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Some integral inequalities for the product of s-convex functions in the fourth sense

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Abstract

In this paper, several novel inequalities are examined for the product of two s-convex functions in the fourth sense. Also, some applications regarding special means and digamma functions are presented.

Keywords: Convex functions, $s-$Convexity, s-Convex functions in the fourth sense, Product two convex functions, Hermite-Hadamard type inequality, Specials means, Digamma function
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1 Introduction

Convex functions play an important role in different areas of mathematics such as theory of optimization, probability and inequality [3, 7, 25]. Convexity is generalized in many aspects, thus, different convexities appeared [4, 5, 9, 19, 32, 34, 37, 38]. For example, Godunova-Levin convex functions, B-convex functions, B^{-1}-convex functions, s-convex functions, p-convex functions, etc. [4, 5, 32]. Many studies have been done on these convexity classes. One of the studies on these functions is to work on inequalities. Many studies on the inequalities based on convexity types have been done so far. Some can be seen in [2, 6, 11, 12, 13, 14, 21, 22, 23, 24, 30, 35, 36, 39] and references therein. There are many studies on integral inequalities involving the products of different convex functions[10, 16, 18, 26, 27, 28, 29, 30, 31].

In this paper, Hermite-Hadamard type inequalities are obtained for the product of s-convex functions in the fourth sense, which is one of these classes.

The s-convex function in the first sense was defined in [28], the s-convex in the second sense was introduced in [8] and the s-convex function in the third sense is studied in [33]. Besides, the s-convex function in the fourth sense is given in this work as follows[15].

A real-valued function $f : I \to \mathbb{R}$ is said to be s-convex function in the fourth sense if the inequality:

$$f(\lambda x + (1-\lambda)y) \leq \lambda^{\frac{1}{s}} f(x) + (1-\lambda)^{\frac{1}{s}} f(y) \tag{1.1}$$

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holds for all $x, y \in I$ and all $\lambda \in [0,1]$, for some fixed $s \in (0,1]$.

The classes of s-convex functions in first, second, third and fourth sense are denoted by K_s^1, K_s^2, K_s^3 and K_s^4 respectively. It can be easily seen that in the case $s=1$, each type of s-convexity is reduced to the ordinary convexity of functions.

Remark 1.1. If $f : I \to \mathbb{R}$ is s-convex function in the second sense, then $f(x) \geq 0$ for all $x \in I$. Some can be seen in [8] and references therein. If $f : I \to \mathbb{R}$ is s-convex function in the fourth sense, then $f(x) \leq 0$ for all $x \in I$.

Indeed, by taking $\lambda = \mu = \frac{1}{2}$ at the inequality 1.1, it turns to the following inequality

$$f(\frac{x+y}{2}) \leqslant \frac{f(x)+f(y)}{2^{\frac{1}{s}}}. \tag{1.2}$$

Then accepting $y = x$ in (1.2), it can be easily seen that the s-convex function in the fourth sense satisfies $f(x) \leq 0$ for all $x \in I$.

If $f : I \to \mathbb{R}$ is a convex function, then the following inequality holds:

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}$$

where $a, b \in I$ with $a < b$. This inequality is known as Hermite-Hadamard inequality. This famous integral inequality is generalized, improved and extended by many mathematicians [6, 14, 39].

Let $f : I \to \mathbb{R}_-$ be s-convex function in the fourth sense and integrable on $[a, b]$, then the following inequality holds:

$$2^{\frac{1}{s}-1} f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{s}{s+1}\left[f(a)+f(b)\right] \tag{1.3}$$

for $s \in (0,1]$ [20].

Both the inequalities hold in reversed direction if f is concave on $[a, b]$. Let us give some notations that we will use in this paper. Throughout this paper $\mathbb{R} = (-\infty, \infty)$, $\mathbb{R}_+ = (0, \infty)$, $\mathbb{R}_- = (-\infty, 0)$, $I = [a, b]$, $a, b \in \mathbb{R}$ with $a < b$ will be considered. We also recall that the Beta function,

$$\beta(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}dt, \qquad \beta(x,y) = \beta(y,x)$$

for $x, y > 0$.

2 Main Results

In this section, some inequalities involving the products of the functions belonging to different s-convex classes in the fourth sense are obtained. Some upper bounds and lower bounds are obtained for the average value of the product of these functions on a closed interval of real numbers. Also, some inequalities related to the image of midpoints of the interval are given. Moreover, some inequalities for the double and triple integral of the products are presented.

Theorem 2.1. Let $f, g : [a, b] \to \mathbb{R}$, and fg be integrable on $[a, b]$. If $f \in K_{s_1}^4$ and $g \in K_{s_2}^4$, then

$$\frac{1}{b-a}\int_a^b f(x)g(x)dx \geq \frac{s_1 s_2}{s_1 + s_1 s_2 + s_2} M(a,b) + \beta\left(1+\frac{1}{s_1}, 1+\frac{1}{s_2}\right) N(a,b) \tag{2.1}$$

where $M(a, b) = f(a)g(a) + f(b)g(b)$ and $N(a, b) = f(a)g(b) + f(b)g(a)$.

Proof . Since f is s_1-convex in the fourth sense and g is s_2-convex in the fourth sense on $[a,b]$, we get

$$f(ta+(1-t)b) \le t^{\frac{1}{s_1}} f(a) + (1-t)^{\frac{1}{s_1}} f(b), \quad g(ta+(1-t)b) \le t^{\frac{1}{s_2}} g(a) + (1-t)^{\frac{1}{s_2}} g(b).$$

We know that f and g are negative, since f and g are s-convex functions in the fourth sense. So, we can write the following inequality;

$$\begin{aligned} f(ta+(1-t)b)g(ta+(1-t)b) &\ge \left(t^{\frac{1}{s_1}} f(a) + (1-t)^{\frac{1}{s_1}} f(b)\right) \\ &\quad \times \left(t^{\frac{1}{s_2}} g(a) + (1-t)^{\frac{1}{s_2}} g(b)\right) \\ &= t^{\frac{1}{s_1}+\frac{1}{s_2}} f(a)g(a) + (1-t)^{\frac{1}{s_1}+\frac{1}{s_2}} f(b)g(b) \\ &\quad + t^{\frac{1}{s_1}}(1-t)^{\frac{1}{s_2}} f(a)g(b) + t^{\frac{1}{s_2}}(1-t)^{\frac{1}{s_1}} f(b)g(a). \end{aligned}$$

If we integrate both sides of this inequality in the interval $[0,1]$ and consider the properties of the Beta function, the following inequality is obtained,

$$\begin{aligned} \int_0^1 f(ta+(1-t)b)g(ta+(1-t)b)dt &\ge \frac{s_1 s_2}{s_1 + s_1 s_2 + s_2} \left[f(a)g(a) + f(b)g(b)\right] \\ &\quad + \beta(1+\frac{1}{s_1}, 1+\frac{1}{s_2}) \left[f(a)g(b) + f(b)g(a)\right], \end{aligned} \tag{2.2}$$

and by substituting $x = ta + (1-t)b$, it is easy to see that

$$\int_0^1 f(ta+(1-t)b)g(ta+(1-t)b)dt = \frac{1}{b-a}\int_a^b f(x)g(x)dx. \tag{2.3}$$

By combining (2.2) and (2.3), we get the desired result. □

Corollary 2.2. If we choose $s_1 = s_2 = 1$, then inequality (2.1) reduces to the following inequality

$$\frac{1}{b-a}\int_a^b f(x)g(x)dx \ge \frac{1}{3}M(a,b) + \frac{1}{6}N(a,b). \tag{2.4}$$

In [30], Pachpatte used the idea of product of functions for nonnegative convex functions to establish the inequality and got the reverse of inequality (2.4).

Corollary 2.3. If we choose $s_1 = s_2 = s$, then inequality (2.1) reduces to the following inequality

$$\frac{1}{b-a}\int_a^b f(x)g(x)dx \ge \frac{s}{2+s}M(a,b) + \beta(1+\frac{1}{s}, 1+\frac{1}{s})N(a,b).$$

In the next theorems, $M(a,b)$ and $N(a,b)$ are the same as in Theorem 2.1.

Theorem 2.4. Let $f, g : [a,b] \to \mathbb{R}$, and fg be integrable on $[a,b]$. If $f \in K^4_{s_1}$ and $g \in K^4_{s_2}$, then

$$\begin{aligned} f\left(\frac{a+b}{2}\right) g\left(\frac{a+b}{2}\right) &\ge \left(\frac{1}{2}\right)^{\frac{s_1 - s_1 s_2 + s_2}{s_1 s_2}} (M(a,b) + N(a,b)) \\ &\quad \times \left(\frac{s_1 s_2}{s_1 + s_1 s_2 + s_2} + \beta(1+\frac{1}{s_1}, 1+\frac{1}{s_2})\right). \end{aligned} \tag{2.5}$$

Proof . Since f is s_1-convex in the fourth sense and g is s_2-convex in the fourth sense on $[a,b]$, we get

$$\begin{aligned} f\left(\frac{a+b}{2}\right) &= f\left(\frac{ta+(1-t)b}{2}+\frac{(1-t)a+tb}{2}\right) \\ &\leq \left(\frac{1}{2}\right)^{\frac{1}{s_1}}\left(f\left(ta+(1-t)b\right)+f\left((1-t)a+tb\right)\right) \\ &\leq \left(\frac{1}{2}\right)^{\frac{1}{s_1}}\left(t^{\frac{1}{s_1}}+(1-t)^{\frac{1}{s_1}}\right)\left(f(b)+f(a)\right), \end{aligned}$$

and

$$\begin{aligned} g\left(\frac{a+b}{2}\right) &= g\left(\frac{ta+(1-t)b}{2}+\frac{(1-t)a+tb}{2}\right) \\ &\leq \left(\frac{1}{2}\right)^{\frac{1}{s_2}}\left(g\left(ta+(1-t)b\right)+g\left((1-t)a+tb\right)\right) \\ &\leq \left(\frac{1}{2}\right)^{\frac{1}{s_2}}\left(t^{\frac{1}{s_2}}+(1-t)^{\frac{1}{s_2}}\right)\left(g(b)+g(a)\right). \end{aligned}$$

If we multiply two inequalities side to side, then we get the following inequality,

$$\begin{aligned} f\left(\frac{a+b}{2}\right)g\left(\frac{a+b}{2}\right) \geq \left(\frac{1}{2}\right)^{\frac{1}{s_1}+\frac{1}{s_2}}&\left(t^{\frac{1}{s_1}}+(1-t)^{\frac{1}{s_1}}\right)\left(t^{\frac{1}{s_2}}+(1-t)^{\frac{1}{s_2}}\right) \\ &\times\left(f(a)g(a)+f(b)g(b)+f(a)g(b)+f(b)g(a)\right). \end{aligned}$$

If we integrate both sides of the inequality over the interval $[0,1]$, we get

$$\begin{aligned} f\left(\frac{a+b}{2}\right)g\left(\frac{a+b}{2}\right) &\geq \left(\frac{1}{2}\right)^{\frac{1}{s_1}+\frac{1}{s_2}}\left[f(a)g(a)+f(b)g(b)+f(a)g(b)+f(b)g(a)\right] \\ &\times\int_0^1\left(t^{\frac{1}{s_1}}+(1-t)^{\frac{1}{s_1}}\right)\left(t^{\frac{1}{s_2}}+(1-t)^{\frac{1}{s_2}}\right)dt \\ &= \left(\frac{1}{2}\right)^{\frac{s_1-s_1s_2+s_2}{s_1s_2}}\left(M(a,b)+N(a,b)\right) \\ &\times\left(\frac{s_1s_2}{s_1+s_1s_2+s_2}+\beta(1+\frac{1}{s_1},1+\frac{1}{s_2})\right). \end{aligned}$$

□

Corollary 2.5. In Theorem 2.4, if we choose $s_1=s_2=1$, then inequality (2.5) reduces to the following inequality,

$$f\left(\frac{a+b}{2}\right)g\left(\frac{a+b}{2}\right) \geq \frac{1}{4}\left(M(a,b)+N(a,b)\right).$$

Theorem 2.6. Let $f,g:[a,b]\to\mathbb{R}$, and fg be integrable on $[a,b]$. If $f\in K_{s_1}^4$ and $g\in K_{s_2}^4$, then

$$\begin{aligned} \frac{1}{b-a}\int_a^b f(x)g(x)dx \leq{}& 2^{\frac{s_1+s_1s_2+s_2}{s_1s_2}}f\left(\frac{a+b}{2}\right)g\left(\frac{a+b}{2}\right) \\ &-\frac{s_1s_2}{s_1+s_1s_2+s_2}N(a,b)-\beta(1+\frac{1}{s_1},1+\frac{1}{s_2})M(a,b). \end{aligned} \tag{2.6}$$

Proof . Since f is s_1-convex in the fourth sense and g is s_2-convex in the fourth sense on $[a,b]$ and using inequalities

$$f\left(\frac{a+b}{2}\right) \leq \left(\frac{1}{2}\right)^{\frac{1}{s_1}}\left(f\left(ta+(1-t)b\right)+f\left((1-t)a+tb\right)\right)$$

and

$$g\left(\frac{a+b}{2}\right) \leq \left(\frac{1}{2}\right)^{\frac{1}{s_2}} \left(g\left(ta+(1-t)b\right)+g\left((1-t)a+tb\right)\right),$$

we get

$$\begin{aligned}
f\left(\frac{a+b}{2}\right) g\left(\frac{a+b}{2}\right) &\geq \left(\frac{1}{2}\right)^{\frac{1}{s_1}+\frac{1}{s_2}} \left(f\left(ta+(1-t)b\right)+f\left((1-t)a+tb\right)\right) \\
&\left(g\left(ta+(1-t)b\right)+g\left((1-t)a+tb\right)\right) \\
&\geq \left(\frac{1}{2}\right)^{\frac{1}{s_1}+\frac{1}{s_2}} \left(f\left(ta+(1-t)b\right)g(ta+(1-t)b)\right. \\
&+f\left((1-t)a+tb\right) g\left((1-t)a+tb\right) \\
&+\left(t^{\frac{1}{s_1}} f(a)+(1-t)^{\frac{1}{s_1}} f(b)\right)\left((1-t)^{\frac{1}{s_2}} g(a)+t^{\frac{1}{s_2}} g(b)\right) \\
&\left.+\left((1-t)^{\frac{1}{s_1}} f(a)+t^{\frac{1}{s_1}} f(b)\right)\left(t^{\frac{1}{s_2}} g(a)+(1-t)^{\frac{1}{s_2}} g(b)\right)\right) \\
&= \left(\frac{1}{2}\right)^{\frac{1}{s_1}+\frac{1}{s_2}} \left(f\left(ta+(1-t)b\right)g(ta+(1-t)b)\right. \\
&+ f\left((1-t)a+tb\right) g\left((1-t)a+tb\right) \\
&+ t^{\frac{1}{s_1}}(1-t)^{\frac{1}{s_2}} f(a)g(a)+(1-t)^{\frac{1}{s_1}+\frac{1}{s_2}} f(b)g(a) \\
&+ t^{\frac{1}{s_1}+\frac{1}{s_2}} f(a)g(b)+t^{\frac{1}{s_2}}(1-t)^{\frac{1}{s_1}} f(b)g(b) \\
&+ t^{\frac{1}{s_2}}(1-t)^{\frac{1}{s_1}} f(a)g(a)+t^{\frac{1}{s_1}+\frac{1}{s_2}} f(b)g(a) \\
&\left.+ (1-t)^{\frac{1}{s_1}+\frac{1}{s_2}} f(a)g(b)+t^{\frac{1}{s_1}}(1-t)^{\frac{1}{s_2}} f(b)g(b)\right)
\end{aligned}$$

We integrate both sides of this inequality over $[0,1]$ and obtain

$$\begin{aligned}
f\left(\frac{a+b}{2}\right) g\left(\frac{a+b}{2}\right) &\geq \left(\frac{1}{2}\right)^{\frac{1}{s_1}+\frac{1}{s_2}+1} \int_0^1 f\left(ta+(1-t)b\right) g(ta+(1-t)b)dt \\
&+ \left(\frac{1}{2}\right)^{\frac{1}{s_1}+\frac{1}{s_2}} \left(\left[\frac{s_1 s_2}{s_1+s_1 s_2+s_2}\left[f(a)g(b)+f(b)g(a)\right]\right.\right. \\
&\left.+\beta(1+\frac{1}{s_1},1+\frac{1}{s_2})\left[f(a)g(a)+f(b)g(b)\right]\right] \\
&+\left[\frac{s_1 s_2}{s_1+s_1 s_2+s_2}\left[f(a)g(b)+f(b)g(a)\right]\right. \\
&\left.\left.+\beta(1+\frac{1}{s_2},1+\frac{1}{s_1})\left[f(a)g(a)+f(b)g(b)\right]\right]\right) \\
&= \left(\frac{1}{2}\right)^{\frac{1}{s_1}+\frac{1}{s_2}+1} \int_0^1 f\left(ta+(1-t)b\right) g(ta+(1-t)b)dt \\
&+ \left(\frac{1}{2}\right)^{\frac{1}{s_1}+\frac{1}{s_2}} \frac{2 s_1 s_2}{s_1+s_1 s_2+s_2} N(a,b)+2\beta\left(1+\frac{1}{s_1},1+\frac{1}{s_2}\right) M(a,b) \\
&= \left(\frac{1}{2}\right)^{\frac{1}{s_1}+\frac{1}{s_2}+1} \left(\frac{1}{b-a}\int_a^b f(x) g(x)dx+\frac{s_1 s_2}{s_1+s_1 s_2+s_2} N(a,b)\right. \\
&\left.+\beta(1+\frac{1}{s_1},1+\frac{1}{s_2}) M(a,b)\right).
\end{aligned}$$

The proof is complete. □

Corollary 2.7. If we choose $s_1 = s_2 = 1$, then inequality (2.6) reduces to the following inequality

$$\frac{1}{b-a}\int_a^b f(x)g(x)dx \le 8f\left(\frac{a+b}{2}\right)g\left(\frac{a+b}{2}\right) - \frac{1}{3}N(a,b) - \frac{1}{6}M(a,b).$$

Corollary 2.8. Using the inequalities (2.1) and (2.6), we have the following inequality

$$\begin{aligned}\frac{s_1s_2}{s_1+s_1s_2+s_2}M(a,b)+\beta\left(1+\frac{1}{s_1},1+\frac{1}{s_2}\right)N(a,b) &\le \frac{1}{b-a}\int_a^b f(x)g(x)dx \\ &\le 2^{\frac{s_1+s_1s_2+s_2}{s_1s_2}} f\left(\frac{a+b}{2}\right)g\left(\frac{a+b}{2}\right) \\ &- \frac{s_1s_2}{s_1+s_1s_2+s_2}N(a,b) - \beta(1+\frac{1}{s_1},1+\frac{1}{s_2})M(a,b).\end{aligned} \tag{2.7}$$

With this inequality, $\frac{1}{b-a}\int_a^b f(x)g(x)dx$ is bounded. Also, if we take $s_1 = s_2 = 1$, the inequality 2.7 reduces to the following inequality,

$$\frac{1}{3}M(a,b) + \frac{1}{6}N(a,b) \le \frac{1}{b-a}\int_a^b f(x)g(x)dx \le 8f\left(\frac{a+b}{2}\right)g\left(\frac{a+b}{2}\right) - \frac{1}{3}N(a,b) - \frac{1}{6}M(a,b).$$

Theorem 2.9. Let $f, g : [a,b] \to \mathbb{R}$, and fg be integrable on $[a,b]$. If $f \in K^4_{s_1}$ and $g \in K^4_{s_2}$, then

$$\begin{aligned}\frac{1}{2}\int_a^b\int_a^b\int_0^1 f(tx+(1-t)y)g(tx+(1-t)y)dtdydx &\ge \frac{(b-a)s_1s_2}{s_1+s_1s_2+s_2}\int_a^b f(x)g(x)dx \\ &+ \beta\left(1+\frac{1}{s_1},1+\frac{1}{s_2}\right)\frac{(b-a)^2}{(1+s_1)(1+s_2)}\left[M(a,b)+N(a,b)\right].\end{aligned} \tag{2.8}$$

Proof . Since f is s_1-convex in the fourth sense and g is s_2-convex in the fourth sense on $[a,b]$, we get

$$\begin{aligned}f(tx+(1-t)y)\,g(tx+(1-t)y) &\ge t^{\frac{1}{s_1}+\frac{1}{s_2}}f(x)g(x) + (1-t)^{\frac{1}{s_1}+\frac{1}{s_2}}f(y)g(y) \\ &+ t^{\frac{1}{s_1}}(1-t)^{\frac{1}{s_2}}f(x)g(y) + t^{\frac{1}{s_2}}(1-t)^{\frac{1}{s_1}}f(y)g(x).\end{aligned}$$

We integrate both sides of this inequality over $[0,1]$ and obtain

$$\begin{aligned}\int_0^1 f(tx+(1-t)y)\,g(tx+(1-t)y)dt &\ge \frac{s_1s_2}{s_1+s_1s_2+s_2}\left[f(x)g(x)+f(y)g(y)\right] \\ &+ \beta(1+\frac{1}{s_1},1+\frac{1}{s_2})\left[f(x)g(y)+f(y)g(x)\right].\end{aligned}$$

We integrate both sides of this inequality on $[a,b]\times[a,b]$ and obtain

$$\begin{aligned}\int_a^b\int_a^b\int_0^1 f(tx+(1-t)y)\,g(tx+(1-t)y)dtdydx &\ge \frac{s_1s_2}{s_1+s_1s_2+s_2}\int_a^b\int_a^b\left[f(x)g(x)+f(y)g(y)\right]dydx \\ &+ \beta\left(1+\frac{1}{s_1},1+\frac{1}{s_2}\right)\int_a^b\int_a^b\left[f(x)g(y)+f(y)g(x)\right]dydx\end{aligned}$$

$$= \frac{s_1 s_2}{s_1 + s_1 s_2 + s_2}(b-a)\left(\int_a^b f(x)g(x)dx + \int_a^b f(y)g(y)dy\right)$$
$$+ \beta\left(1+\frac{1}{s_1}, 1+\frac{1}{s_2}\right)\left(\left(\int_a^b f(x)dx\right)\left(\int_a^b g(y)dy\right)\right.$$
$$\left. + \left(\int_a^b f(y)dy\right)\left(\int_a^b g(x)dx\right)\right). \tag{2.9}$$

From inequality (1.3), we have the following inequalities,

$$\frac{1}{b-a}\int_a^b f(x)dx \leq \frac{s_1}{s_1+1}\left[f(a)+f(b)\right], \quad \frac{1}{b-a}\int_a^b g(x)dx \leq \frac{s_2}{s_2+1}\left[f(a)+f(b)\right].$$

In inequality (2.9), using the above inequalities the following inequality is obtained

$$\int_a^b \int_a^b \int_0^1 f\left(tx+(1-t)y\right) g(tx+(1-t)y)dtdydx$$
$$\geq \frac{2(b-a)s_1 s_2}{s_1 + s_1 s_2 + s_2}\int_a^b f(x)g(x)dx + \beta\left(1+\frac{1}{s_1}, 1+\frac{1}{s_2}\right)$$
$$\left((b-a)^2 \frac{s_1 s_2}{(1+s_1)(1+s_2)}\left(f\left(a\right)+f(b)\right)\left(g(a)+g(b)\right)\right.$$
$$\left. + (b-a)^2 \frac{s_1 s_2}{(1+s_1)(1+s_2)}\left(f\left(a\right)+f(b)\right)\left(g(a)+g(b)\right)\right)$$
$$= \frac{2(b-a)}{s_1 + s_1 s_2 + s_2}\int_a^b f(x)g(x)dx + \beta\left(1+\frac{1}{s_1}, 1+\frac{1}{s_2}\right)\frac{2\left(b-a\right)^2 s_1 s_2}{(1+s_1)(1+s_2)}$$
$$\times \left(f\left(a\right)g(a) + f(b)g(b) + f\left(a\right)g(b) + f(b)g(a)\right) .$$

□

Corollary 2.10. If we choose $s_1 = s_2 = 1$, then inequality (2.8) reduces to the following inequality

$$\int_a^b \int_a^b \int_0^1 f\left(tx+(1-t)y\right) g(tx+(1-t)y)dtdydx \geq \frac{2\left(b-a\right)}{3}\int_a^b f(x)g(x)dx + \frac{\left(b-a\right)^2}{12}\left(M(a,b)+N(a,b)\right).$$

Theorem 2.11. Let $f, g : [a,b] \to \mathbb{R}$, and fg be integrable on $[a,b]$. If $f \in K^4_{s_1}$ and $g \in K^4_{s_2}$, then

$$\int_a^b \int_0^1 f\left(tx+(1-t)\frac{a+b}{2}\right) g\left(tx+(1-t)\frac{a+b}{2}\right)dtdx \geq \frac{s_1 s_2}{(s_1 + s_1 s_2 + s_2)}\int_a^b f(x)g(x)dx$$
$$+ \left[M(a,b)+N(a,b)\right]\left\{\beta(1+\frac{1}{s_1}, 1+\frac{1}{s_2})\right.$$
$$\left(\frac{s_1}{1+s_1}2^{\frac{s_2-1}{s_2}} + \frac{s_2}{1+s_2}2^{\frac{s_1-1}{s_1}}\right)$$
$$\left. + \left(b-a\right)2^{\frac{2s_1 s_2 - s_2 - s_1}{s_1 s_2}}\frac{s_1 s_2}{s_1 + s_1 s_2 + s_2}\right\} \tag{2.10}$$

Proof . Using convexity of f and g, we have

$$f\left(tx+(1-t)\frac{a+b}{2}\right)g\left(tx+(1-t)\frac{a+b}{2}\right)\geq t^{\frac{1}{s_1}+\frac{1}{s_2}}f(x)g(x)+(1-t)^{\frac{1}{s_1}+\frac{1}{s_2}}f\left(\frac{a+b}{2}\right)g\left(\frac{a+b}{2}\right)$$

$$+t^{\frac{1}{s_1}}(1-t)^{\frac{1}{s_2}}f(x)g\left(\frac{a+b}{2}\right)+t^{\frac{1}{s_2}}(1-t)^{\frac{1}{s_1}}f\left(\frac{a+b}{2}\right)g(x).$$

We integrate both side of this inequality over $[0,1]$ and obtain

$$\int_0^1 f\left(tx+(1-t)\frac{a+b}{2}\right)g\left(tx+(1-t)\frac{a+b}{2}\right)dt\geq$$

$$\geq\frac{s_1s_2}{s_1+s_1s_2+s_2}f(x)g(x)+\frac{s_1s_2}{s_1+s_1s_2+s_2}f\left(\frac{a+b}{2}\right)g\left(\frac{a+b}{2}\right)$$

$$+\beta\left(1+\frac{1}{s_1},1+\frac{1}{s_2}\right)f\left(\frac{a+b}{2}\right)g(x)+\beta\left(1+\frac{1}{s_2},1+\frac{1}{s_1}\right)g\left(\frac{a+b}{2}\right)f\left(x\right).$$

Now integrating both sides of this inequality over $[a,b]$, using the inequality (1.3), the s-convexity in the fourth sense of f, g we observe that

$$\int_a^b\int_0^1 f\left(tx+(1-t)\frac{a+b}{2}\right)g\left(tx+(1-t)\frac{a+b}{2}\right)dtdx$$

$$\geq\frac{s_1s_2}{s_1+s_1s_2+s_2}\left(\int_a^b f(x)g(x)dx+f\left(\frac{a+b}{2}\right)g\left(\frac{a+b}{2}\right)(b-a)\right)$$

$$+\beta\left(1+\frac{1}{s_1},1+\frac{1}{s_2}\right)\left(f\left(\frac{a+b}{2}\right)\int_a^b g(x)dx+g\left(\frac{a+b}{2}\right)\int_a^b f(x)dx\right)$$

$$\geq\frac{s_1s_2}{s_1+s_1s_2+s_2}\left(\int_a^b f(x)g(x)dx+f\left(\frac{a+b}{2}\right)g\left(\frac{a+b}{2}\right)(b-a)\right)$$

$$+\beta\left(1+\frac{1}{s_1},1+\frac{1}{s_2}\right)\frac{s_1}{1+s_1}g\left(\frac{a+b}{2}\right)(f(a)+f(b))$$

$$+\beta\left(1+\frac{1}{s_2},1+\frac{1}{s_1}\right)\frac{s_2}{1+s_2}f\left(\frac{a+b}{2}\right)(g(a)+g(b))$$

$$=\frac{s_1s_2}{s_1+s_1s_2+s_2}\left(\int_a^b f(x)g(x)dx+(b-a)\,2^{\frac{2s_1s_2-s_2-s_1}{s_1s_2}}\left(\left(f(a)+f(b)\right)\left(g(a)+g(b)\right)\right)\right)$$

$$+\beta(1+\frac{1}{s_1},1+\frac{1}{s_2})\left(\frac{s_1}{1+s_1}2^{\frac{s_2-1}{s_2}}\left(\left(f(a)+f(b)\right)\left(g(a)+g(b)\right)\right)\right.$$

$$\left.+\frac{s_2}{1+s_2}2^{\frac{s_1-1}{s_1}}\left(\left(f(a)+f(b)\right)\left(g(a)+g(b)\right)\right)\right)$$

$$=\frac{s_1s_2}{s_1+s_1s_2+s_2}\int_a^b f(x)g(x)dx$$

$$+\left[M(a,b)+N(a,b)\right]\left[\beta(1+\frac{1}{s_1},1+\frac{1}{s_2})\left(\frac{s_1}{1+s_1}2^{\frac{s_2-1}{s_2}}+\frac{s_2}{1+s_2}2^{\frac{s_1-1}{s_1}}\right)\right.$$

$$\left.+\,(b-a)\,\frac{s_1s_2}{s_1+s_1s_2+s_2}2^{\frac{2s_1s_2-s_2-s_1}{s_1s_2}}\right].$$

□

Remark 2.12. In Theorem 2.11, if we choose $s_1 = s_2 = 1$, then inequality (2.10) reduces to the following inequality

$$\int_a^b \int_0^1 f\left(tx + (1-t)\frac{a+b}{2}\right) g\left(tx + (1-t)\frac{a+b}{2}\right) dtdx \geq \frac{1}{3}\int_a^b f(x)g(x)dx + \frac{2(b-a)+1}{6}\left[M(a,b) + N(a,b)\right].$$

Theorem 2.13. Let $f, g : [a, b] \to \mathbb{R}$, and fg be integrable on $[a, b]$. If $f \in K^4_{s_1}$ and $g \in K^4_{s_2}$, then

$$\begin{aligned}
&\frac{s_1 s_2}{s_1 + s_1 s_2 + s_2} M(a,b) + \beta\left(\frac{1}{s_1} + 1, \frac{1}{s_2} + 1\right) N(a,b) + \frac{1}{b-a}\int_a^b f(x)g(x)dx \\
&\geq \frac{1}{(b-a)^{\frac{1+s_1}{s_1}}}\int_a^b \left[(b-x)^{\frac{1}{s_1}} f(a) + (x-a)^{\frac{1}{s_2}} f(b)\right] g(x)dx \\
&+ \frac{1}{(b-a)^{\frac{1+s_2}{s_2}}}\int_a^b \left[(b-x)^{\frac{1}{s_1}} g(a) + (x-a)^{\frac{1}{s_2}} g(b)\right] f(x)dx. \qquad (2.11)
\end{aligned}$$

Proof . Since $f \in K^4_{s_1}$, $g \in K^4_{s_2}$, we have the following inequalities,

$$t^{\frac{1}{s_1}} f(a) + (1-t)^{\frac{1}{s_1}} f(b) - f(ta + (1-t)b) \geq 0$$

and

$$t^{\frac{1}{s_2}} g(a) + (1-t)^{\frac{1}{s_2}} g(b) - g(ta + (1-t)b) \geq 0.$$

If we multiply two inequalities side to side, then we get the following inequality,

$$\begin{aligned}
&\left(t^{\frac{1}{s_1}} f(a) + (1-t)^{\frac{1}{s_1}} f(b)\right)\left(t^{\frac{1}{s_2}} g(a) + (1-t)^{\frac{1}{s_2}} g(b)\right) + f(ta + (1-t)b)g(ta + (1-t)b) \\
&\geq \left(t^{\frac{1}{s_1}} f(a) + (1-t)^{\frac{1}{s_1}} f(b)\right) g(ta + (1-t)b) + \left(t^{\frac{1}{s_2}} g(a) + (1-t)^{\frac{1}{s_2}} g(b)\right) f(ta + (1-t)b).
\end{aligned}$$

Integrating the above inequality with respect to t over $[0, 1]$, we get

$$\begin{aligned}
&\int_0^1 \left(t^{\frac{1}{s_1}} f(a) + (1-t)^{\frac{1}{s_1}} f(b)\right)\left(t^{\frac{1}{s_2}} g(a) + (1-t)^{\frac{1}{s_2}} g(b)\right) dt + \int_0^1 f(ta + (1-t)b)g(ta + (1-t)b)dt \\
&\geq \int_0^1 \left(t^{\frac{1}{s_1}} f(a) + (1-t)^{\frac{1}{s_1}} f(b)\right) g(ta + (1-t)b)dt + \int_0^1 \left(t^{\frac{1}{s_2}} g(a) + (1-t)^{\frac{1}{s_2}} g(b)\right) f(ta + (1-t)b)dt. \qquad (2.12)
\end{aligned}$$

Directly computing, we obtain

$$\begin{aligned}
&\int_0^1 \left(t^{\frac{1}{s_1}} f(a) + (1-t)^{\frac{1}{s_1}} f(b)\right)\left(t^{\frac{1}{s_2}} g(a) + (1-t)^{\frac{1}{s_2}} g(b)\right) dt \\
&= \int_0^1 \left(t^{\frac{1}{s_1}+\frac{1}{s_2}} f(a)g(a) + (1-t)^{\frac{1}{s_1}+\frac{1}{s_2}} f(b)g(b)\right) dt \\
&+ \int_0^1 \left(t^{\frac{1}{s_1}}(1-t)^{\frac{1}{s_2}} f(a)g(b) + (1-t)^{\frac{1}{s_1}} t^{\frac{1}{s_2}} f(b)g(a)\right) dt
\end{aligned}$$

$$
\begin{aligned}
&= \frac{s_1 s_2}{s_1 + s_1 s_2 + s_2} \left(f(a)g(a) + f(b)g(b)\right) \\
&+ \beta(1 + \frac{1}{s_1}, 1 + \frac{1}{s_2}) \left(f(a)g(b) + f(b)g(a)\right) \\
&= \frac{s_1 s_2}{s_1 + s_1 s_2 + s_2} M(a,b) + \beta(1 + \frac{1}{s_1}, 1 + \frac{1}{s_2}) N(a,b)
\end{aligned} \tag{2.13}
$$

and

$$
\begin{aligned}
&\int_0^1 \left(t^{\frac{1}{s_1}} f(a) + (1-t)^{\frac{1}{s_1}} f(b)\right) g(ta + (1-t)b)dt \\
&= \frac{1}{b-a} \int_a^b \left(\left(\frac{b-x}{b-a}\right)^{\frac{1}{s_1}} f(a) + \left(\frac{x-a}{b-a}\right)^{\frac{1}{s_1}} f(b)\right) g(x)dx \\
&= \frac{1}{(b-a)^{\frac{1+s_1}{s_1}}} \int_a^b \left((b-x)^{\frac{1}{s_1}} f(a) + (x-a)^{\frac{1}{s_1}} f(b)\right) g(x)dx,
\end{aligned} \tag{2.14}
$$

and

$$
\begin{aligned}
&\int_0^1 \left(t^{\frac{1}{s_2}} g(a) + (1-t)^{\frac{1}{s_2}} g(b)\right) f(ta + (1-t)b)dt \\
&= \frac{1}{b-a} \int_a^b \left(\left(\frac{b-x}{b-a}\right)^{\frac{1}{s_2}} g(a) + \left(\frac{x-a}{b-a}\right)^{\frac{1}{s_2}} g(b)\right) f(x)dx \\
&= \frac{1}{(b-a)^{\frac{1+s_2}{s_2}}} \int_a^b \left((b-x)^{\frac{1}{s_2}} g(a) + (x-a)^{\frac{1}{s_2}} g(b)\right) f(x)dx,
\end{aligned} \tag{2.15}
$$

Substituting (2.3), (2.13), (2.14), (2.15) in (2.12), we get the desired result. □

Corollary 2.14. If we choose $s_1 = s_2 = 1$, then inequality (2.11) reduces to the following inequality

$$
\begin{aligned}
\frac{1}{3}M(a,b) + \frac{1}{6}N(a,b) + \frac{1}{b-a}\int_a^b f(x)g(x)dx &\geq \frac{1}{(b-a)^2}\int_a^b \left((b-x)f(a) + (x-a)f(b)\right) g(x)dx \\
&+ \frac{1}{(b-a)^2}\int_a^b \left((b-x)g(a) + (x-a)g(b)\right) f(x)dx.
\end{aligned}
$$

3 Applications

We consider the application of our theorems to special means as well as beta and digamma functions. Let us recall the following means for positive real numbers a, b.

Let a, b, p be positive number with $a \neq b$ and $p \neq 1$,

$$
\begin{aligned}
A(a,b) &= \frac{a+b}{2}, \\
M_p(a,b) &= \left(\frac{a^p + b^p}{2}\right)^{\frac{1}{p}}, \\
L_p(a,b) &= \begin{cases} a & , \quad \text{if } a = b \\ \left(\frac{a^p - b^p}{p(a-b)}\right)^{1/(p-1)} & , \quad \text{else} \end{cases}
\end{aligned}
$$

are called Arithmetic mean, Power mean and Stolarsky mean (Generalized logarithmic mean) respectively.

Proposition 3.1. Let $a, b \in \mathbb{R}_+$ with $a < b$. The inequality holds:

$$\left[L_{\frac{s_1+s_1s_2+s_2}{s_1s_2}}(a,b)\right]^{\frac{s_1s_2}{s_1+s_2}} \geq \frac{2s_1s_2}{s_1+s_1s_2+s_2}\left[M_{\frac{1}{s_1}+\frac{1}{s_2}}(a,b)\right]^{\frac{1}{s_1}+\frac{1}{s_2}} + 2\beta\left(1+\frac{1}{s_1}, 1+\frac{1}{s_2}\right)G^{\frac{2}{s_1}}(a,b)\left[M_{\frac{1}{s_2}-\frac{1}{s_1}}(a,b)\right]^{\frac{1}{s_2}-\frac{1}{s_1}}.$$

Proof . The assertion follows from Theorem 2.1 applied to s-convex functions in the fourth sense $f(x) = -x^{\frac{1}{s_1}}$ and $g(x) = -x^{\frac{1}{s_2}}$, $x \in [a,b]$

$$\frac{1}{b-a}\int_a^b \left(-x^{\frac{1}{s_1}}\right)\left(-x^{\frac{1}{s_2}}\right)dx \geq \frac{s_1s_2}{s_1+s_1s_2+s_2}\left(\left(-a^{\frac{1}{s_1}}\right)\left(-a^{\frac{1}{s_2}}\right)+\left(-b^{\frac{1}{s_1}}\right)\left(-b^{\frac{1}{s_2}}\right)\right) + \beta\left(\frac{1}{s_1}+1, \frac{1}{s_2}+1\right)\left(\left(-a^{\frac{1}{s_1}}\right)\left(-b^{\frac{1}{s_2}}\right)+\left(-a^{\frac{1}{s_2}}\right)\left(-b^{\frac{1}{s_1}}\right)\right).$$

After simple operations, the following inequality is obtained,

$$\frac{\left(b^{\frac{1}{s_1}+\frac{1}{s_2}+1}-a^{\frac{1}{s_1}+\frac{1}{s_2}+1}\right)}{(b-a)\left(\frac{1}{s_1}+\frac{1}{s_2}+1\right)} \geq \frac{s_1s_2}{s_1+s_1s_2+s_2}\left(a^{\frac{1}{s_1}+\frac{1}{s_2}}+b^{\frac{1}{s_1}+\frac{1}{s_2}}\right) + \beta\left(1+\frac{1}{s_1}, 1+\frac{1}{s_2}\right)a^{\frac{1}{s_1}}b^{\frac{1}{s_1}}\left(b^{\frac{1}{s_2}-\frac{1}{s_1}}+a^{\frac{1}{s_2}-\frac{1}{s_1}}\right). \tag{3.1}$$

□

Corollary 3.2. If we choose $s_1 = s_2 = 1$, then inequality (3.1) reduces to the following inequality,

$$[L_3(a,b)]^{\frac{1}{2}} \geq \frac{1}{3}M(a,b) + \frac{1}{6}N(a,b),$$

where $M(a,b)$ and $N(a,b)$ are the same as in Theorem 2.1.

By means of Theorem 2.4, we can find an symmetric upper bound function for beta function.

Proposition 3.3. For $u, v \geq 1$,

$$\beta(u,v) \leq \frac{u+v}{2uv}\left(\frac{\left(2^{u+v+1}-1\right)(v+1)(u+1)(u+v+1)}{(u+v+2)(2u+2v+uv+2)} - 2\right).$$

Proof . Assuming $f(x) = -x^{\frac{1}{s_1}}$ and $g(x) = -x^{\frac{1}{s_2}}$, we apply Theorem 2.4 on $[t,1]$ with $0 < t < 1$ and have

$$\left(\frac{1+t}{2}\right)^{\frac{1}{s_1}}\left(\frac{1+t}{2}\right)^{\frac{1}{s_2}} \geq \left(\frac{1}{2}\right)^{\frac{s_1-s_1s_2+s_2}{s_1s_2}}\left(1+t^{\frac{1}{s_1}}+t^{\frac{1}{s_2}}+t^{\frac{1}{s_1}+\frac{1}{s_2}}\right)\left(\frac{s_1s_2}{s_1+s_1s_2+s_2}+\beta\left(1+\frac{1}{s_1}, 1+\frac{1}{s_2}\right)\right).$$

Integrating both sides with respect to t on $[0,1]$, then, making some simplifications and using symmetry and following property of beta function

$$\beta(u+1,v) = \frac{u}{u+v}\beta(u,v),$$

we have

$$\frac{(s_1+1)(s_2+1)\left(2^{\frac{1}{s_1}+\frac{1}{s_2}+1}-1\right)}{2(s_1+s_2+2s_1s_2)(2s_1+2s_2+2s_1s_2+1)} \geq \frac{s_1s_2}{s_1+s_1s_2+s_2} + \frac{1}{(s_1+s_2)(s_1+s_1s_2+s_2)}\beta\left(\frac{1}{s_1}, \frac{1}{s_2}\right).$$

Replacing $\frac{1}{s_1} = u$ and $\frac{1}{s_2} = v$ yields to

$$\beta(u,v) \leq \frac{u+v}{2uv}\left(\frac{\left(2^{u+v+1}-1\right)(v+1)(u+1)(u+v+1)}{(u+v+2)(2u+2v+uv+2)} - 2\right) \tag{3.2}$$

for $u, v \geq 1$. □

Corollary 3.4. In Proposition 3.3, if we choose $u = v$, then inequality (3.2) reduces to the following inequality,

$$\beta(u,u) \leq \frac{1}{u}\left(\frac{\left(2^{2u+1}-1\right)(u+1)(2u+1)}{2\left(4u+u^2+2\right)} - 2\right).$$

Also, using the Theorem 2.4 given in main results, we can obtain some inequalities involving digamma function.

Proposition 3.5. Let $u, v > 1$. Then

$$\Psi(u+v) + \gamma \geq \frac{1}{u+v-1} + \beta(u,v)(u+v-1)\left(\frac{1}{u} + \frac{1}{v}\right)$$

where $\Psi(u)$ is digamma function, i.e.

$$\Psi(u) = \frac{\Gamma'(u)}{\Gamma(u)} \text{ for } u > 0$$

and γ is Euler-Mascheroni constant i.e. $\gamma \approx 0.5772156649....$ Moreover,

$$\Psi(u) + \gamma \geq \frac{1}{u} + \frac{4(u-1)}{u}\beta\left(\frac{u}{2}, \frac{u}{2}\right) + 1$$

for $u \geq 4$.

Proof . Let us write $t = \frac{a}{b}$ and simplify the expression in (3.1). Then we have

$$\begin{aligned}\frac{s_1 s_2}{s_1 + s_1 s_2 + s_2}\frac{1 - t^{\frac{1}{s_1}+\frac{1}{s_2}+1}}{1-t} \geq{}& \frac{s_1 s_2}{s_1 + s_1 s_2 + s_2}\left(-t^{\frac{1}{s_1}}\right)\left(-t^{\frac{1}{s_2}}\right) + 1 \\ &+ \beta\left(1 + \frac{1}{s_1}, 1 + \frac{1}{s_2}\right)\left(t^{\frac{1}{s_1}} + t^{\frac{1}{s_2}}\right).\end{aligned}$$

Let us integrate the expression with respect to t on $[0, 1]$,

$$\begin{aligned}\int_0^1 \frac{1 - t^{\frac{1}{s_1}+\frac{1}{s_2}+1}}{1-t}dt \geq{}& \int_0^1 \left(t^{\frac{1}{s_1}+\frac{1}{s_2}} + 1\right)dt \\ &+ \beta\left(1 + \frac{1}{s_1}, 1 + \frac{1}{s_2}\right)\frac{s_1 + s_1 s_2 + s_2}{s_1 s_2}\int_0^1 \left(t^{\frac{1}{s_1}} + t^{\frac{1}{s_2}}\right)dt,\end{aligned}$$

hence, this inequality is obtained

$$\begin{aligned}\int_0^1 \frac{1 - t^{\frac{1}{s_1}+\frac{1}{s_2}+1}}{1-t}dt \geq{}& \frac{s_1 s_2}{s_1 + s_1 s_2 + s_2} + 1 \\ &+ \beta\left(1 + \frac{1}{s_1}, 1 + \frac{1}{s_2}\right)\frac{s_1 + s_1 s_2 + s_2}{s_1 s_2}\left(\frac{s_1}{1+s_1} + \frac{s_2}{1+s_2}\right).\end{aligned}$$

Using the following integral representation of digamma function

$$\Psi(r) = \int_0^1 \frac{1 - t^{r-1}}{1-t}dt - \gamma$$

where $r > 0$. We have

$$\begin{aligned}\Psi\left(2 + \frac{1}{s_1} + \frac{1}{s_2}\right) + \gamma &= \int_0^1 \frac{1 - t^{\frac{1}{s_1}+\frac{1}{s_2}+1}}{1-t}dt \\ &\geq \frac{s_1 s_2}{s_1 + s_1 s_2 + s_2} + \beta\left(1 + \frac{1}{s_1}, 1 + \frac{1}{s_2}\right)\frac{s_1 + s_1 s_2 + s_2}{s_1 s_2}\left(\frac{s_1}{1+s_1} + \frac{s_2}{1+s_2}\right) + 1.\end{aligned}$$

The substitution $u = 1 + \frac{1}{s_1}$ and $v = 1 + \frac{1}{s_2}$ above yields to

$$\Psi(u+v) + \gamma \geq \frac{1}{u+v-1} + \beta(u,v)(u+v-1)\left(\frac{1}{u} + \frac{1}{v}\right) + 1$$

for $u, v \geq 2$.
In case $s_1 = s_2 = s$, we have $u = v$. Then

$$\Psi(2u) + \gamma \geq \frac{1+2u}{2u} + \beta(u,u)\,\frac{2(2u-1)}{u}$$

for $u \geq 1$. Changing $2u$ to u gives the desired result. □

For more information about the Digamma function, see [1][17].

4 Conclusion

In this study, by taking two functions from class K_s^4, integral inequalities containing their products are obtained. While obtaining these inequalities, the convex properties of the functions and Hermite-Hadamard inequality are used. In addition, an upper bound function for the Beta function and a lower bound function for the Digamma function were obtained.

Researchers interested in this topic can do a similar study for different function classes.

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