdot B \notin G \subseteq s \cdot A \notin G\). Then there exists \(s \in \tau\) such that \(s \cdot B \notin G\), and since \(A \subseteq B\), \(\text{Int}_l(e)(A) \subseteq \text{Int}_l(e)(B)\). So \(s \cdot \text{Int}_l_G(B) \notin G\), hence \(B \in G^\ast\alpha\)-open set. □

Remark 2.8. The two concepts \(G^\ast\alpha\)-open set and \(\alpha\)-open set are independent. See Examples 2.9 and 2.10.

Example 2.9. Let \((Q, \tau, G)\) be any grill topology and \(Q = \{q_1, q_2, q_3\}\), \(\tau = \{Q, \phi, \{q_1\}\}\), \(G = \{s \in Q\}^\ast\alpha(o( Q ) = P(Q)\) and \(o( Q ) = \{Q, \phi, \{q_1\}\} \cup \phi\), hence it is clear that \(o(q_1) = G^\ast\alpha(o( Q ))\) but \(\{q_2\} \notin o(Q)\).

Example 2.10. Let \((N, \tau), (q_1, \{q_1\})\), \(G = \{s \subseteq N ; s\) is an infinite set \}, it is clear that \(O\) (odd number) is an \(\alpha\)-open set, but it is not a \(G^\ast\alpha\)-open set.

Corollary 2.11. Suppose that \((Q, \tau, G)\) is a grill topological space and \(A\) subset of \(Q\). If \(G = p(q) \setminus \{\emptyset\}\). Then \(A\) is \(G^\ast\alpha\)-open set only if and only if \(A\) is \(\alpha\)-open set.

Proof. Let \(A\) be \(G^\ast\), so there exists \(s \in \tau\); \(s - A \notin G\) and \(A - \text{Int}_l_G(s) \notin G\); \(\emptyset \cdot A = \emptyset \notin G\) and \(A - \text{Int}_l_G(s) = \emptyset \notin G\).

Conversely, it is clear that \(A \subseteq B \subseteq \text{Int}_l_G(cs), u = A \notin G\) and \(A - \text{Int}_l_G(s) = \emptyset \notin G\). Then \(\emptyset \cdot A \notin G\) and \(A - \text{Int}_l(s) = \emptyset \notin G\). □

Definition 2.12. Let \((Q, \tau, G)\) be a grill topological space. A subset \(A\) in \(Q\) is called \(G^\ast\alpha\)-open if \(A \subseteq \Psi(\text{Int}(A))\).

Proposition 2.13. Every \(\Psi\alpha\)-open is a \(G^\ast\alpha\)-open.

Proof. Let \(A\) be \(\Psi\alpha\)-open, so there exists \(s \in \tau\) such that \(s \subseteq A \subseteq \Psi(s)\), so \(s \cdot A = \emptyset \land A - \Psi(s) = \emptyset \land s \cdot A \notin G \land A - \text{Int}_l_G(s) \notin G\). □

Proposition 2.14. Every \(\Psi\alpha\)-open is an \(\alpha\)-open.

Proof. Since \(\Psi\alpha\)-open, there exists \(s \in \tau\) such that \(s \subseteq A \subseteq \text{Int}_l_G(s) \subseteq \text{Int}_l(s)\). Thus, \(\text{Int}_l_G(s) \subseteq \text{Int}_l(s)\) and \(s \subseteq A \subseteq \text{Int}_l_G(s) \subseteq \text{Int}_l(s)\), so \(s \subseteq A \subseteq \text{Int}_l(s)\). □

Theorem 2.15. A subset \(A\) of a grill \((Q, \tau, G)\) is a \(G^\ast\alpha\)-open set if and only if there exists \(s \in \tau\) in order for \(s \subseteq A \subseteq \Psi(s)\).

Proof. If \(A\) is a \(G^\ast\alpha\)-open set, so \(A \subseteq \Psi(\text{Int}(A))\). Conversely, let \(s \subseteq A \subseteq \Psi(s)\) for \(s \in \tau\), therefore \(s \subseteq A \subseteq \text{Int}(A)\) as well as \(\Psi(s) \subseteq \Psi(\text{Int}(A))\), as a result, \(A \subseteq \Psi(\text{Int}(A))\). □
Lemma 2.16. \( \bigcup_{i \in A} \text{Int}(\varsigma\ell_i(A_i)) \subseteq \text{Int}(\varsigma\ell_i(\bigcup_{i \in A} A_i)). \)

Proof. We have \( A_i \subseteq \bigcup_{i \in A} A_i \). Then \( \varsigma\ell_i(A_i) \subseteq \varsigma\ell_i(\bigcup_{i \in A} A_i), \text{Int}(\varsigma\ell_i(A_i)) \subseteq \text{Int}(\varsigma\ell_i(\bigcup_{i \in A} A_i)). \) Thus,
\[
\bigcup_{i \in A} \text{Int}(\varsigma\ell_i(A_i)) \subseteq \text{Int}(\varsigma\ell_i(\bigcup_{i \in A} A_i)).
\]
\( \square \)

Theorem 2.17. The union of any family of \( G^\ast\alpha \) open sets is a \( G^\ast\alpha \) open set.

Proof. For any \( A_i \in G^\ast\alpha \) open set, we show that \( \bigcup_{i \in A} A_i \in G^\ast\alpha \) open set. Since, \( A_i \in G^\ast\alpha \) is a open set, there exists \( \varsigma \in \tau \) such that \( (\varsigma_i^{-1}, A_i) \notin \mathbf{C} \) and \( \text{Int}(\varsigma\ell_i(\bigcup_{i \in A} A_i)) \notin \mathbf{C} \) (by Lemma 2.16). But \( (\varsigma_i^{-1}, A_i) \subseteq (\bigcup_{i \in A} A_i) \in \mathbf{C} \supset (\bigcup_{i \in A} A_i) \notin \mathbf{C} \). So there exists an open set \( w = \bigcup_{i \in \varsigma} \mathbf{C} \) such that \( (w \cup \bigcup_{i \in A}) \notin \mathbf{C} \). Now we prove \( (\bigcup_{i \in A} A_i - \text{Int}(\varsigma\ell_i(\bigcup_{i \in \varsigma} A_i)) \notin \mathbf{C} \). we have \( A_i - \text{Int}(\varsigma\ell_i(\bigcup_{i \in \varsigma} A_i)) \notin \mathbf{C} \) and \( \text{Int}(\varsigma\ell_i(\bigcup_{i \in \varsigma} A_i)) \subseteq \bigcup_{i \in A} A_i - \text{Int}(\varsigma\ell_i(\bigcup_{i \in \varsigma} A_i)) \notin \mathbf{C} \). So \( \bigcup_{i \in A} A_i - \text{Int}(\varsigma\ell_i(\bigcup_{i \in \varsigma} A_i)) \notin \mathbf{C} \). Since, \( \text{Int}(\varsigma\ell_i(\bigcup_{i \in \varsigma} A_i)) \subseteq \bigcup_{i \in A} A_i - \text{Int}(\varsigma\ell_i(\bigcup_{i \in \varsigma} A_i)) \notin \mathbf{C} \). So \( \bigcup_{i \in A} A_i - \text{Int}(\varsigma\ell_i(\bigcup_{i \in \varsigma} A)) \notin \mathbf{C} \). Thus, \( \bigcup_{i \in A} A_i \in G^\ast\alpha \) open set. \( \square \)

Remark 2.18. The collection of all \( G^\ast\alpha \) open sets is represented supra topology.

3. Several sorts of open functions

Definition 3.1. A function \( f : (\mathbb{Q}, \tau, \mathbf{C}) \rightarrow (Y, \tau', \mathbf{C}) \) is said to be:

1. \( G^\ast\alpha \) open function, symbolize by “\( G^\ast\alpha \) o function” if \( f(\varsigma) \in G^\ast\alpha o(y) \), whenever \( \varsigma \in G^\ast\alpha o(q) \).

2. \( G^\ast\alpha \) – o function, symbolize by “\( G^\ast\alpha \) – o function” if \( f(\varsigma) \in G^\ast\alpha o(y) \), whenever \( \varsigma \in \tau \).

3. \( G^\ast\alpha \) – o function, symbolize by “\( G^\ast\alpha \) – o function” if \( f(\varsigma) \in G^\ast\alpha o(y) \), whenever \( \varsigma \in \tau \).

Proposition 3.2. let \( f : (\mathbb{Q}, \tau, \mathbf{C}) \rightarrow (Y, \tau', \mathbf{C}) \) be a function

(i) If \( f \) is a \( G^\ast\alpha \) – o function, so \( f \) is an open function.

(ii) If \( f \) is a \( G^\ast\alpha \) – o function so \( f \) is \( G^\ast\alpha \) – o function.

(iii) If \( f \) is a \( G^\ast\alpha \) – o function so \( f \) is \( G^\ast\alpha \) – o function.

(iv) If \( f \) is an open function so \( f \) is \( G^\ast\alpha \) – o function.

Proof. (i) Let \( \varsigma \in \tau \). By Remark 2.3(i), \( \varsigma \in G^\ast\alpha \) – o(q). Since \( f \) is a \( G^\ast\alpha \) – open, \( f(\varsigma) \) is open in \( (Y, \tau') \). Therefore, \( f \) is an open function.

(ii) If \( \varsigma \in G^\ast\alpha \) – o(q). Since \( f \) is a \( G^\ast\alpha \) – o function, \( f(\varsigma) \in \tau' \). By Remark 2.3(i), \( f(\varsigma) \in G^\ast\alpha o(y) \). So \( f \) is a \( G^\ast\alpha \) – o function.

(iii) Let \( \varsigma \in \tau \). By Remark 2.3(i), \( \varsigma \in G^\ast\alpha \) – o(q). because \( f \) is a \( G^\ast\alpha \) – o function, so \( f(\varsigma) \in G^\ast\alpha o(y) \). So \( f \) is a \( G^\ast\alpha \) – o function.

(iv) Suppose that \( \varsigma \in \tau \). Since \( f \) is an open function, \( f(\varsigma) \in \tau' \). By Remark 2.3(i), \( \varsigma \in G^\ast\alpha o(y) \). So \( f \) is a \( G^\ast\alpha \) – o function. \( \square \)
The following diagram, created to explain the connections that exist between numerous nations were presented in Definition 3.1.

**Diagram 1**

Example 3.3. Let \( Q = \{ q_1, q_2, q_3 \} \), \( \tau = \{ Q, \phi, \{ q_1 \} \} \), \( \mathcal{G} = \mathcal{P}(Q) \setminus \{ \phi \} \). Define \( f : (Q, \tau, G) \to (Y, \tau, \mathcal{G}) \), \( f(q) = q \). It is clear that \( f \) is an open function, \( \mathcal{G}^{o-o}(Q) = \{ \sigma \subseteq Q : q_1 \in \sigma \} \cup \{ \phi \} \). So there exists \( \{ q_1, q_2 \} \in \mathcal{G}^{o-o}(Q) \) such that \( f(\{ q_1, q_2 \}) \notin \tau \). Then we observe that \( f \) is not \( \mathcal{G}^{o-o} \) function.

Example 3.4. Let \( Q = \{ q_1, q_2, q_3 \} \), \( \tau = \{ Q, \phi, \{ q_1, q_2 \} \} \), \( \mathcal{G} = \{ \sigma \subseteq Q : q_3 \in \sigma \} \), \( \tau_\phi = \mathcal{P}(Q) \), \( \mathcal{G}^{o-o}(Q) = \mathcal{P}(Q) \), \( f : (Q, \tau, G) \to (Q, \tau, \mathcal{G}) \). Define \( f(q) = q_2 \). Then \( \tau_\phi = \{ Q, \phi, \{ q_2, q_1 \}, \{ q_3, q_2 \}, \{ q_2, q_1, q_1 \} \} \), \( \mathcal{G}^{o}(Q) = \{ Q, \phi, \{ q_2 \}, \{ q_3, q_2 \}, \{ q_2, q_1, q_1 \} \} \). So, \( f \) is a \( \mathcal{G}^{o} \) function, but \( f \) is not a \( \mathcal{G}^{*o} \) function, because there exist \( \{ q_3, q_2 \} \in \mathcal{G}^{*o} \alpha - o(\sigma) \), but \( f(\{ q_3, q_2 \}) = \{ q_1, q_3 \} \notin \mathcal{G}^{*o} \alpha - o(\phi). \)

Definition 3.6. The function \( f : (Q, \tau, G) \to (Y, \tau', \mathcal{G}) \) is called

(i) \( \mathcal{G}^{*} \alpha \)-closed function, symbolize by "\( \mathcal{G}^{*} \alpha \)-c function" if \( f(\sigma) \in \mathcal{G}^{*} \alpha \ c(y) \), whenever \( \sigma \in \mathcal{G}^{*} \alpha \).

(ii) \( \mathcal{G}^{*} \alpha \)-closed, symbolize by "\( \mathcal{G}^{*} \alpha \)-c function" if \( f(\sigma) \in \mathcal{G}^{*} \alpha \ c(y) \), whenever \( \sigma \) is a closed set in \((Q, \tau)\).

(iii) \( \mathcal{G}^{***} \alpha \)-closed function, symbolize by "\( \mathcal{G}^{***} \alpha \)-c function" if \( f(\sigma) \) is closed set in \((Y, \tau', \mathcal{G})\), whenever \( \sigma \in \mathcal{G}^{***} \alpha \).

Proposition 3.7. Let \( f : (Q, \tau, G) \to (Y, \tau', \mathcal{G}) \) be a function, then

(i) \( f \) is an closed function, when \( f \) is a \( \mathcal{G}^{***} \alpha - c \) function.

(ii) \( f \) is \( \mathcal{G}^{*} \alpha - c \) function, when \( f \) is a \( \mathcal{G}^{***} \alpha - c \) function.

(iii) \( f \) is \( \mathcal{G}^{*} \alpha - c \) function when \( f \) is a \( \mathcal{G}^{*} \alpha - c \) function.

(iv) \( f \) is \( \mathcal{G}^{**} \alpha - c \) function when \( f \) is a closed function.
Proof. By Remark 2.3(i) and Definition 3.6, the prove holds. □ The inverse of Proposition 3.7 is not true. See Examples 3.3 and 3.4.

Remark 3.8. When \( f \) is onto so:

(i) \( \mathcal{G}^*\alpha\)-open and \( \mathcal{G}^*\alpha\)-closed functions are identical.

(ii) \( \mathcal{G}^*\alpha\)-open and \( \mathcal{G}^{**}\alpha\)-closed functions are identical.

(iii) \( \mathcal{G}^{***}\alpha\)-open and \( \mathcal{G}^{***}\alpha\)-closed functions are identical.

Proof. Considering that \( f \) is an onto function together with Definitions 3.1, 3.6, prove the above statements. □

The following diagram explain the ties that bind these two types of closed functions

![Diagram 2](image)

4. Some types of continuous functions

In the following, new type of continuous functions will present their definitions and the relationships between those functions will explain.

Definition 4.1. The function \( f : (\mathcal{Q}, \tau, \mathcal{G}) \rightarrow (\mathcal{Q}, \tau, \mathcal{G}) \) is called

1. \( \mathcal{G}^*\alpha\)-continues function, shortly \( \mathcal{G}^*\alpha\)-continues function, if \( f^{-1}(\xi) \in \mathcal{G}^*\alpha_{oo}(\mathcal{Q}) \), for all \( \xi \in \tau \).

2. strongly \( \mathcal{G}^*\alpha\)-continues function shortly strongly \( \mathcal{G}^*\alpha\)-continuous function, if \( f^{-1}(\xi) \in \tau \), for every \( \xi \in \mathcal{G}^*\alpha_{oo}(Y) \).

3. \( \mathcal{G}^*\alpha\)-irresolute function, shortly \( \mathcal{G}^*\alpha\)-irresolute function, if \( f^{-1}(\xi) \in \mathcal{G}^*\alpha_{oo}(\mathcal{Q}) \), for every \( \xi \in \mathcal{G}^*\alpha_{oo}(Y) \).

Proposition 4.2. Let \( f : (\mathcal{Q}, \tau, \mathcal{G}) \rightarrow (\mathcal{Q}, \tau, \mathcal{G}) \) be a function. Then

1. \( f \) \( \alpha \)-is \( \mathcal{G}^*(\mathcal{Q})\alpha \)-irresolute function, when \( f \) is strongly-\( \mathcal{G}^*(\mathcal{Q})\alpha \)-continuous function.

2. \( f \) is continuous function, when \( f \) is strongly-\( \mathcal{G}^*(\mathcal{Q})\alpha \)-continuous function.
(3) $f$ is $\mathcal{G}^*(\mathcal{Q})\alpha$–continuous function, when $f$ is continuous function.

(4) if $f$ is a $\mathcal{G}^*(\mathcal{Q})\alpha$–irresolute function, then $f$ is $\mathcal{G}^*(\mathcal{Q})\alpha$–continuous function.

Proof. (1) Let $\xi \in \mathcal{G}^\alpha\alpha_o(Y)$, since $f$ is a strongly $\mathcal{G}^*(\mathcal{Q})\alpha$–continuous function, $f^{-1}(\xi) \in \tau$. By Remark 3.8, $f^{-1}(\xi) \in \mathcal{G}^\alpha\alpha_o(Q)$, this implies that $f$ is $\mathcal{G}^*(\mathcal{Q})\alpha$–irresolute function.

(2) Let $\xi \in \tau$. By Remark 2.3(i), $\xi \in \mathcal{G}^\alpha\alpha_o(Y)$. Since $f$ is strongly-$\mathcal{G}^*(\mathcal{Q})\alpha$–continuous function, $f^{-1}(\xi)$ is an open set in $(\mathcal{Q}, \tau)$, this implies that $f$ is a continues function.

(3) Let $\xi \in \tau$. Since $f$ is a continues function, $f^{-1}(\xi)$ is an open set in $(\mathcal{Q}, \tau)$. By Remark 2.3(i) $f^{-1}(\xi) \in \mathcal{G}^\alpha\alpha_o(Q)$, so, $f$ is $\mathcal{G}^*(\mathcal{Q})\alpha$–continuous function.

(4) Let $\xi \in \tau$. By Remark 2.3(i), $\xi \in \mathcal{G}^\alpha\alpha_o(Y)$. Since $f$ is $\mathcal{G}^*(\mathcal{Q})\alpha$–irresolute function, $f^{-1}(\xi) \in \mathcal{G}^\alpha\alpha_o(Q)$, so, $f$ is $\mathcal{G}^*(\mathcal{Q})\alpha$–continuous function. □

The inverse of Proposition 4.2 is not true in general.

Example 4.3. The function $f : (\mathcal{Q}, \tau, \mathcal{G}) \to (\mathcal{Q}, \tau, \mathcal{G}''')$ such that $f(q_1) = q_2$, for each $q_1 \in \mathcal{Q}$, where $\mathcal{Q} = \{q_1, q_2, q_3\}$, $\tau = \{\phi, \{q_1\}\}$, $\mathcal{G} = P(\mathcal{Q}) \setminus \{\phi\}$, $\mathcal{G}''' = \{\xi; q_1 \in \xi\}$, $\mathcal{G}^\alpha\alpha_o(\mathcal{Q}) = \{\xi; q_1 \in \xi\} \cup \{\phi\}$, $\mathcal{G}''''\alpha_o(\mathcal{Q}) = P(q_1)$, so that, $f$ is $\mathcal{G}^\alpha\alpha_o(\mathcal{Q})$ continuous function and continuous function but it is not irresolute and not strongly because, there exists $\{q_2, q_3\} \in \mathcal{G}''''\alpha_o(\mathcal{Q})$, $f^{-1}(\{q_2, q_3\}) = \{q_2\} \notin \tau$.

Example 4.4. Consider the function $f : (\mathcal{Q}, \tau, \mathcal{G}) \to (\mathcal{Q}, \tau, \mathcal{G}''')$ such that $f(\{q_1\}) = \{q_2\} f(\{q_2\}) = \{q_1\}, f(\{q_3\}) = \{q_3\}$, where $\mathcal{Q} = \{q_1, q_2, q_3\}$, $\tau = \{\phi, \{q_1\}\}$, $\mathcal{G} = P(\mathcal{Q}) \setminus \{\phi\}$, $\mathcal{G} = \{\xi; q_1 \in \xi\}$, $\mathcal{G}^\alpha\alpha_o(\mathcal{Q}) = P(q_1)$, $\mathcal{G}''''\alpha_o(\mathcal{Q}) = \{\xi; q_1 \in \xi\} \cup \{\phi\}$. Then $f$ is $\mathcal{G}^\alpha\alpha_o(\mathcal{Q})$ continuous function and irresolute function but it is not continues and not strongly function since $f^{-1}(\{q_1\}) = \{q_2\} \notin \tau$.

The following diagram, explains the relations between the concept in Definition 4.1.

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![Diagram 3](attachment:image.png)
References